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NEUTRON TRANSPORT THEORY IN THE MULTIGROUP ENERGETICAL APPROXIMATION

(Preliminary communication)

Ivo MAREK , Praha

1. Introduction and formulation of the problem

Our purpose in this paper is to establish the system of neutron transport equations in the multigroup energetical approximation. We shall study the case of fast reactors i.e. the case, when all energetical groups of neutrons participate in the fission passing through. We shall use definitions and denotations of the monography [3]. As to the mathematical formulation of the problem and to some assumptions we will use the papers of Vladimirov [6], [7], in which the mathematical theory of one-group approximation of neutron transport equation is given.

The system of multigroup-approximation equations for neutron transport can be symbolically written as

$$L\varkappa = B\varkappa + \lambda C\varkappa + s \,, \tag{1.1}$$

where $\varkappa = (\varkappa_1, \dots, \varkappa_m)$ is the solution-vector and the operators L, B, C will be defined below.

Let G be a bounded open and connected set in the Euclidean space R_n $(n \geq 1)$; let $\dot{G}$ be the boundary of G . By the symbol Ω will be denoted the set of normed vectors $\vec{\Omega} = (\Omega_1, \dots, \Omega_n), |\vec{\Omega}| = 1$. Let us put $\mathcal{U} = G \times \Omega$. We define the operators L_j , $j = 1, \dots, m$ on the set $\mathcal{U}$ as follows:

$$L_j \varkappa_j \equiv \vec{\Omega} \nabla \varkappa_j + \Sigma_j \varkappa_j \,, \quad \Sigma_j \geq c > 0$$

with the boundary conditions

(1.2) $$\varkappa_j(\vec{r},\vec{\Omega}) = 0 \quad \text{for} \quad \vec{r} \in \dot{G}\,, \quad (\vec{\Omega},\vec{n}) < 0\,,$$

where the symbol $\vec{n}$ denotes the direction of the exterior normal to the boundary $\dot{G}$ and $(\vec{\Omega},\vec{n})$ denotes the cosinus of the angle between the vectors $\vec{\Omega},\vec{n}$.

Remark. The condition (1.2) can be written in the given form only if some further conditions with regard to the boundary $\dot{G}$ are satisfied. In the following paragraphs we shall formally use (1.2) in the general case, but we mean by this the exact formulation of the boundary conditions given in [6].

We define the operators

(1.3) $$B_{jk}\varkappa_k = \int_\Omega \Sigma_k^s w_{kj}^s(\vec{\Omega},\vec{\Omega}')\varkappa_k\, d\vec{\Omega}'\,,$$

(1.4) $$C_{jk}\varkappa_k = \int_\Omega \Sigma_k^f \nu_k^f w_{kj}^f(\vec{\Omega},\vec{\Omega}')\varkappa_k\, d\vec{\Omega}'$$

and we put

(1.5) $$\nu_{kj}^s = \Sigma_k^s w_{kj}^s\,, \quad \nu_{kj}^f = \Sigma_{kj}^f \nu_k^f w_{kj}^f\,.$$

We shall suppose that the kernels (1.5) have the properties (a) - (d) of the paper [7] p. 683.

The operators L, B, C will be defined as operator-matrices

(1.6) $$L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_m \end{pmatrix}, \quad B = (B_{jk}), \quad C = (C_{jk})\,.$$

Besides the assertions on the existence of the solutions of the homogeneous ($s = \mathfrak{A}$) and the inhomogeneous ($s \neq \mathfrak{A}$) systems (1.1, we shall give similar assertions for the adjoint system

(1.7) $$L^*x^* = B^*x^* + \bar{\lambda}\, C^*x^* + s^*\,,$$

where L^*, B^*, C^* are the adjoint operators to the L, B, C

in the Hilbert space $X = L_2(\Omega) \times \dots \times L_2(\Omega)$ with the scalar product $(x, y) = \sum_{j=1}^{m} (x_j, y_j)_j$, where $(x_j, y_j)_j$ is the scalar product in the space $L_2(\Omega)$ for the j-th coordinate of vectors $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$. We shall give general iteration methods for the construction of the mentioned solutions of the systems (1.1) and (1.7).

2. Auxiliary lemmas

The symbol $[X]$ denotes the space of linear bounded operators mapping X into itself. Let $K \subset X$ be the cone [2] of non negative vector-functions on Ω. The spectrum of the linear operator T will be denoted by $\sigma(T)$.

To an arbitrary vector $x = (x_1, \dots, x_m)$ we shall define a vector $|x|$ with the coordinates $(|x_1|, \dots, |x_m|)$, where $|x_j|$ is the absolute value of the function x_j, so that

$$|x_j(\vec{r}, \vec{\Omega})| = x_j(\vec{r}, \vec{\Omega}) \cdot \exp\{-i \arg x_j(\vec{r}, \vec{\Omega})\}.$$

If $x \in X$, $y \in X$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, the inequality

$$x < y$$

means that

$$x_j(\vec{r}, \vec{\Omega}) \leqq y_j(\vec{r}, \vec{\Omega})$$

for almost all $(\vec{r}, \vec{\Omega}) \in \Omega$ and the sharp inequality holds at least for one index j_0 in a set having positive measure.

Linear bounded operator $T \in [X]$ is called K-positive, if the vector $Tx = y \in K$ for any $x \in K$. K-positive operator T is called absolutely K-positive, if T has the following properties:

(α) For any vector $x \in K$, $x \neq \theta$ and for any $\varepsilon > 0$ there exists a natural N so that the measure of the set of

the zero-points of the vector-function $T^N x$ is not greater than ε .

(β) For any x , for which $\arg x_j \neq const$ at least for one index j_0 the inequality

(2.1) $$|Tx| < T|x|$$

holds.

Lemma 2.1 (fundamental lemma). An absolutely K-positive compact operator T with the positive spectral radius $r(T)$ has a positive simple eigenvalue μ_0 and an eigenvector x_0 positive almost everywhere in Ω corresponds to this value. An almost everywhere positive in Ω eigenvector x_0^* of the adjoint operator T^* corresponds to the same value μ_0. The following relations

$$Tx_0 = \mu_0 x_0 \ , \ T^* x_0^* = \mu_0 x_0^* \ ,$$

(2.2) $$|\lambda| < \mu_0 \ , \quad |\lambda^*| < \mu_0$$

are correct for $\lambda \in \sigma(T)$, $\lambda \neq \mu_0, \lambda^* \in \sigma(T^*), \lambda^* \neq \mu_0$. If the eigenvectors y_0, y_0^* of the operators T, T^* lie in K , then $y_0 = c_0 x_0$, $y_0^* = c_0^* x_0^*$, where c_0 , c_0^* are positive constants.

The application of the fundamental lemma to the systems (1.1) and (1.7) consists in using of the mentioned lemma for the operators

(2.3) $$T = (L-B)^{-1} C \ , \ T^* = C^* (L^* - B^*)^{-1}$$

(see chapter 3).

Let Γ be an interval of real numbers. The operator-function $T = T(\gamma), \gamma \in \Gamma, T(\gamma) \in [X]$ is called continuous in the point $\gamma_0 \in \Gamma$, if for any $\varepsilon > 0$ such a $\delta > 0$ exists

so that

$$\|T(\gamma) - T(\gamma_0)\| < \varepsilon$$

for $|\gamma - \gamma_0| < \sigma$. If $T = T(\gamma)$ is continuous in every point $\gamma \in \Gamma$, we shall call T continuous in Γ .

Lemma 2.2. Let us assume that

1. For any $\gamma \in \Gamma$, $T(\gamma) \in [X]$ is an absolutely K - positive compact operator.
2. Operator-function $T = T(\gamma)$ is continuous in Γ .
3. A vector $u_1 \in K$ exists such that $u - u_1 \in K$, where u ($\|u\| = 1$) is the eigenvector of the operator $T(\gamma)$ corresponding to the dominant eigenvalue $\mu_0(\gamma)$ and the relation

$$\{[T(\gamma') - T(\gamma)]u_1 - \nu(\gamma, \gamma')u_1\} \in K$$

holds for $\gamma \in \Gamma, \gamma' \in \Gamma, \gamma < \gamma'$ where $\nu(\gamma, \gamma') > 0$.

Then a positive dominant eigenvalue-function $\mu_0 = \mu_0(\gamma)$ of the operator-function $T = T(\gamma)$ is continuous and purely monotonous in Γ i.e. the inequality

$$\mu_0(\gamma) < \mu_0(\gamma')$$

holds for $\gamma < \gamma'$.

The value $\gamma_0 \in \Gamma$ is called critical parameter of the compact absolutely K -positive operator-function $T = T(\gamma)$ if the identity $\mu_0(\gamma_0) = 1$ holds for the corresponding dominant eigenvalue $\mu_0(\gamma_0)$ of the operator $T(\gamma_0)$.

From the lemma 2.2 follows the existence or the non existence of the critical parameters. If they do exist, their unicity follows from the same lemma.

3. Existence of the solutions of homogeneous systems

In this chapter we shall give some assertions from which the validity of the fundamental lemma 2.1 for the operators

(2.3) follows. Thus we obtain the fundamental result - the existence of a positive dominant simple eigenvalue of the systems (1.1), (1.7).

Lemma 3.1. The operator $L^{-1}C$ is absolutely K-positive, compact and its spectral radius $r(L^{-1}C)$ is positive.

From this lemma immediately follows

Theorem 3.1. Let be $r(L^{-1}B) < 1$. Then a positive dominant simple eigenvalue $\mu_0 = \lambda_0^{-1}$ of the homogeneous systems (1.1) (1.7) ($s = \theta$, $s^* = \theta$) exists. An eigenvector $x_0 \in K$ of the system (1.1) and an eigenvector $x_0^* \in K$ of the system (1.7) correspond to this eigenvalue. Besides the mentioned eigenvectors the homogeneous systems (1.1), (1.7) do not have other eigenvectors in K .

Remark. It is easy to show that the assumption $r(L^{-1}B) < 1$ in theorem 3.1 is natural and that the relation $r(L^{-1}B) \geq 1$ is physically impossible.

4. Existence of the solutions of the inhomogeneous systems

From the results of the third chapter it follows that all λ , for which

(4.1) $$|\lambda| < \mu_0^{-1}$$

are regular points of the systems (1.1) and (1.7). We can obtain the solutions of the mentioned systems in the following form

(4.2) $$x = R(1, \lambda T)(L - B)^{-1} s ,$$

(4.3) $$C^* x^* = R(1, \bar{\lambda} T^*) C^* (L^* - B^*)^{-1} s^*$$

where $R(\lambda, T) = (\lambda I - T)$, $R(\lambda, T^*) = (\lambda I - T^*)$ and s, s^* are arbitrary vectors in X .

Theorem 4.1. For λ , for which (4.1) holds, the system (1.1) or (1.7) has one and only one solution x or x^* for arbitrary $s \in X$ or $s^* \in X$.

For arbitrary λ the system (1.1) or (1.7) has a solution x or x^* then and only then, if

$$(s, y^*) = 0 \quad \text{or} \quad (s^*, y) = 0$$

for any solution y^* or y of the homogeneous system

$$L^* y^* = B^* y + \bar{\lambda} C^* y \quad \text{or} \quad L y = B y + \lambda C y .$$

5. Iteration methods

Among the numerical methods for solving the systems (1.1) and (1.7) the iteration methods are very important. In the case of homogeneous systems (1.1), (1.7) one of the often used iteration methods is the so called method of source-iteration. This method is a certain mathematical model of the physical processes described by systems (1.1) (1.7). Mathematically the source-iteration method is a modification of Kellogg's method for the construction of eigenvalues and eigenvectors of linear operators. The iteration formulae presented here contain or generalize most of the used iteration processes. The convergence of the mentioned processes can be derived from a single general principle (see [4]).

Generally we can define the source iteration method as follows:

(5.1) $\quad L x^{(n)} = B x^{(n)} + C x^{(n-1)}, \quad x^{(0)} \in K, \quad x^{(0)} \neq \theta ,$

(5.2) $\quad L^* x^{*(n)} = B^* x^{*(n)} + C^* x^{*(n-1)}, \quad x^{*(0)} \in K, \quad x^{*(0)} \neq \theta ,$

(5.3) $\quad x_{(n)} = \dfrac{x^{(n)}}{(x^{(n)}, x^{(0)})} , \quad x^*_{(n)} = \dfrac{x^{*(n)}}{(x^{*(n)}, C x^{(0)})} ,$

(5,4) $$\lambda_{(n)} = \mu_{(n)}^{-1} = \frac{(Cx^{(n)}, y'_n)}{(Cx^{(n+1)}, z'_n)},$$

where y'_n, z'_n form sequences of vectors in X, which converge weakly to the same vector $y' \in X$:

(5,5) $(x, y'_n) \to (x, y'), (x, z'_n) \to (x, y')$ for $x \in X$.

Let be

(5,6) $$(x^{(0)}, B_1^* y) \neq 0,$$

where $$B_1^* = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T^*)\, d\lambda ;$$

C_0 denotes a circle with its center in μ_0 and with the radius ρ_0 such that for $H = \{\lambda \,|\, |\lambda| \leqq \rho_0\}$ the relation

$$H \cap \sigma(T^*) = \{\mu_0\}$$

holds.

Theorem 5.1 The source iteration method defined by formulae (5.1) - (5.4) converges i.e. the following relations hold in the norm of the space X

$$x_{(n)} \to x_0, \quad x^*_{(n)} \to x_0^*$$
$$\lambda_{(n)} \to \lambda_0 = \mu_0^{-1},$$

where x_0, x_0^* are positive solutions of the homogeneous systems (1.1) and (1.7) corresponding to the positive dominant eigenvalue μ_0.

If (4.1) holds, the following iteration processes [5], [6] can be used for the construction of the solutions of the inhomogeneous systems (1.1) and (1.7) besides the usual iterations

(5.7) $$x_{n+1} = \frac{1}{1 - \lambda \mu_{(n)}} \lambda^{n+1} x^{(n+1)} + \sum_{k=1}^{n} \lambda^k x^{(k)} + (L-B)^{-1} \delta,$$

(5.8) $$x^*_{n+1} = \frac{1}{1-\bar{\lambda}(\mu_{(n)})}\,\bar{\lambda}^{n+1}x^{*(n+1)} + \sum_{k=1}^{n}\bar{\lambda}^k x^{*(k)} + (L^*-B^*)^{-1}s^*,$$

where $x^{(k)}$, $x^{*(k)}$, $\mu_{(n)}$ are defined by formulae (5.1), (5.2), (5.4) and $x^{(0)} = (L-B)^{-1}s$, $x^{*(0)} = (L^*-B^*)^{-1}s^*$.

Theorem 5.2. If the assumption (4.1) holds, the sequence (5.7) or (5.8) converges to the single solution x or x^* of the inhomogeneous system (1.1) or (1.7) i.e.

$$x_{n+1} \longrightarrow x \quad \text{or} \quad x^*_{n+1} \longrightarrow x^*$$

in the norm of the space X.

6. Critical parameters

In the case of nuclear reactors the operators L, B, C, L^*, B^*, C^* in systems (1.1), (1.7) depend on some parameters. We shall illustrate this on an example.

Let us suppose that the operators mentioned above depend on a parameter $\gamma \in \Gamma$, where Γ is an interval of real numbers.

The nuclear reactor is called critical, overcritical or subcritical, depending on whether the corresponding dominant eigenvalue $\mu_0(\gamma)$ of the operator $[L(\gamma)-B(\gamma)]^{-1}C(\gamma)$ is equal, greater or smaller than 1. It is evident that only a single critical parameter can exist. Formally this follows from the fact that $\mu_0 = \mu_0(\gamma)$ is a purely monotonous function of the argument $\gamma \in \Gamma$.

We shall give an example of the dependence of the criticality of the nuclear reactor on the enriching of the fuel by a more fissionable isotope.

Let us consider the system (1.1) and the following system

(6.1) $$L'x' = B'x' + \lambda' C'x' ,$$

where

$$\Sigma_k = \Sigma_k(\gamma),\ \Sigma_k^f = \Sigma_k^f(\gamma),\ \nu_k^f = \nu_k^f(\gamma),$$

(6.2) $$\Sigma_k' = \Sigma_k(\gamma'),\ \Sigma_k'^f = \Sigma_k^f(\gamma'),\ \nu_k'^f = \nu_k^f(\gamma'),$$

$$\Sigma_k' > \Sigma_k,\ \Sigma_k'^f > \Sigma_k^f,\ \nu_k'^f > \nu_k^f .$$

Theorem 6.1. If the assumptions(6.2) are fulfilled the inequality

$$\lambda_0' < \lambda_0$$

holds for the values $\lambda_0 = \lambda_0(\gamma)$, $\lambda_0' = \lambda_0(\gamma')$.

If an enriching $\gamma' \in \Gamma$ exists such that the corresponding $\lambda_0' = \lambda_0(\gamma') < 1$, then one and only one value of enriching $\gamma_0 \in \Gamma$ exists, for which $\lambda_0(\gamma_0) = 1$.

In conclusion we call the readers' attention to the fact that many similar problems having the form (1.1), where L, B, C are linear operators such that $T = (L - B)^{-1}$ is absolutely K-positive compact operator, can be considered by our method.

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ON NONLINEAR NUMERICAL ITERATION PROCESSES

Stanislav MALOŇ , Praha

I

Let Y be a Banach space and F its closed subset. Let K be a Lipschitz operator mapping F into Y , i.e. there exists a positive constant β such that

(1) $$\| Ku - Kv \| \leq \beta \| u - v \|$$

holds for any u, $v \in F$.

Let us consider the iteration process

(2) $$y_0 \in F,$$
$$y_{n+1} = Ky_n , \quad n = 0, 1, 2, \ldots .$$

This process is convergent if the well known conditions given in the Banach theorem (see e.g. [2]) are satisfied, i.e.

(3a) $$\beta < 1 ,$$

(3b) $$y_0 \in F \Rightarrow y_1 = Ky_0 \in F ,$$

(3c) $$S(y_1, r) \subset F ,$$

where S is a closed sphere whose centre is y_1 and radius

$$r = \frac{\beta}{1-\beta} \| y_1 - y_0 \| .$$

We shall suppose that the sequence (2) converges to the limit y^*.

In practice, if the digital computation technique be used, there often will be necessary to transfer the problem of realising the sequence $\{y_n\}$, defined by (2), into a space different from the original one, so that the elements y_n might be numerically interpreted [1]. To this effect it

is necessary to replace the original process (2) by another subsidiary process which is easy to be realised as to using the numerical technique; moreover, we must naturally desire the original process (2) to be approximated sufficiently accurately by the subsidiary process.

In agreement with Kantorovič [1], let us transfer the problem into space $\bar{Y}$ isomorphe with Y, the isomorphism being realised by the linear bounded operation φ; it is natural to assume that the elements $\bar{y} \in \bar{F}$ can be numerically interpreted. The construction of the subsidiary iteration process will be done by a suitable operator $\bar{K}$ approximating the operator K. Let us assign the analogical process

$$\bar{y}_0 = \varphi y_0$$

$$\bar{y}_{n+1} = \bar{K}\bar{y}_n, \quad n = 0, 1, 2, \dots \tag{4}$$

to the process (2).

In this paper we study the approximative solution of the equation $y = Ky$ when the iteration process (2) is replaced by the process (4).

First, we shall deal with the question what sufficient conditions are to be required from the operator $\bar{K}$ to get the limit $\bar{y}^*$ of the process (4) sufficiently near to the limit y^* of the sequence (2) in terms of the definition

$$\rho(y^*, \bar{y}^*) < \varepsilon, \qquad \text{where} \tag{5}$$

$$\rho(u, \bar{u}) = \|\varphi u - \bar{u}\|_{\bar{Y}}, \quad u \in Y, \ \bar{u} \in \bar{Y}$$

and ε is a given positive number.

Next, we consider the question of the influence of the rounding-off errors.

We assume that the approximating operator $\bar{K}$ is Lipschitz bounded, i.e. for $\bar{u}, \bar{v}$ being two arbitrary elements belonging to $\bar{F}$,

(6) $$\| \bar{K}\bar{u} - \bar{K}\bar{v} \|_{\bar{Y}} \leq \bar{\beta} \| \bar{u} - \bar{v} \|_{\bar{Y}}$$

holds.

The process (4) will converge to a certain limit $\bar{y}^*$, if analogical conditions, the same as for the process (2), i.e.

(7a) $$\bar{\beta} < 1 ,$$

(7b) $$\bar{y}_0 \in \bar{F} \Rightarrow \bar{y}_1 = \bar{K}\bar{y}_0 \in \bar{F} ,$$

(7c) $$\bar{S}_1(\bar{y}_1, \bar{r}_1) \subset \bar{F}, \quad \bar{r}_1 = \frac{\bar{\beta}}{1-\bar{\beta}} \| \bar{y}_1 - \bar{y}_0 \|_{\bar{Y}}$$

hold.

What other conditions are to be satisfied by the approximation $\bar{K}$, so that the limits y^* and $\bar{y}^*$ may be sufficiently near in terms of definition (5)? The answer to this question is given by the following theorem:

Theorem 1: Let the following assumptions be fulfilled:

1) The conditions (1), (3a,b,c), (7a,b,c) and (6) are satisfied.

2) Approximating operator $\bar{K}$ is such that for any element $u \in F$ the inequality

(8) $$\| \varphi K u - \bar{K} \varphi u \|_{\bar{Y}} \leq \alpha \| u \|_{Y}$$ x)

x) The condition of L.V.Kantorovič, [1], p.107

holds.

3) $\bar{S}(\varphi y_1, \varkappa) \subset \bar{S}_1(\bar{y}_1, \bar{\varkappa}_1)$

holds, where y_1, $\varkappa$ are defined by the process (2) and by the formula (3c),

4) $\|\varphi\| \leq 1$.

Then

(9) $$\lim_{n\to\infty} \|\varphi y^* - \bar{y}_n\|_{\bar{Y}} \leq \frac{\alpha c}{1-\beta}$$

holds, where $c = \sup_{0\leq i<\infty} \|y_i\|_Y$.

Proof. As both sequences $\{y_n\}, \{\bar{y}_n\}$ are convergent according to our assumptions and the operator φ is continuous, the limit in (9) exists; it remains only to prove the inequality.

1) First, we shall prove that for any positive integer n the inequality

(10) $$\|\varphi y_{n+1} - \bar{y}_{n+1}\|_{\bar{Y}} \leq \frac{\alpha c_n}{1-\beta}$$

holds, where

$$c_n = \max \|y_i\|_Y .$$

Evidently, for any $u \in S$

$$\|\varphi u - \varphi y_1\| \leq \|\varphi\| \|u - y_1\| \leq \|u - y_1\| \leq \varkappa$$

holds, i.e. $\varphi u \in \bar{S} \subset \bar{S}_1$.

Then it follows from our assumptions that

$$\|\varphi y_1 - \bar{y}_1\|_{\bar{Y}} = \|\varphi K y_0 - \bar{K}\varphi y_0\|_{\bar{Y}} \leq \alpha \|y_0\|_Y$$

and

$$\|\varphi y_{n+1} - \bar{y}_{n+1}\|_{\bar{Y}} = \|\varphi K y_n - \bar{K}\bar{y}_n\|_{\bar{Y}} \leq$$

$$\leq \|\varphi K y_n - \bar{K}\varphi y_n\|_{\bar{Y}} + \|\bar{K}\varphi y_n - \bar{K}\bar{y}_n\|_{\bar{Y}} .$$

As $y_n \in S \subset F$, (8) can be applied on the last by one term further, $\varphi y_n \in \bar{S}$, $\bar{y}_n \in \bar{S}_1$, consequently φy_n and $\bar{y}_n$ belong to $\bar{F}$ and (6) can be applied on the last term; consequently

$$\|\varphi y_{n+1} - \bar{y}_{n+1}\|_{\bar{Y}} \leqq \alpha \|y_n\|_Y + \bar{\beta} \|\varphi y_n - \bar{y}_n\|_{\bar{Y}} \leqq$$

$$\leqq \alpha \|y_n\|_Y + \bar{\beta} [\alpha \|y_{n-1}\|_Y + \bar{\beta} \|\varphi y_{n-1} - \bar{y}_{n-1}\|_{\bar{Y}}] \leqq \cdots$$

$$\leqq \alpha \sum_{i=0}^{n} \bar{\beta}^i \|y_{n-i}\|_Y < \frac{\alpha c_n}{1-\bar{\beta}} .$$

2) It is evident that

$$\|\varphi y^* - \bar{y}_n\|_{\bar{Y}} \leqq \|\varphi y^* - \varphi y_n\|_{\bar{Y}} + \|\varphi y_n - \bar{y}_n\|_{\bar{Y}} \leqq$$

$$\leqq \|y^* - y_n\|_Y + \frac{\alpha c_n}{1-\bar{\beta}}$$

holds. As $\|y^* - y_n\| \to 0$ and $c_n \leqq c$, the inequality (9) follows immediately.

Note. If the operator K and its approximation $\bar{K}$ are linear bounded operators mapping complete spaces Y resp. $\bar{Y}$ into themselves, then we can put $F = Y$, $\bar{F} = \bar{Y}$, $\beta = \|K\|$, $\bar{\beta} = \|\bar{K}\|$ and the assumptions (3b), (3c), (7b), (7c) are to be dropped.

III

In practice however, as the actual computation is realised by digital numbers, rounding-off errors in the process (4) arise. Consequently, the computation procedure is not defined by the iteration formula (4) but in general by the following one:

$$\tilde{y}_0 = \varphi y_0 + \bar{\eta}$$

(11)

$$\tilde{y}_{n+1} = \tilde{K} \tilde{y}_n , \quad n = 0, 1, 2, \ldots$$

where $\bar{\eta} \in \bar{Y}$ and the operator $\tilde{K}$ is defined on the same subspace $\bar{F}$ as $\bar{K}$ and approximates $\bar{K}$ in terms of definition

(12) $$\| \bar{K}\bar{u} - \tilde{K}\bar{u} \|_{\bar{Y}} \leq \xi , \quad \bar{u} \in \bar{Y},$$

where ξ is a nonnegative number (the upper bound of error caused by accumulation of rounding-off errors when computing the value $\bar{K}\bar{u}$).

It is the process (11) only which allways can be realised.

Theorem 2. Let the following conditions be fulfilled: x)

1) The inequality (12) holds for any $\bar{u} \in \bar{Y}$,

2) (7a),(7b),(7c) hold,

3) $\tilde{y}_0 \in \bar{F}$,

4) $\bar{U}(\tilde{y}_1 , \tilde{r} + \gamma) \subset \bar{F}$,

where $\bar{U}(\tilde{y}_1, \tilde{r} + \gamma)$ is the closed sphere, $\tilde{y}$ its centre, $\tilde{r} + \gamma$ its radius,

$$\gamma = \frac{\bar{\beta}\|\bar{\eta}\| + \xi}{1 - \bar{\beta}} , \quad \tilde{r} = \frac{\bar{\beta}}{1-\bar{\beta}} [\|\tilde{y}_1 - \tilde{y}_0\| + \|\bar{\eta}\|] + \gamma .$$

Then 1) $\tilde{y}_i \in \bar{U}$ for all $i = 1, 2, \ldots$

2) The estimation

(13) $$\| \bar{y}_n - \tilde{y}_n \| \leq \bar{\beta}^n \|\bar{\eta}\| + \frac{\xi}{1-\bar{\beta}}$$

holds.

Proof: 1) We shall prove that the sphere $\bar{S}_1$ defined by the formula (7c) is contained in the sphere

$$\bar{\Sigma}(\tilde{y}_1 , \tilde{r}),$$

---

x) In the following we omit to designate spaces when writing norms.

i.e. that $h \in \bar{S}_1$ implies $\| h - \tilde{y}_1 \| \leq \tilde{\varkappa}$.

Really, for any $\tilde{y}_0 \in \bar{F}$

$$\| h - \tilde{y}_1 \| \leq \| h - \bar{y}_1 \| + \| \bar{y}_1 - y_1 \|$$

$$\| h - \bar{y}_1 \| \leq \frac{\bar{\beta}}{1-\bar{\beta}} \| \bar{y}_1 - \bar{y}_0 \| \leq \frac{\bar{\beta}}{1-\bar{\beta}} (\| \bar{y}_1 - \tilde{y}_1 \| +$$

$$+ \| \tilde{y}_1 - \bar{y}_0 \|)$$

$$\| \bar{y}_1 - \tilde{y}_1 \| \leq \| \bar{K} \bar{y}_0 - \bar{K} \tilde{y}_0 \| + \| \bar{K} \tilde{y}_0 - \tilde{K} \tilde{y}_0 \| \leq \bar{\beta} \| \bar{y}_0 - \tilde{y}_0 \| + \xi .$$

From these inequalities we get simply

$$\| h - \tilde{y}_1 \| \leq \frac{\bar{\beta} \| \bar{\eta} \| + \xi}{1 - \bar{\beta}} + \frac{\bar{\beta}}{1-\bar{\beta}} (\| \tilde{y}_1 - \tilde{y}_0 \| + \| \tilde{\eta} \|) = \tilde{\varkappa} .$$

2) Evidently γ -neighborhood of $\bar{S}_1$ is contained in γ -neighborhood of $\bar{\Sigma}$ and consequently in $\bar{U}$. We shall prove by induction that if $\tilde{y}_i \in \bar{U}$, then also $\tilde{y}_{i+1} \in \bar{U}$. For $i = 1$ it is proved; for $i \geq 1$ we have

$$\| \bar{y}_i - \tilde{y}_i \| = \| \bar{K} \bar{y}_{i-1} - \tilde{K} \tilde{y}_{i-1} \| \leq$$

$$\| \bar{K} \bar{y}_{i-1} - \bar{K} \tilde{y}_{i-1} \| + \| \bar{K} \tilde{y}_{i-1} - \tilde{K} \tilde{y}_{i-1} \| \leq \bar{\beta} \| \bar{y}_{i-1} - \tilde{y}_{i-1} \| + \xi \leq \cdots$$

$$\leq \bar{\beta} \{ \bar{\beta} [\cdots (\bar{\beta} \| \bar{y}_1 - \tilde{y}_1 \| + \xi) + \cdots + \xi \} + \xi =$$

$$= \bar{\beta}^i \| \bar{\eta} \| + \xi \cdot (1 + \bar{\beta} \cdots + \bar{\beta}^{i-1}) < \bar{\beta}^i \| \bar{\eta} \| + \frac{\xi}{1-\bar{\beta}} ,$$

i.e. the estimation (13) holds. From it follows that

$$\| \bar{y}_i - \tilde{y}_i \| \leq \frac{\bar{\beta}^i \| \bar{\eta} \| + \xi}{1 - \bar{\beta}} < \gamma .$$

As $\bar{y}_i \in \bar{S}_1$, the first part of the assertion is also proved.

Note. The influence of truncation and of rounding errors was studied by M. Urabe. Theorem 2 is slightly generalized result of his paper [3], whose formula (2.5) p. 481 is

a special case of our formula (13) when $\bar{y}_0 = \tilde{y}_0$.

IV

Conclusion. Summing up the results of items 2 and 3, we get the following theorem:

Theorem 3. Let the conditions of the 1st and 2nd theorem be satisfied. Then the following assertions hold:

1) All elements of the sequence (11) belong to $\bar{F}$.

2) For the distance of the n-th approximation $\tilde{y}_n$ from the element φy^* the estimation

$$(14)\quad \|\varphi y^* - \tilde{y}_n\|_{\bar{Y}} < \|y^* - y_n\|_Y + \bar{\beta}^n \|\bar{\eta}\|_{\bar{Y}} + \frac{\alpha c_n + \delta}{1-\bar{\beta}}$$

holds, where c_n is defined in the theorem 1 .

Proof: Evidently

$$\|\varphi y^* - \tilde{y}_n\| \leqq \|\varphi y^* - \varphi y_n\| + \|\varphi y_n - \bar{y}_n\| + \|\bar{y}_n - \tilde{y}_n\| .$$

The first term of the right is at most equal to the error of the n-th approximation in the process (2). For the second term we use the estimation (10) and for the third the estimation (13).

Note. The influence of errors $\|y^* - y_n\|$ and $\|\bar{\eta}\|$ diminishes with $n \to \infty$ to zero. But as δ is a fixed positive number, it does not follow from (14), that the process (11) should converge in current sense. We can assert only, that a number V being given,

$$V > \frac{\alpha c + \delta}{1-\bar{\beta}} , \quad c = \sup_n c_n ,$$

such an integer n_o exists that $\tilde{y}_n$ belongs to the sphere $\bar{S}_V(\varphi y^*, V)$ for all $n \geq n_o$. In practice, as a rule, $\bar{Y}$ will be an Euclidean m-dimensional space R^m. In every finite part of R^m there is a finite number of vectors whose components are digital numbers with given number of figures; let us assume that the sphere $\bar{S}_V$ contains just N elements. Then evidently, if the sequence $\{\tilde{y}_n\}$ does not converge, it will be periodic beginning from a certain $n_1 \geq n_o$, with the period N at most. In other words, the sequence $\{\tilde{y}_n\}$ will reach the state of numerical convergence in the sense of M. Urabe [3]. An arbitrary element $\tilde{y}_{n_1+i}$, $i = 0, 1, \ldots$ can be accepted as an approximation of y^* the error of which does not exceed the number V.

The state of numerical convergence need not take place when especially from the conditions of theorem 2 it is the 2nd condition only which is not fulfilled. However, in the case mentioned in the Note of item II the 2nd condition is to be dropped.

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ON THE HOMOLOGY THEORY OF DISCRETE SPACES

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In present paper the homology theory from category of couples of discrete spaces is constructed. It is shown that there exist sufficiently different homology theories from the category of couples of finite discrete spaces, satisfying all seven Eilenberg-Steenrod axioms. On the other hand, we get the uniqueness theorem adding another axiom.

§ 1.

1.1. Definition: A topological space X is said to be a discrete space if $\overline{\bigcup_\iota A_\iota} = \bigcup_\iota \overline{A}_\iota$ for arbitrary system $\{A_\iota\}$ of subsets of X.

1.2. The following statement is obvious:

Theorem: Let X be a discrete space (discrete T_0-space, respectively). Then a function $\mathcal{Cl}: X \to \exp X$, defined by formula $\mathcal{Cl}(x) = (\overline{x})$, has the following properties:

I) $x \in \mathcal{Cl}(x)$

II) $x \in \mathcal{Cl}(y),\ u \in \mathcal{Cl}(x) \Rightarrow u \in \mathcal{Cl}(y)$

(III) $x \in \mathcal{Cl}(y),\ y \in \mathcal{Cl}(x) \Rightarrow x = y$, respectively).

On the contrary, for every function $F: X \to \exp X$, satisfying the conditions I, II there exists one and only one discrete topology over the set X such that $F(x) = (\overline{x})$ for every x. If F satisfies the condition III, this topolo is T_0.

1.3. Let us denote

$$St(x) = \{y \mid x \in Cl(y)\}, \quad St(A) = \bigcup_{x \in A} St(x) .$$

Obviously $St(A)$ is always an open set and it is the least open set containing A.

1.4. Theorem: Let X, Y be discrete spaces, $f: X \to Y$ be a mapping. Then f is a continuous mapping iff $f(Cl(x)) \subset Cl(f(x))$ for every $x \in X$.

Proof: Obviously f is a continuous mapping iff $f(St(x)) \subset St(f(x))$ for every x. Let $y \in f(Cl(x))$. Then there exists a $u \in Cl(x)$, such that $y = f(u)$. Because $x \in St(u)$, we have $f(x) \in St(f(u))$ and hence $y = f(u) \in Cl(f(x))$. The rest of the proof is obvious.

§ 2.

2.1. Definition: A sequence $(x_0, \ldots, x_n)$ of elements of X such that $x_i \in Cl(x_{i+1})$, $i = 0, \ldots, n-1$, is said to be an elementary n-chain over the discrete space X. Let G be an abelian group; n-chain over X is every formal combination of finite number of elementary n-chains with coefficients from G. The set of all n-chains over X with obviously defined structure of an abelian group is said to be the group of n-chains of X (with coefficients from G and will be denoted $C_n(X; G)$ (or simply $C_n(X)$), if there is no danger of misunderstanding). We define yet $C_n(\emptyset) = 0$.

2.2. Remark: Let us call simple n-chains such n-chains $\sum g_\alpha \sigma$, that at most one g_α is non-zero element of G. It is easy to prove that for every additive function from the

set of all simple n-chains into some abelian group there exists just one homomorphism which is its extension over $C_n(X)$. Therefore it is sufficient to define a homomorphism from $C_n(X)$ in its simple n-chains only.

2.3. Definition: Let X, Y be discrete spaces, $f: X \to Y$ be a continuous mapping. Because of 1.4 we can define the homomorphism $f_n : C_n(X) \to C_n(Y)$ by formula

$$f_n(g \cdot (x_0, \dots, x_n)) = g \cdot (f(x_0), \dots, f(x_n)) .$$

Homomorphism $d_n : C_n(X) \to C_{n-1}(X)$ is defined by formula

$$d_n(g \cdot (x_0, \dots, x_n)) = \sum_{i=0}^{n} (-1)^i g \cdot (x_0, \dots, \hat{x}_i, \dots, x_n) .$$

2.4. Definition: Let (X, A) be a couple of discrete spaces (in the ordinary sense; i.e. A is a subspace of X). Then we define

$$C_n(X, A; G) = C_n(X; G) / C_n(A; G)$$

Obviously $d_n(C_n(A)) \subset C_{n-1}(A)$ and therefore we can define $\bar{d}_n : C_n(X, A) \to C_n(X, A)$ by formula $\bar{d}_n([a]) = [d_n(a)]$. For a mapping $f: (X, A) \to (Y, B)$ we have $f_n(C_n(A)) \subset C_n(B)$ and therefore we can define $\bar{f}_n : C_n(X, A) \to C_n(Y, B)$ in an analogous way.

2.5. It is easy to prove

Lemma: $\bar{d}_{n-1} \circ \bar{d}_n = 0$, $(\bar{g} \circ \bar{f})_n = \bar{g}_n \circ \bar{f}_n$,

$$\bar{f}_{n-1} \circ \bar{d}_n = \bar{d}_n \circ \bar{f}_n .$$

2.6. Corollary: The system $\{C_n(X, A), \bar{d}_n\}$ (if we define $C_k(X, A) = 0$, $\bar{d}_k = 0$ for $k < 0$) is a chain complex in the sense of [E-S, chapter V]. If $f: (X, A) \to (Y, B)$ is a continuous mapping, the system $\{\bar{f}_n\}$ is a mapping of

$\{C_n(X,A)\}$ into $\{C_n(Y,B)\}$. We define homology groups of couples of discrete spaces and induced homomorphisms of their continuous mappings as homology groups and induced mappings of corresponding chain complexes and their mappings, respectively. We use the denotations $H_n(X,A)$, f_{*n} , and $\partial(n,X,A)$ for homology groups, induced homomorphisms and boundary homomorphisms, respectively. For groups of cycles and boundaries we use the denotations $Z_n(X,A)$, $B_n(X,A)$, respectively. In this way we get a homology theory from the category of couples of discrete spaces which satisfies obviously first four Eilenberg-Steenrod axioms and, as it is easy to see, also the axiom of dimension. In the following paragraph we are going to prove that remaining axioms of homotopy and excision are satisfied, too (with a slight change in the definition of homotopy).

2.7. Remark: If we define $C_{-1}(X,A;G) = G$ instead of $C_{-1}(X,A) = 0$, $d_{-1}(g(x_0)) = g$, f_{-1} identical homomorphism of G , we get (2.4 holds obviously, too) the "reduced homology theory". All things said about homology theory defined in 2.6 , except of dimension axiom in the ordinary form, are true for reduced homology theory, too (the same for homotopy and excision axiom).

§ 3.

3.1. Definition: Let $\alpha \in C_n(X)$, $\alpha = \sum a_i(x_{i0},\dots,x_{in})$. The set $\{x_{ij}\}$ is said to be the carrier of α .

3.2. Definition: Let $\alpha \in C_n(X)$, let R be its carrier. Let φ, ψ be mappings from R into Y such that $\varphi(x) \in Cl(\psi(x))$ for every $x \in R$. Then we define the

homomorphisms $D_n: C_n(R) \to C_{n+1}(Y)$ as follows:
$D_n(a(x_0,\dots,x_n)) = \sum_{k=0}^{n} (-1)^k (\varphi(x_0),\dots,\varphi(x_k),\psi(x_k),\dots,\psi(x_n))$.

3.3. Lemma: $d_{n+1} D_n(\alpha) = \psi_n(\alpha) - \varphi_n(\alpha) - D_{n-1} d_n(\alpha)$.

Proof: Is a matter of counting.

3.4. Definition: Mappings $f, g: X \to Y$ are said to be homotopical if there exists a continuous mapping $h: X \times I \to Y$ such that

a) $h(x,0) = f(x)$, $h(x,1) = g(x)$ for every $x \in X$.

b) The set $h((x) \times I)$ is finite for every $x \in X$.

In the analogous way the relation of homotopy between two mappings of couples of discrete spaces is defined.

3.5. Remark: If Y is a star-finite space (i.e. $St(x)$ is finite for every $x \in X$), the condition b) is satisfied automatically. In general, it is possible to prove that image of any compact space in a star-finite space is finite.

3.6. Lemma: Let T be a topological space, X be a discrete space. Let $M \subset T$ $a \in \overline{M}$, let $f: T \to X$ be a continuous mapping. Then there exists an element $y \in M$ such that $f(a) \in Cl(f(y))$.

Proof: Because of continuity of f there exists a neighborhood V of the point a such that $f(V) \subset St(a)$. $a \in \overline{M}$ and therefore $V \cap M \neq \emptyset$. Let us take some $y \in V \cap M$. We have $f(y) \in St(a)$ and hence $f(a) \in Cl(f(y))$.

3.7. Lemma: Let X, Y be discrete spaces, $f, g: X \to Y$ be homotopical mappings. Let R be a finite subspace of X. Then there exists a finite sequence of mappings

$$h_0, h_1, \dots, h_m: R \to Y$$

such that $h_0 = f|R$, $h_m = g|R$ and for $i = 1, \dots, m$ the following alternative holds

either $h_i(x) \in Cl(h_{i-1}(x))$ for every $x \in R$,

or $h_{i-1}(x) \in Cl(h_i(x))$ for every $x \in R$.

Proof: Let us take a homotopy mapping $h: X \times I \to Y$, $h(x,0) = f(x)$, $h(x,1) = g(x)$. Obviously $h(R \times I) = \bigcup_{x \in R} h((x) \times I)$ and hence finite. Let us define mappings $h_\lambda: R \to Y$ by formula $h_\lambda(x) = h(x, \lambda)$. Because of finiteness of $h(R \times I)$ we have only finite number of different mappings h_λ. Let $I_1, \dots, I_n$ be equivalence classes in I with respect to the equivalence relation defined as follows:

λ equivalent λ' iff $h_\lambda = h_{\lambda'}$.

Let I_{k_0} be the class containing zero and let us define $h_0 = h_\lambda, \lambda \in I_{k_0}$. Let us denote $a_0 = \sup I_{k_0}$. Then either $a_0 \in I_{k_0}$ or $a_0 \notin I_{k_0}$. In the first case either $a_0 = 1$ or $a_0 < 1$. If $a_0 = 1$ it is $f|R = g|R$. If $a_0 < 1$ there exists a class I_{k_1} such that a_0 is its condensation point and that there exists a point in I_k, lying behind a_0. Let us denote $h_1 = h_\lambda$, $\lambda \in I_{k_1}$. According to 3.6 we have $h_0(x) \in Cl(h_1(x))$ for every $x \in R$.

Let $a_0 \notin I_{k_0}$; in that case let us denote I_{k_1} the class containing a_0 and $h_1 = h_\lambda$, $\lambda \in I_{k_1}$. Because of $a_0 \in \overline{I}_{k_0}$ we get immediately that $h_1(x) \in Cl(h_0(x))$ for every $x \in R$.

Now let us assume h_{l-1} to be found, $h_{l-1} = h_\lambda$, $\lambda \in I_{k_{l-1}}$, and denote $a_{l-1} = \sup I_{k_{l-1}}$.

If $a_{l-1} \in I_{k_{l-1}}$, we have the two possibilities: either $a_{l-1} = 1$ and there is no need of further proof, or $a_{l-1} < 1$.

In the second case there exists a class I_{k_l} such that $a_{l-1} \in \overline{I}_{k_l}$, and I_{k_l} contains a point which lies behind a_{l-1}. We define $h_l = h_\lambda$, $\lambda \in I_{k_l}$ and get $h_{l-1}(x) \in Cl(h_l(x))$ for every $x \in R$.
If $a_{l-1} \notin I_{k_{l-1}}$ let us denote I_{k_l} the class containing a_{l-1}. Let us define $h_l = h_\lambda$, $\lambda \in I_{k_l}$ and we get $h_l(x) \in Cl(h_{l-1}(x))$ for every $x \in R$. It is easy to see that our h_i never repeat (because of choice of h_i such that I_{k_i} contains always an element lying behind all elements of $I_{k_{i-1}}$). Hence we must, after a finite number of steps, get $h_m = g \mid R$.

3.8. Theorem: Let $f, g : X \to Y$ be homotopical mappings. Then $f_{*n} = g_{*n}$ for every n.

Proof: Let us take an $[\alpha] \in H_n(X), \alpha \in [\alpha], \alpha \in Z_n(X)$. Let R be the carrier of α, h_i mappings from the lemma 3.7. Because of $\alpha \in Z_n(R)$ we can use the lemma 3.3 and we get

either $d_n D_n(\alpha) = (h_i)_n(\alpha) - (h_{i+1})_n(\alpha)$

or $d_n D_n(\alpha) = (h_{i+1})_n(\alpha) - (h_i)_n(\alpha)$.

Anyway, the cycles $(h_i)_n(\alpha)$ and $(h_{i+1})_n(\alpha)$ are homological for every i and we get immediately $f_{*n}([\alpha]) = g_{*n}([\alpha])$.

3.9. Theorem: Let $f, g : (X, A) \to (Y, B)$ be homotopical mappings of couples of discrete spaces. Then $f_{*n} = g_{*n}$ for every n.

Proof: Let us denote f', g' corresponding mappings $X \to Y$. f' and g' are also homotopical. Further let us denote $i : X \subset (X, A)$, $i' : Y \subset (Y, B)$. We have (see 3.8) $f'_n(\beta) - g'_n(\beta) \in B_n(Y)$ for every $\beta \in Z_n(X)$. i_n are epimorphisms and hence

$$f_{*n}([\alpha]) - g_{*n}([\alpha]) = [f_n(\alpha) - g_n(\alpha)] =$$

$$= [f_n\, i_n(\beta) - g_n\, i_n(\beta)] = [i'_n(f'_n(\beta) - g'_n(\beta))] =$$

$$= [i'_n\, d_{n+1}(\gamma)] = [d_{n+1}\, i_{n+1}(\gamma)] = 0 \ .$$

3.10. Lemma: Let $U \subset A \subset X$, $f: (X-U, A-U) \subset (X, A)$. Then $f_n : C_n(X-U, A-U) \longrightarrow C_n(X, A)$ are monomorphisms. If $\mathcal{U}\, St(U) \subset A$, f_n are isomorphisms.

Proof: Let $[\alpha] \in C_n(X-U, A-U)$, $\alpha = \sum a_i(x_{i0}, \ldots, x_{in}) \in C_n(X-U)$. We have $f'_n(\alpha) = \sum a_i(x_{i0}, \ldots, x_{in})$, where $f': X-U \subset X$. Let $f_n([\alpha]) = 0$. Hence $f'_n(\alpha) \in C_n(A)$ and therefore $x_{ij} \in A$ for every i, j. Because of $\alpha \in C_n(X-U)$, no x_{ij} is an element of U. Hence $x_{ij} \in A-U$ and we get $\alpha \in C_n(A-U)$ and therefore $[\alpha] = 0$. Now, let $\mathcal{U}\, St(U) \subset A$. We are going to prove that f_n are epimorphisms. At first we show that every elementary chain containing some element of U contains elements of A only. Let in $(x_0, \ldots, x_m)$ $x_k \in U$.

If $i \leqq k$, $x_i \in \mathcal{U}(x_k) \subset \mathcal{U}(U) \subset \mathcal{U}\, St(U) \subset A$.

If $i \geqq k$, $x_i \in St(x_k) \subset St(U) \subset \mathcal{U}\, St(U) \subset A$.

Now let us take an $\alpha = \sum a_i(x_{i0}, \ldots, x_{in})$ and decompose it in $\alpha' + \alpha''$, $\alpha' = \sum a'_i(x'_{i0}, \ldots, x'_{in})$, $\alpha'' = \sum a''_i(x''_{i0}, \ldots, x''_{in})$, where $(x'_{i0}, \ldots, x'_{in})$ are all elementary chains of α which do not contain any element of U, $(x''_{i0}, \ldots, x''_{in})$ are remaining ones. Hence $\alpha'' \in C_n(A)$ and therefore $[\alpha'] = [\alpha]$. Let us denote β the chain α' as an element of $C_n(X-U)$. We have $f'_n(\beta) = \alpha'$ and therefore $f_n([\beta]) = [f'_n(\beta)] = [\alpha'] = [\alpha]$.

3.11. Theorem: Let

$Cl\,St(U) \subset A \subset X$, $f: (X-U, A-U) \subset (X, A)$.

Then f_{*n} are isomorphisms.

Proof: f_{*n} are monomorphisms:

Let $f_{*n}([\alpha]) = 0$, $\alpha \in [\alpha] \in H_n(X-U, A-U)$, $\alpha \in Z_n(X-U, A-U)$, i.e. $f_n(\alpha) \in B_n(X, A)$. Hence there exists an element $\beta \in C_{n+1}(X, A)$ such that $f_n(\alpha) = \bar{d}_{n+1}(\beta)$. Hence

$$f_n(\alpha) = \bar{d}_{n+1} f_{n+1} f_{n+1}^{-1}(\beta) = f_n \bar{d}_{n+1}(f_{n+1}^{-1}(\beta))$$

and therefore $\alpha = \bar{d}_{n+1}(f_{n+1}^{-1}(\beta))$, $[\alpha] = 0$.

f_{*n} are epimorphisms:

Let $[\gamma] \in H_n(X, A)$, $\gamma \in Z_n(X, A)$, $\gamma \in [\gamma]$. Let us denote $\alpha = f_n^{-1}(\gamma)$. It is $\alpha \in C_n(X-U, A-U)$, $f_n(\alpha) = \gamma$. Because of $f_{n-1}\bar{d}_n(\alpha) = \bar{d}_n f_n(\alpha) = \bar{d}_n(\gamma) = 0$, we have $\alpha \in Z_n(X-U, A-U)$ and therefore $f_{*n}([\alpha]) = [f_n(\alpha)] = [\gamma]$.

3.12. Remark: We see that we got the "excission axiom" in a slightly stronger form. We can define excission as an imbedding $f: (X-U, A-U) \subset (X, A)$, where $Cl\,St(U) \subset A$, though in formal translation of Eilenberg-Steenrod excission axiom we get the assumption U open and $St\,Cl(U) \subset A$.

§ 4.

4.1. Definition: Let X be a discrete space. The space $\mathcal{C}(X)$ is the set of all finite subsets $\{x_i\}$ of sets of type $Cl(y)$, $y \in X$, with topology defined by inclusion (i.e. $\alpha \in Cl(\beta) \Longleftrightarrow \alpha \subset \beta$).

Let $f: X \to Y$ be a continuous mapping. We define the mapping $\mathcal{C}(f): \mathcal{C}(X) \to \mathcal{C}(Y)$ by formula $\mathcal{C}(f)(\{x_i\}) = \{f(x_i)\}$. (It is correct, because if $x_i \in Cl(y)$, $f(x_i) \in Cl(f(y))$ according to the continuity of f.)

If $A \subset X$ we have obviously $\mathcal{C}(A) \subset \mathcal{C}(X)$ and we can define $\mathcal{C}(X,A) = (\mathcal{C}(X), \mathcal{C}(A))$. Let $f: (X,A) \to (Y,B)$ be a continuous mapping. Then we have $\mathcal{C}(f')(\mathcal{C}(A)) \subset \mathcal{C}(B)$, where f' is the corresponding mapping $X \to Y$ and we can define $\mathcal{C}(f)$ in the natural way.

4.2. Theorem: $\mathcal{C}$ is a covariant h-functor from the category of couples of finite discrete spaces into itself.

Proof: The fact that $\mathcal{C}(f)$ is continuous (for f continuous) is obvious. The same for the facts about $\mathcal{C}$ being a covariant functor preserving pairs of mappings of a type $j: A \subset X$, $i: X \subset (X,A)$ and preserving one-point spaces. It remains to prove the homotopy and excission preserving.

Homotopy: Let $f, g: X \to Y$ be homotopical mappings. Because of 3.7 we can assume without loss of generality

(1) $f(x) \in \mathcal{U}(g(x))$ for every $x \in X$.

We define $h: \mathcal{C}(X) \times I \to \mathcal{C}(Y)$ by formulae

$$h(\{x_i\}, 0) = \{f(x_i)\}$$

$$h(\{x_i\}, 1) = \{g(x_i)\}$$

$$h(\{x_i\}, t) = \{f(x_i)\} \cup \{g(x_i)\} \quad \text{for } 0 < t < 1.$$

$\{f(x_i)\} \cup \{g(x_i)\}$ is an element of $\mathcal{C}(Y)$, for according to (1), $\{f(x_i)\} \cup \{g(x_i)\} \subset \mathcal{U}(g(y))$, where y is such a point of X, that $\{x_i\} \subset \mathcal{U}(y)$). The continuity of h is obvious.

Excission: Let $\mathcal{U}\,St(U) \subset A \subset X$. We are going to prove that then $\mathcal{U}\,St(\mathcal{C}(U)) \subset \mathcal{C}(A)$. Let $\{x_i\}_{i=1,\dots,m} \in \mathcal{U}\,St(\mathcal{C}(U))$. Hence there is $\{x_i\}_{i=1,\dots,m} \in St(\mathcal{C}(U)), m \geq n$. Hence at first there exists some element $y \in X$, $x_i \in \mathcal{U}(y)$ for every $i = 1, \dots, m$. Because of $\{x_i\}_{i=1,\dots,m} \in St(\mathcal{C}(U))$, there exists an

i_o , $x_{i_o} \in U$. Hence $\{x_i\}_{i=1,\dots,m} \subset Cl(y)$ and $y \in St(x_{i_o}) \subset A$ and therefore $\{x_i\}_{i=1,\dots,m} \in \mathcal{C}(A)$.

4.3. Corollary: We can define new homology theory from the category of couples of finite discrete spaces by formulae

$$H'_n(X, A; G) = H_n(\mathcal{C}(X), \mathcal{C}(A); G), \quad f'_{*n} = (\mathcal{C}(f))_{*n},$$
$$\partial'(n, X, A) = \partial(n, \mathcal{C}(X), \mathcal{C}(A)).$$

4.4. Theorem: There exist non-isomorphical homology theories from the category of couples of finite discrete spaces satisfying all seven Eilenberg-Steenrod axioms and having the same coefficient group.

Proof: The homology theory defined in 2.6 and the one defined in 4.3 satisfy Eilenberg-Steenrod axioms. Let us construct the space X , consisting of four points a, b, c and d with the topology defined as follows:

$$Cl(a) = \{a, c, d\}, \; Cl(b) = \{b, c, d\}, \; Cl(c) = (c), \; Cl(d) = (d).$$

It is only a matter of counting to prove that

$$H_o(X; G) \approx G, \; H'_o(X; G) \approx G, \; H_1(X; G) \approx G, \text{ but } H'_1(X, G) = 0.$$

4.5. Remark: In the following paragraphs we shall prove further important property of the homology theory from 2.6 . We shall prove that the theorem of uniqueness for homology theories, satisfying Eilenberg-Steenrod axioms and having this property, does hold for the category of couples of finite discrete spaces.

§ 5.

5.1. Definition: Let X be a discrete space. In the set of all finite sequences of elements of X of a type $(x_o, \dots, x_m)$ with $x_i \in Cl(x_{i+1})$, $x_i \neq x_{i+1}$, we define

a topology by inclusion. Let us denote $\mathcal{B}(X)$ the space obtained this way. Let us define a mapping $\varepsilon : \mathcal{B}(X) \to X$ (more precisely ε_X) by formula $\varepsilon((x_0, \ldots, x_n)) = x_n$.

5.2. Lemma: ε - is a continuous mapping onto.

Proof: The fact that ε is onto is obvious, because $\mathcal{B}(X)$ contains the sequences consisting of one element, too. Continuity: Let $\xi = (x_0, \ldots, x_n) \in \mathcal{B}(X)$. Let $\xi' \in Cl(\xi)$. Hence $\xi' = (x_{k_0}, \ldots, x_{k_l})$, $k_l \leq n$ and therefore

$$\varepsilon(\xi') = x_{k_l} \in Cl(x_n) = Cl(\varepsilon(\xi))$$

5.3. Definition: Let $f : X \to Y$ be a continuous mapping. Let us define $\mathcal{B}(f) : \mathcal{B}(X) \to \mathcal{B}(Y)$ by formula $\mathcal{B}(f)(x_0, \ldots, x_n) = (f(x_0), \ldots, f(x_n))'$ ($(y_0, \ldots, y_n)'$ means the maximal strictly monotone subsequence of $(y_0, \ldots, y_n)$; according to the continuity of f , $(f(x_0), \ldots, f(x_n))'$ is really an element of $\mathcal{B}(Y)$). If $A \subset X$, we have $\mathcal{B}(A) \subset \mathcal{B}(X)$ and therefore we can define $\mathcal{B}(X, A) = (\mathcal{B}(X), \mathcal{B}(A))$. The mapping $\varepsilon_{(X,A)} : \mathcal{B}(X, A) \to (X, A)$ is defined in the obvious way. If $f : (X, A) \to (Y, B)$ is a continuous mapping of couples of discrete spaces, we have $\mathcal{B}(f)(\mathcal{B}(A)) \subset \mathcal{B}(B)$ and therefore we can define $\mathcal{B}(f)$ for such mappings in the natural way.

5.4. Theorem: $\mathcal{B}$ is a covariant functor from the category of couples of discrete spaces into itself. The system $\{\varepsilon_{(X,A)}\}$ is a transformation of functor $\mathcal{B}$ into identical functor of our category.

Proof: Continuity of $\mathcal{B}(f)$ for continuous f and functorial properties of $\mathcal{B}$ are obvious. It remains to prove the commutativity of diagrams of a type

$$\begin{array}{ccc} \mathcal{B}(X,A) & \xrightarrow{\mathcal{B}(f)} & \mathcal{B}(Y,B) \\ \downarrow{\scriptstyle \varepsilon_{(X,A)}} & & \downarrow{\scriptstyle \varepsilon_{(Y,B)}} \\ (X,A) & \xrightarrow{f} & (Y,B) \end{array}$$

Let us take a $\xi = (x_0, \ldots, x_n) \in B(X)$. We have

$$f\,\varepsilon_{(X,A)}(\xi) = f(x_n),$$

$$\varepsilon_{(Y,B)}\,B(f)(\xi) = \varepsilon_{(Y,B)}(f(x_0), \ldots, f(x_n))' = f(x_n) \quad \text{q.e.d.}$$

5.5. It is very easy to prove

Lemma: $\varepsilon_n : C_n(\mathcal{B}(X,A)) \to C_n(X,A)$ is an epimorphism for every n .

5.6. Lemma: Let us define a homomorphism

$$s_n : C_n(\mathcal{B}(X)) \longrightarrow C_n(\mathcal{B}(X))$$

by formula

$$s_n(g(\xi_0, \ldots, \xi_n)) = g((\varepsilon(\xi_0))', (\varepsilon(\xi_0), \varepsilon(\xi_1))', \ldots, (\varepsilon(\xi_0), \ldots, \varepsilon(\xi_n))')$$

Then $\operatorname{Ker} s_n = \operatorname{Ker} \varepsilon_n$.

Proof: We must at first prove that the definition of s_n is correct, i.e. to prove the fact that $(\varepsilon(\xi_0), \ldots, \varepsilon(\xi_k))' \in \mathcal{B}(X)$. Let us take a non-negative integer $i < n$. $(\xi_0, \ldots, \xi_n)$ is an elementary chain, hence $\xi_i \in \mathcal{C}l(\xi_{i+1})$, i.e. $\xi_i \subset \xi_{i+1}$ and therefore $\varepsilon(\xi_i) \in \mathcal{C}l(\varepsilon(\xi_{i+1}))$.

It holds:

(1) $\varepsilon_n(g(\xi_0, \ldots, \xi_n)) = \varepsilon_n(g(\eta_0, \ldots, \eta_n))$

iff (2) $s_n(g(\xi_0,\dots,\xi_n)) = s_n(g(\eta_0,\dots,\eta_n))$

(because both (1) and (2) are equivalent with the assertion $\mathfrak{ae}(\xi_i) = \mathfrak{ae}(\eta_i)$ for $i = 0,\dots,n$).

We have to prove the equivalence:

(1′) $\mathfrak{ae}_n(\sum_{i=1}^{m} g_i(\xi_{i0},\dots,\xi_{in})) = 0$,

iff (2′) $s_n(\sum_{i=1}^{m} g_i(\xi_{i0},\dots,\xi_{in})) = 0$.

In the set $\{1,\dots,m\}$ let us define the equivalence relation by formula

(3) $i \sim j$ iff there exists a $g \neq 0$ such that

$$\mathfrak{ae}_n(g(\xi_{i0},\dots,\xi_{in})) = \mathfrak{ae}_n(g(\xi_{j0},\dots,\xi_{jn})).$$

Let us denote $I_0,\dots,I_k$ the equivalence classes. The same equivalence classes will be obtained if we substitute $\mathfrak{ae}_n$ by s_n in (3) (see (1) and (2)).

Now, both (1′) and (2′) are equivalent with the assertion

$$\sum_{i\in I_j} a_i = 0 \qquad \text{for every } j.$$

5.7. Lemma: Let $(\xi_0,\dots,\xi_m)$ be an elementary chain. Let us define $\xi'_k = (\mathfrak{ae}(\xi_0),\dots,\mathfrak{ae}(\xi_k))'$. Then $\xi'_k \in Cl(\xi_k)$ for every k and $\mathfrak{ae}(\xi'_l) = \mathfrak{ae}(\xi_k)$.

Proof: I. For $k = 0$, $(\mathfrak{ae}(\xi_0))$ is obviously a subset of ξ_0.

II. Let $\xi'_{k-1} \subset \xi_{k-1}$. Hence $\xi'_{k-1} \subset \xi_k$ and therefore $\xi'_k = (\xi'_{k-1}, \mathfrak{ae}(\xi_k))' \subset \xi_k$ (because of $\mathfrak{ae}(\xi_k) \in \xi_k$).

5.8. Lemma: Let us preserve the denotation of preceding lemma. According to this lemma we can define a homomorphism $D_n : C_n(\mathfrak{B}(X)) \longrightarrow C_{n+1}(\mathfrak{B}(X))$ by formula

$$D_n(g(\xi_0,\dots,\xi_n)) = \sum_{i=0}^{n} (-1)^i g(\xi'_0,\dots,\xi'_i,\xi_i,\dots,\xi_n).$$

Then we have

$$d_1 D_0(\alpha) = \alpha - s_0(\alpha),$$

$$d_{n+1} D_n(\alpha) = \alpha - s_n(\alpha) - D_{n-1} d_n(\alpha) \quad (n \geq 1),$$

and

$$\mathscr{e}_n D_{n-1} d_n(\alpha) = - d_{n+1} \mathscr{e}_{n+1} D_n(\alpha) \quad (n \geq 1).$$

Proof: Is a matter of counting.

5.9. Theorem: $\mathscr{e}_{*n} : H_n(\mathcal{B}(X,A)) \to H_n(X,A)$ are isomorphisms.

Proof: I. $\mathscr{e}_{*n}$ is a monomorphism:

In this proof let us denote $\mathscr{e} = \mathscr{e}_{(X,A)}$, $\mathscr{e}' = \mathscr{e}_X$, $\mathscr{e}'' = \mathscr{e}_A$.

Let $\mathscr{e}_{*n}([a]) = 0$, i.e. $\mathscr{e}_n(a) \in B_n(X,A)$.

Hence there exists a $b \in C_{n+1}(X,A)$, $b = [\beta]$ such that $\overline{d}_{n+1}(b) = \mathscr{e}_n(a)$. $a \in Z_n(\mathcal{B}(X,A))$; let $a = [\alpha]$;

hence we have $[d_{n+1}(\beta)] = [\mathscr{e}'_n(\alpha)]$ and hence

$d_{n+1}(\beta) - \mathscr{e}'_n(\alpha) = \gamma \in C_n(A)$.

$\mathscr{e}_n$ is an epimorphism and hence there exists a $b' \in C_n(\mathcal{B}(X,A))$, $b' = [\beta']$ such that $b = \mathscr{e}_{n+1}(b')$ and hence

$[\beta] = [\mathscr{e}'_{n+1}(\beta')]$ and therefore

(1) $\quad d_{n+1} \mathscr{e}'_{n+1}(\beta') - \mathscr{e}'_n(\alpha) - d_{n+1}(\gamma') - \gamma = 0.$

$\mathscr{e}''_n$ is an epimorphism and therefore there exists a $\sigma \in C_n(\mathcal{B}(A))$ such that $d_{n+1}(\gamma') - \gamma = \mathscr{e}''_n(\sigma') = \mathscr{e}'_n(\sigma)$. We get the formula (1) in the form

$$\mathscr{e}'_n(d_{n+1}(\beta') - \alpha - \sigma') = 0$$

and therefore $\lambda = d_{n+1}(\beta') - \alpha - \sigma \in \operatorname{Ker} \mathscr{e}'_n = \operatorname{Ker} s_n$. Hence we have

(A) if $n = 0$:

$d_1 D_0(\lambda) = \lambda$ and hence

$\overline{d}_1([D_0(\lambda)]) = [\lambda] = \overline{d}_1([\beta]) - [\alpha] \quad ([\sigma] = 0)$.

Hence $a = \overline{d}_1(b - [D_0(\lambda)])$.

B) If $n > 0$:

$$d_{n+1} D_n(\lambda) = \lambda - D_{n-1} d_n(\lambda) \quad \text{and hence}$$

$$\bar{d}_{n+1}([D_n(\lambda)]) = [\lambda] - [D_{n-1} d_n(\lambda)] =$$

$$= [d_{n+1}(\beta)] - [\alpha] - [D_{n-1}(d_{n+1}(\beta) - \alpha - \delta'))] =$$

$$= \bar{d}_{n+1}(b) - a - [D_{n-1}(d_n(\alpha) - d_n(\delta'))] .$$

Because of $a \in Z_n(\mathcal{B}(X, A))$, we have $d_n(\alpha) \in C_{n-1}(\mathcal{B}(A))$; obviously $d_n(\delta') \in C_{n-1}(\mathcal{B}(A))$, too. Because $D_i(C_i(\mathcal{B}(A)) \subset C_{i+1}(\mathcal{B}(A))$, we have $a = \bar{d}(b - [D_n(\lambda)])$ and hence $[a] = 0$.

II) $\mathfrak{se}_{*n}$ is an epimorphism:

Let $[b] \in H_n(X, A)$, $b \in Z_n(X, A)$, $b \in [b]$.

$\mathfrak{se}_n$ is an epimorphism and hence there is an $a \in C_n(\mathcal{B}(X, A))$ such that $\mathfrak{se}_n(a) = b$. Let $n = 0$. Then $C_0(\mathcal{B}(X, A)) = Z_0(\mathcal{B}(X, A))$ and hence

$$[a] \in H_0(\mathcal{B}(X, A)), \quad \mathfrak{se}_{*0}([a]) = [\mathfrak{se}_0(a)] = [b] .$$

Let $n > 0$, $a = [\alpha]$, $b = [\beta]$. We have $\bar{d}_n(b) = 0$, i.e. $d_n(\beta) \in C_{n-1}(A)$.
We have $\mathfrak{se}_{n-1}(d_n(a)) = \bar{d}_n \mathfrak{se}_n(a) = 0$ and hence $\mathfrak{se}'_{n-1} d_n(\alpha) \in C_{n-1}(A)$.
Because $\mathfrak{se}''_{n-1}$ is an epimorphism, there exists a $\gamma \in C_{n-1}(\mathcal{B}(A))$ such that

$$d_n(\alpha) - \gamma \in \operatorname{Ker} \mathfrak{se}'_{n-1} = \operatorname{Ker} s_{n-1} .$$

Hence we have for $n = 1$:

$d_1 D_0(d_1(\alpha) - \gamma) = d_1(\alpha) - \gamma$ and therefore

$\bar{d}_1([D_0(d_1(\alpha) - \gamma)]) = d_1(a)$. Let us denote

$a' = a - [D_0(d_1(\alpha) - \gamma)]$. It is $\bar{d}_1(a') = 0$, i.e.

$a' \in Z_1(\mathcal{B}(X, A))$, and $\mathfrak{se}_1(a') = \mathfrak{se}_1(a) - [\mathfrak{se}_1 D_0 d_1(\alpha)] =$

$= b - [d_2 \mathfrak{se}_2 D_1(\alpha)] = b - \bar{d}_2([\mathfrak{se}_2 D_1(\alpha)])$ and hence $\mathfrak{se}_{*1}([a']) = [b]$.

If $n > 1$, we have $d_n(D_{n-1}(d_n(\alpha) - \gamma)) =$

$= d_m(\alpha) - \gamma - D_{m-2}(d_{m-1}(d_m(\alpha) - \gamma)) = d_m(\alpha) - \gamma + D_{m-2}(d_{m-1}(\gamma))$.

On the other hand we have $\gamma \in C_{m-1}(\mathcal{B}(A))$, hence $d_{m-1}(\gamma) \in C_{m-2}(\mathcal{B}(A))$ and hence $D_{m-2}\, d_{m-1}(\gamma) \in C_{m-1}(\mathcal{B}(A))$. We get $\bar{d}_m([D_{m-1}(d_m(\alpha) - \gamma)]) = \bar{d}_m(a)$ and hence (because of $D_{m-1}(\gamma) \in C_m(\mathcal{B}(A))$) $\bar{d}_m(a - [D_{m-1}(d_m(\alpha))]) = 0$.

Now let us denote $a' = a - [D_{m-1}(d_m(\alpha))]$ and we get easily $\mathcal{H}_{*m}([a']) = [b]$.

5.10. Remark: Because of 5.4 we can formulate the preceding result in the following way: For every n, the system $\{(\mathcal{H}_{(X,A)})_{*n}\}$ is a natural isomorphism between the functors $H_n \circ \mathcal{B}$ and H_n.

5.11. Theorem: For every couple (X, A) of discrete spaces the commutativity holds in the diagram

$$\begin{array}{ccc} H_m(\mathcal{B}(X,A)) & \xrightarrow{(\mathcal{H}_{(X,A)})_{*n}} & H_m(X,A) \\ \downarrow \partial(\mathcal{B}(X,A),n) & & \downarrow \partial(X,A,n) \\ H_{m-1}(\mathcal{B}(A)) & \xrightarrow{(\mathcal{H}_A)_{*m-1}} & H_{m-1}(A) \end{array}$$

Proof: It is an easy consequence of commutativity of the diagrams

$$\begin{array}{ccc} C_m(\mathcal{B}(A)) & \xrightarrow{(\mathcal{B}j)_m} & C_m(\mathcal{B}(X)) \\ \downarrow (\mathcal{H}_A)_m & & \downarrow (\mathcal{H}_X)_m \end{array} \qquad \begin{array}{ccc} C_m(\mathcal{B}(X)) & \xrightarrow{(\mathcal{H}_X)_m} & C_m(X) \\ \downarrow d_m & & \downarrow d_m \end{array}$$

and

$$\begin{array}{ccc} C_m(A) & \xrightarrow{j_m} & C_m(X) \end{array} \qquad \begin{array}{ccc} C_{m-1}(\mathcal{B}(X)) & \xrightarrow{(\mathcal{H}_X)_{m-1}} & C_{m-1}(X) \end{array}$$

(where $j : A \subset X$), and the definition of ∂ (see, for example [E-S]).

5.12. Remark: Let us agree to call, for further purposes, "the property (B)" the property of homology theory $\{H_{n,*}, \partial\}$, formulated in 5.10 and 5.11.

§ 6.

6.1. Definition: Let X be a triangulable space, T some triangulation of X. Let us denote $\mathcal{D}^T(X)$ the set of all simplexes of this triangulation with the topology defined by formula:

$s' \in Cl(s)$ iff s' is a face of s.

Let A be a subspace of X triangulated by T. Then $\mathcal{D}^T(A) \subset \mathcal{D}^T(X)$ and hence we can define $\mathcal{D}^T(X,A) = (\mathcal{D}^T(X), \mathcal{D}^T(A))$. Let $f: X \to Y$ be a simplicial mapping with respect to the triangulations T, U of X, Y respectively. We define $\mathcal{D}^{T,U}(f): \mathcal{D}^T(X) \to \mathcal{D}^U(Y)$ by formula $\mathcal{D}^{T,U}(f)(s) = f(s)$ (the image of the set). Obviously, $\mathcal{D}^{T,U}(f)$ is a continuous mapping. $\mathcal{D}^{T,U}(f)$ for a mapping f of couples of triangulated spaces is defined in the obvious way.

6.2. Theorem: Let $\{H_n, *, \partial\}$ be some homology theory from the category of couples of finite discrete spaces, satisfying all Eilenberg-Steenrod axioms and having the property Then there exists a homology theory $\{H'_n, *', \partial'\}$ from the category of triangulable couples which agree with $\{H_n, *, \partial\}$, i.e. for every triangulable couple (X,A) and for its every triangulation T there can be defined isomorphisms

$$i(X,A;T;n): H'_n(X,A) \approx H_n(\mathcal{D}^T(X,A))$$

such that the system $\{i(X,A;T,n)\}$ has the following properties:

(1) For every (X,A), T the commutativity holds in the rectangle

$$\begin{array}{ccc} H'_n(X,A) & \xrightarrow{i(X,A;T)} & H_n(\mathcal{D}^T(X,A)) \\ \downarrow \partial' & & \downarrow \partial \\ H'_{n-1}(A) & \xrightarrow{i(A;T)} & H_{n-1}(\mathcal{D}^T(A)) \end{array}$$

2) For every $f:(X,A)\to(Y,B)$ simplicial with respect to triangulations T, U the commutativity holds in the rectangle

$$\begin{array}{ccc} H'_n(X,A) & \xrightarrow{i(X,A;T)} & H_n(\mathcal{D}^T(X,A)) \\ \downarrow{\scriptstyle f_{*n}} & & \downarrow{\scriptstyle (\mathcal{D}^{T,U}(f))_{*n}} \\ H'_n(Y,B) & \xrightarrow{i(Y,B;U)} & H_n(\mathcal{D}^U(Y,B)) \end{array}$$

Proof: Let us construct the Čech homology theory from the category of triangulable couples in the way that we use our homology theory for nerves (of finite coverings) taken as discrete spaces. It is easy to show that the nerve of covering of (X,A) by stars of edges in the triangulation T is homeomorphic with $\mathcal{D}^T(X,A)$. Further, it is easy to show that we have $\mathcal{D}^U(X,A)$ homeomorphic with $\mathcal{B}\mathcal{D}^T(X,A)$ if U is the barycentrical subdivision of T, and that the corresponding projection, induced by projection of nerves, is homotopical to the mapping $\varkappa$ from § 5. Finally, it is obvious that the system of coverings by stars of edges in barycentrical subdivisions of a given triangulation is confinal in the directed set of all coverings. For wanted isomorphism i we can take the projection from the limit-group in the construction to the group of covering by stars of edges. The commutativity relations are not difficult to prove.

6.3. Theorem: For every two homology theories $\{H_n,*,\partial\}$, $\{\bar{H}_n,*,\bar{\partial}\}$ from the category of couples of finite discrete spaces, satisfying all Eilenberg-Steenrod axioms and having the property (B), and for every homomorphism $h_0: H_0(P)\to\bar{H}_0(P)$ (where P is a one-point space) there exists a system of homomorphisms

$$h(n;X,A) : H_n(X,A) \longrightarrow \bar{H}_n(X,A)$$

such that:

(1) $h(0,P) = h_0$

(2) For every continuous mapping $f : (X,A) \longrightarrow (Y,B)$ the commutativity holds in the rectangle

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{h(n,X,A)} & \bar{H}_n(X,A) \\ \downarrow f_{*n} & & \downarrow f_{\#n} \\ H_n(Y,B) & \xrightarrow{h(n,Y,B)} & \bar{H}_n(Y,B) \end{array}$$

(3) For every couple (X,A) the commutativity holds in the rectangle

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{h(n,X,A)} & \bar{H}_n(X,A) \\ \downarrow \partial & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow{h(n-1,A)} & \bar{H}_{n-1}(A) \end{array}$$

If h_0 is an isomorphism, so is every $h(n,X,A)$.

Proof: For every couple (X,A) of finite discrete spaces let us take some triangulated couple $\mathcal{K}(X,A)$ with triangulation T such that $\mathcal{D}^T(\mathcal{K}(X,A))$ is homoemorphical with $\mathcal{B}(X,A)$. (Such a $\mathcal{K}(X,A)$ obviously exists, moreover, for every continuous mapping f the mapping $\mathcal{B}(f)$ can be represented as a simplicial mapping of corresponding triangulated pairs.)

Now, let us construct for $\{H_n, *, \partial\}$, $\{\bar{H}_n, \#, \bar{\partial}\}$ the homology theories $\{H'_n, *', \partial'\}, \{\bar{H}'_n, \#', \bar{\partial}'\}$ (see 6.2) from the category of triangulable couples agreeing with the given ones. According to the theorem 10.1 from [E-S, chapter III] there exists a system of homomorphisms $\{h'(n,X,A)\}$

for $h'_o = \mathfrak{ae}_{\#o}^{-1} (\bar{\tau}(P_o; 0))^{-1} h_o (i(P_o; 0)) \mathfrak{ae}_*$ (i , $\bar{\tau}$ are the isomorphisms from 6.2), having corresponding properties.

If we take now

$$h(n, X, A) = (\mathfrak{ae}_{(X,A)})_{\#n} \bar{\tau}(\mathcal{H}(X,A), n)(h'(\mathcal{H}(X,A), n))(i(\mathcal{H}(X,A), n)^{-1} (\mathfrak{ae}_{(X,A)})_{\#n}^{-1}$$

all assertions can be easily verified.

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I. M A R E K, Neutron transport theory in the multigroup energetical approximation

S. M A L O Ň , On nonlinear numerical iteration processes

A. P U L T R , On the homology theory of discrete spaces