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GENERALIZED POINTWISE SYMMETRIC SPACES

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Abstract: In this paper we give an example of a Riemannian s-manifold (with a discontinuous s-structure) which does not admit any regular s-structure in the sense of A.J. Ledger (x).

Key words: Homogeneous manifolds, Riemannian manifolds, symmetric spaces.

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1. Introduction. Let (M,g) be a differentiable Riemannian manifold. An isometry s_x of (M,g) for which $x \in M$ is an isolated fixed point is called a symmetry of M at x, ([7]). An s-structure on (M,g) is a family $\{s_x : x \in M\}$ of symmetries of (M,g) (one symmetry at each point). Here the map $s: M \longrightarrow I(M)$ need not be even continuous. According to a theorem by F. Brickel, if (M,g) admits an s-structure, then the group $I(M)$ of isometries is transitive ([7]), and thus M is a homogeneous Riemannian manifold.

An s-structure $\{s_x\}$ is called regular if for every two points $x, y \in M$

(x) I wish to thank to A.W. Deicke, who provided the basic "model", and also to A. Gray and H. Samelson, who kindly answered my questions concerning the transformation groups on spheres.

$$s_x \circ s_y = s_z \circ s_x \ , \quad z = s_x(y) \qquad \text{(Cf. [3]).}$$

If $\{s_x\}$ is regular, then the map $s: M \longrightarrow I(M)$ is always differentiable (cf. [5], Theorem 1).

An s-structure $\{s_x\}$ is called <u>of order</u> k if $(s_x)^k =$ = identity for all $x \in M$, and k is the least integer of this property. Following A.W. Deicke, if (M,g) admits an s-structure, then it always admits an s-structure of finite order. Further, if (M,g) admits a regular s-structure then (M,g) admits a regular s-structure of finite order. (Cf.[5], Lemma 3 and Theorem 2).

<u>A generalized symmetric Riemannian space</u> is a Riemannian manifold (M,g) admitting a regular s-structure (cf [5]). Now, we shall introduce a more general

<u>Definition</u>. <u>A generalized pointwise symmetric Riemannian space</u> is a Riemannian manifold (M,g) admitting an s-structure.

<u>Order</u> of a generalized symmetric (or generalized pointwise symmetric) Riemannian space (M,g) is the minimum order of a regular s-structure on (M,g) (or the minimum order of an s-structure on (M,g), respectively).

It is easy to show that a generalized pointwise symmetric Riemannian space of order 2 is a usual Riemannian (globally) symmetric space. Moreover, the canonical s-structure consisting of geodesic symmetries is always regular (see [3]). Thus, for order 2, the concepts "pointwise symmetric" and "symmetric" are equivalent.

The existence of generalized symmetric Riemannian spa-

ces of order greater than two is shown in [7], and many examples of such spaces (of orders 3, 4 and 6) are given in [4] and [6].

The purpose of this paper is to present a family of generalized pointwise symmetric Riemannian spaces which are <u>not</u> generalized symmetric. This example seems to be non-trivial as it uses the classification of compact connected Lie groups acting transitively and effectively on spheres, due to D. Montgomery, H. Samelson and A. Borel.

2. <u>The main theorem</u>. Consider the Hermitean manifold $(C^{2n+1}[z^1,\dots,z^{n+1}], g_\lambda)$ with the metric $g_\lambda = \sum_{i=1}^{2n+1} dz^i\, d\bar{z}^i + \lambda \left(\sum_{i=1}^{2n+1} z^i\, d\bar{z}^i\right)\left(\sum_{j=1}^{2n+1} \bar{z}^j\, dz^j\right)$ where $\lambda \neq 0$, $\lambda > -1$ is a constant. Let us consider the sphere S^{4n+1} defined by $\sum_{i=1}^{2n+1} z^i \bar{z}^i = 1$, and the real Riemannian metric $\hat{g}_\lambda$ on S^{4n+1} induced by g_λ. (Here the real coordinates are introduced putting $z^j = x^j + iy^j$, $j = 1,\dots,2n+1$.)

<u>Theorem</u>. <u>For</u> $n \geq 2$, <u>the Riemannian manifold</u> $(S^{4n+1}, \hat{g}_\lambda)$ <u>is generalized pointwise symmetric of order</u> 4 <u>but it is not generalized symmetric</u>.

<u>Proof</u>. Let us define the <u>origin</u> of S^{4n+1} to be the point $o = (0,\dots,0,1)$ of C^{2n+1}. The transformation of C^{2n+1} given by $(z^{2i-1})' = -\bar{z}^{2i}$, $(z^{2i})' = \bar{z}^{2i-1}$ $(i = 1,\dots,n)$, $(z^{2n+1})' = \bar{z}^{2n+1}$, induces a transformation $\tilde{s}_o$ of S^{4n+1} with a fixed point o. Clearly, $\tilde{s}_o$ is an isometry of $(S^{4n+1}, \hat{g}_\lambda)$. We can see easily that the tangent map

$(\tilde{s}_o)_{*o}$ has no nonzero fixed vectors in the tangent space $(S^{4n+1})_o$, and hence o is an isolated fixed point of $\tilde{s}_o$. Moreover, we have $(\tilde{s}_o)^4 = \text{identity}$.

The group $U(2n + 1)$ of all unitary transformations of C^{2n+1} (with respect to its natural structure of a linear Hermitean space) preserves the metric g_λ and it acts transitively and effectively on S^{4n+1}. Thus $U(2n + 1)$ can be considered as a group of isometries of the Riemannian manifold $(S^{4n+1}, \hat{g}_\lambda)$.

Define an isometry $\tilde{s}_x$ of $(S^{4n+1}, \hat{g}_\lambda)$ for every $x \in S^{4n+1}$ as follows: let $A \in U(2n + 1)$ be such that $A(0) = x$, and put $\tilde{s}_x = A \circ \tilde{s}_o \circ A^{-1}$. (The transformation $\tilde{s}_x$ depends, in general, on the choice of A). Then x is an isolated fixed point of $\tilde{s}_x$. Thus $(S^{4n+1}, \hat{g}_\lambda)$ is a generalized pointwise symmetric space. (This example was pointed out by A.W. Deicke.)

Let us remark that $(S^{4n+1}, \hat{g}_\lambda)$ is not locally symmetric and it is of odd dimension. Thus, the order of the space cannot be 2 or 3 and hence $k = 4$.

We shall now prove the second part of the Theorem. In the following, $SO(4n + 2)$, $U(2n + 1)$ and $SU(2n + 1)$ will always denote the transformation groups of S^{4n+1} which are induced by the corresponding transformation groups of the given real space R^{4n+2} and of the complex space C^{2n+1}.

Lemma. Let K be a connected group of isometries of $(S^{4n+1}, \hat{g}_\lambda)$ acting transitively on S^{4n+1}. Then

$K \supseteq SU(2n+1)$.

Proof. According to Montgomery - Samelson [8], and Borel [1],[2], each compact connected Lie transformation group acting transitively on S^{4n+1} is isomorphic to one of the following groups: $SO(4n+2)$, $U(2n+1)$, $SU(2n+1)$. Let G be the component of unity of the full isometry group $I(S^{4n+1}, \hat{g}_\lambda)$, then $G \supseteq U(2n+1)$. G cannot be isomorphic to $SO(4n+2)$; otherwise $\hat{g}_\lambda$ would be a metric of constant curvature. Thus $G = U(2n+1)$.

Let K be an arbitrary connected and transitive group of isometries of $(S^{4n+1}, \hat{g}_\lambda)$; then $K \subseteq U(2n+1)$. If K is isomorphic to $U(2n+1)$, then $K = U(2n+1)$ and Lemma is proved. Let now K be isomorphic to $SU(2n+1)$. Then the Lie algebra $\underline{k}$ is isomorphic to $\underline{su}(2n+1)$, and $\underline{k} \subset \underline{u}(2n+1)$. On the other hand, we have $\underline{u}(2n+1) = \underline{su}(2n+1) \oplus \mathbb{R}$ (direct sum), and the subalgebra $\underline{su}(2n+1)$ is simple. Hence it follows $\underline{k} = \underline{su}(2n+1)$, and consequently, $K = SU(2n+1)$. This completes the proof.

Let now $\{s_x\}$ be a regular s-structure on $(S^{4n+1}, \hat{g}_\lambda)$, and let K denote the component of unity of the automorphism group of the Riemannian s-manifold $(S^{4n+1}, \hat{g}_\lambda, \{s_x\})$. (Here, by automorphisms we mean isometries $A \in G$ such that $A\, s_x = s_{A(x)} \circ A$ for all $x \in M$.) According to [3], Theorem 5.6, K is a closed subgroup of G acting transitively on M . According to the Lemma, $K \supseteq SU(2n+1)$. For the stability group K_o of K at the origin o we have $K_o \supseteq SU(2n)$ (= the subgroup of $SU(2n+1)$ leaving all points $(0,\ldots$

$\ldots,0,e^{ig})$ of S^{4n+1} fixed). The transformation s_o commutes with each element of K_o and particularly, it commutes with each element of $SU(2n)$.

Consider the tangent space $(S^{4n+1})_o$. It is generated by the vectors

$e_i = \left(\frac{\partial}{\partial x^i}\right)_o$, $f_j = \left(\frac{\partial}{\partial y^j}\right)_o$, where $i = 1,\ldots, 2n$, $j = 1,\ldots, 2n + 1$..

Here f_{2n+1} is orthogonal to the 4n-dimensional subspace V generated by e_i , f_i for $i = 1,\ldots, 2n$.

Let H denote the real isotropy representation of $SU(2n)$ in the tangent space $(S^{4n+1})_o$, and $S_o = (s_{o*})_o$. All linear transformations $h \in H$, and also S_o , are orthogonal transformations of $(S^{4n+1})_o$ with respect to the scalar product $(\hat{g}_\lambda)_o$. H acts transitively on the subspace V , and all fixed vectors with respect to H are of the form λf_{2n+1} . S_o commutes with each $h \in H$ and hence $S_o(f_{2n+1})$ is a fixed vector with respect to H . Thus $S_o(f_{2n+1}) = \pm f_{2n+1}$, and since S_o does not admit non-zero fixed vectors, $S_o(f_{2n+1}) = - f_{2n+1}$. Also, the subspace V is invariant with respect to S_o .

Let $\underline{h}$ denote the Lie algebra of H . For every pair (r,s) , $1 \leq r \neq s \leq 2n$, consider the endomorphisms B_{rs} , $C_{rs} \in \underline{h}$ defined as follows:

$B_{rs}(e_r) = e_s$, $B_{rs}(f_r) = f_s$, $B_{rs}(e_s) = - e_r$, $B_{rs}(f_s) = - f_r$,

$C_{rs}(e_r) = - f_s$, $C_{rs}(f_r) = e_s$, $C_{rs}(e_s) = - f_r$, $C_{rs}(f_s) = c_r$,

$$B_{rs}(e_i) = B_{rs}(f_i) = C_{rs}(e_i) = C_{rs}(f_i) = 0 , \qquad i \neq r, s .$$

Let S_o satisfy

$$\begin{aligned} S_o(e_i) &= \sum_{j=1}^{2n} a_i^j e_j + b_i^j f_j \\ S_o(f_i) &= \sum_{j=1}^{2n} c_i^j e_j + d_i^j f_j \end{aligned} \qquad i = 1,\dots, 2n .$$

From the relations $(B_{rs} \circ S_o)(e_i) = (S_o \circ B_{rs})(e_i)$

$$(B_{rs} \circ S_o)(f_i) = (S_o \circ B_{rs})(f_i) \qquad i \neq r, s$$

we get $a_i^j = b_i^j = c_i^j = d_i^j = 0$, for all i, j such that $1 \leq i \neq j \leq$. (For this step, the inequality $n > 1$ is decisive.)

From the relations

$$(B_{rs} \circ S_o)(e_r) = (S_o \circ B_{rs})(e_r)$$

$$(B_{rs} \circ S_o)(f_r) = (S_o \circ B_{rs})(f_r)$$

we get

$$a_r^r = a_s^s , \; b_r^r = b_s^s , \; c_r^r = c_s^s , \; d_r^r = d_s^s \qquad 1 \leq r, s \leq 2n .$$

Finally, from the relation

$$(C_{rs} \circ S_o)(e_r) = (S_o \circ C_{rs})(e_r) \quad \text{we get}$$

$$a_r^r = d_s^s = a , \; b_r^r = - c_s^s = b , \qquad 1 \leq r, s \leq 2n .$$

We have obtained

$$\begin{aligned} S_o(e_j) &= a e_j + b f_j \\ S_o(f_j) &= - b e_j + a f_j \\ S_o(f_{2n+1}) &= - f_{2n+1} . \end{aligned} \qquad \begin{aligned} &1 \leq j \leq 2n , \\ &a^2 + b^2 = 1 . \end{aligned}$$

In the complex form,

$$S_o\left(\left(\frac{\partial}{\partial z^j}\right)_o\right) = e^{i\varphi}\left(\left(\frac{\partial}{\partial z^j}\right)_o\right) \qquad j = 1,\dots,2n, \; e^{i\varphi} = a + bi$$

$S_o(f_{2n+1}) = - f_{2n+1}$.

Now, let us denote by $Z_1,\dots,Z_{2n+1}$ the complex vector fields on S^{4n+1} which are tangent components of the vector fields $\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^{2n+1}}$ respectively. Let ∇, R denote the Riemannian connection and the curvature tensor field of the metric $\hat{g}_\lambda$ respectively. After a long but routine calculation we derive

$(\nabla_{Z_2} R)_o (Z_1,\bar{Z}_1,Z_{2n+1},\bar{Z}_2) \neq 0$, i.e.,

$$(\nabla_{\frac{\partial}{\partial z^2}} R)_o((\frac{\partial}{\partial z^1})_o,(\frac{\partial}{\partial \bar{z}^1})_o, f_{2n+1},(\frac{\partial}{\partial \bar{z}^2})_o) \neq 0 .$$

$(\nabla R)_o$ being invariant with respect to S_o , we come to a contradiction.

Remark. For $n = 1$, the Riemannian manifold $(S^5,\hat{g}_\lambda)$ is generalized symmetric of order 4 (cf. [6]) .

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