EXISTENCE OF MULTIPLE SOLUTIONS FOR A THIRD-ORDER THREE-POINT REGULAR BOUNDARY VALUE PROBLEM

MARTIN ŠENKYŘÍK, Olomouc

(Received December 27, 1990)

Summary. In the paper we prove an Ambrosetti-Prodi type result for solutions u of the third-order nonlinear differential equation, satisfying $u'(0) = u'(1) = u(\eta) = 0$, $0 \leqslant \eta \leqslant 1$.

Keywords: Boundary value problem, lower and upper solutions, coincidence degree, Nagumo functions, Ambrosetti-Prodi results

AMS classification: 34B15

1. INTRODUCTION

In a recent paper, Fabry, Mawhin and Nkashama [3] have considered periodic problems of the form

$$u'' + f(x, u) = s,$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0$$

and have proved that if

$$f(x, u) \to \infty \quad \text{as } |u| \to \infty$$

uniformly in $x \in [0, 2\pi]$, an Ambrosetti-Prodi type result [1] holds, namely, there exists s_1 such that the above problem has no solution if $s < s_1$, at least one solution if $s = s_1$, and at least two solutions if $s > s_1$. A similar result holds for

$$u' + f(x, u) = s,$$
$$u(0) = u(2\pi)$$

(see [5]) and the corresponding proofs rely on a combination of the techniques of lower and upper solutions and the degree theory.

In [2] a somewhat weakened Ambrosetti-Prodi-like [1] result is given only for the following special case of a higher order boundary value problem (BVP):

$$u^{(n)} + g(u) = s + e(x, u),$$
$$u(0) - u(2\pi) = \ldots = u^{(n-1)}(0) - u^{(n-1)}(2\pi) = 0.$$

In this paper we prove an Ambrosetti-Prodi-like result [1] for the third-order BVP

$$(1)_s \qquad u''' + f(t, u, u', u'') = s,$$
$$(2) \qquad u'(0) = u'(1) = u(\eta) = 0, \quad 0 \leqslant \eta \leqslant 1.$$

This problem models the static deflection of a three-layered elastic beam.

The proofs in this chapter are based on a combination of the techniques of lower and upper solutions and the degree theory.

2. Notations and definitions

$$\|x\| = \max \{|x(t)|, \ t \in [0, 1]\}.$$

Functions σ_1 and $\sigma_2 \in C^3(0, 1)$ satisfying

$$\sigma_1''' \geqslant s - f(t, x, \sigma_1'(t), \sigma_1''(t)),$$
$$\sigma_2''' \leqslant s - f(t, x, \sigma_2'(t), \sigma_2''(t))$$

for $t \in [0, 1]$, $x \in [\min\{\sigma_1(t), \sigma_2(t)\}, \max\{\sigma_1(t), \sigma_2(t)\}]$ and

$$\sigma_1(\eta) = \sigma_2(\eta) = 0,$$
$$\sigma_1'(0) \leqslant 0, \quad \sigma_1'(1) \leqslant 0,$$
$$\sigma_2'(0) \geqslant 0, \quad \sigma_2'(1) \geqslant 0,$$

will be called a lower and an upper solution of the BVP $(1)_s$, (2), respectively.

By replacing the above inequalities with strict inequalities we obtain the definition of a strict lower and a strict upper solution of the BVP $(1)_s$, (2).

The BVP $(1)_s$, (2) is equivalent to

$$Lu + N_s u = 0,$$

where

$$L\colon \operatorname{dom} L \to C^0(0,1),\ Lu = u''',$$
$$X = \{x \in C^2(0,1),\ x \text{ satisfies (2)}\},\ \operatorname{dom} L = C^3(0,1) \cap X,$$
$$N_s\colon X \to C^0(0,1),\quad N_s u = f(t,u,u',u'') - s,\quad s \in \mathbb{R}.$$

It can be easily proved (see [4]) that $L+N_s$ is L-compact on $\overline{\Omega}$ (with $\overline{\Omega}$ the closure of Ω), where Ω is an open bounded subset of X.

3. Lemmas and theorems

Lemma 1. (On a priori estimates) *Let u be a solution of* $(1)_s$, (2) *and let* $\|u'\| \leqslant R$, $R \in \mathbb{R}$, $R > 0$. *Assume that for every $R \in \mathbb{R}$, $R > 0$ there exists a continuous function $h_R\colon \mathbb{R}^+ \to [a_R, \infty)$ ($a_R > 0$) such that*

$$|f(t,x,y,z)| \leqslant h_R(|z|) \tag{3}$$

for $x, y \in [-R, R]$, $t \in [0,1]$, $z \in \mathbb{R}$, where

$$\int_0^\infty \frac{t\,\mathrm{d}t}{h_R(t)} = \infty. \tag{4}$$

Then there exists r^* (depending only on s, R, h_R) such that

$$\|u''\| \leqslant r^*.$$

Proof. Let u be a solution of $(1)_s$, (2) and $\|u'\| \leqslant R$. We define

$$\Omega(x) = \int_0^x \frac{t\,\mathrm{d}t}{h_R(|t|) + |s|}.$$

From (4) it folows that Ω is a bijective mapping of $\mathbb{R}^+$ onto itself. From (2) it follows that there exists $a_0 \in (0,1)$ such that $u''(a_0) = 0$. Let $r^* = \Omega^{-1}\big(\Omega(1) + 2R\big)$ and assume that $|u''(t_1)| > r^*$, where $t_1 \in (a_0, 1]$. Let $[a_1, b_1] \subset [a_0, 1]$ be the maximal interval containing t_1 in which $|u''(t)| \geqslant 1$ and let $s_1 \in (a_1, b_1]$ be such that

$$|u''(s_1)| = \varrho_1 = \max\big\{|u''(t)|\colon a_1 \leqslant t \leqslant b_1\big\}. \tag{5}$$

From (3) and $(1)_s$ it follows that

$$|u'''| = |s - f(t,u,u',u'')| \leqslant h_R(|u''|) + |s|. \tag{6}$$

If $u''(t) \geqslant 1$, then

$$\int_{a_1}^{s_1} \frac{u''u'''}{h_R(u'') + |s|} \leqslant \int_{a_1}^{s_1} u'' \,\mathrm{dt}.$$

The last inequality implies that $\Omega(\varrho_1) - \Omega(1) \leqslant 2R$ and $\varrho_1 \leqslant r^*$ which contradicts (5). We can obtain a similar contradiction if $u''(t) \leqslant -1$ on $[a_1, s_1]$. For $t_1 \in [0, a_0]$ the proof is analogous. Lemma 1 is proved. □

Theorem 2. *Let σ_1 be a lower solution and σ_2 an upper solution of the BVP* $(1)_s$, (2) *and let $\sigma_1'(t) \leqslant \sigma_2'(t)$ for every $t \in [0,1]$. If the function f satisfies* (3), *then the BVP* $(1)_s$, (2) *has a solution u such that*

$$\sigma_1'(t) \leqslant u'(t) \leqslant \sigma_2'(t) \quad \textit{for each } t \in [0,1].$$

Proof. The theorem follows from Lemma 1 (On a priori estimates) and from the results given in [6]. □

Remark. [6] deals with the BVP

$$u''' = f(t, u, u', u''), \quad (2).$$

The existence of a solution u satisfying

$$\sigma_1'(t) \leqslant u'(t) \leqslant \sigma_2'(t),$$

where σ_1, σ_2 is a lower and an upper solution, respectively, is proved under a more general growth condition than (3).

Theorem 3. *Let f be nonincreasing (or nondecreasing) for $t \in [0, \eta]$ (for $t \in [\eta, 1]$) as a function of x for every fixed $y, z \in \mathbb{R}$. Further suppose there exist R_1, $s_1 \in \mathbb{R}$, $R_1 > 0$ such that*

$$f(t, R_1(t-\eta), 0, 0) < s_1 \quad \textit{for } t \in [0,1], \tag{7}$$

and for any $r_1 \geqslant R_1$ the inequality

$$s_1 < f(t, -r_1(t-\eta), y, 0) \quad \textit{for } t \in [0,1],\ y \leqslant -r_1, \tag{8}$$

is valid. If the function f satisfies (3), *then there exists $s_0 < s_1$ (with the posibility that $s_0 = -\infty$) such that for $s < s_0$ the BVP* $(1)_s$, (2) *has no solution and for $s \in (s_0, s_1]$ the BVP* $(1)_s$, (2) *has at least one solution.*

Proof. Let $s^* = \max\{f(t,0,0,0);\ t \in [0,1]\}$. From (7) and (8) it follows that $s^* - f(t,x,0,0) \geqslant 0$ and $s^* - f(t,x,-R_1,0) \leqslant 0$ for $t \in [0,1]$, $x \in \bigl[\min\{0,-R_1(t-\eta)\}, \max\{0,-R_1(t-\eta)\}\bigr]$. From the last two inequalities we get that $\sigma_1 = -R_1(t-\eta)$ is a lower solution of $(1)_{s^*}$, (2) and $\sigma_2 = 0$ is an upper solution of the BVP $(1)_{s^*}$, (2), so Theorem 2 implies that the BVP $(1)_{s^*}$, (2) has a solution.

Next we show that if the BVP $(1)_s$, (2) has a solution u for $s = \mathrm{s} < s_1$ then it also has a solution for $s \in [\mathrm{s}, s_1]$. If $s \in [\mathrm{s}, s_1]$ then $u''' = \mathrm{s} - f(t,u,u',u'')$ and $u''' \leqslant s - f(t,x,u',u'')$ for $t \in [0,\eta]$, $x \geqslant u$ or for $t \in [\eta,1]$, $x \leqslant u$. It is easily seen that for $s \leqslant s_1$ all solutions of $(1)_s$, (2) satisfy the relation $-R_1 \leqslant u'$. If $u'(t_0) \leqslant -R_1$ for some $t_0 \in (0,1)$, then there exists $t_1 \in (0,1)$ such that $\min\{u'(t),\ t \in (0,1)\} = u'(t_1)$, $u''(t_1) = 0$, $u'''(t_1) \geqslant 0$. If $t_1 \in [\eta,1)$ then $u'(t_1) = -r_1 \leqslant -R_1$, $u'(t) \geqslant -r_1$ for $t \in [\eta,1)$ and $u(t_1) \geqslant -r_1(t_1-\eta)$. From (8) it follows that $s_1 < f(t_1,u(t_1),-r_1,0)$, $u'''(t_1) < 0$ and this contradicts our assumption. A similar contradiction can be obtained for $t_1 \in (0,\eta]$.

(8) implies that $s - f(t,x,-R_1,0) \leqslant 0$ for $t \in [0,1]$, $x \in \bigl[\min\{u(t),-R_1(t-\eta)\}, \max\{u(t),-R_1(t-\eta)\}\bigr]$. Setting $\sigma_1 = -R_1(t-\eta)$, $\sigma_2 = u$ and using Theorem 2 we can see that the BVP $(1)_s$, (2) has a solution.

Taking $s_0 = \inf\{s \in \mathbb{R}\colon (1)_s, (2) \text{ has a solution}\}$ with $s_0 = -\infty$ if the BVP $(1)_s$, (2) has a solution for any $s \leqslant s_1$, it follows from the above discussion that $s_0 \leqslant s^* < s_1$ and that $(1)_s$, (2) has a solution for any $s \in (s_0, s_1]$. Theorem 3 is proved. □

Lemma 4. *Let $\Omega = \{x \in \operatorname{dom} L\colon \sigma_1'(t) < x'(t) < \sigma_2'(t),\ \|x''\| < k\}$, where $\sigma_1 < \sigma_2$, σ_1 is a strict lower solution and σ_2 is a strict upper solution of $(1)_s$, (2). If f satisfies (3) then there exists $k \in \mathbb{R}$ such that the coincidence degree of $L + N_s$ in Ω relative to L (see [4]) satisfies*

$$d_L(L+N_s,\Omega) = \pm 1 \pmod 2.$$

Proof. We define

$$g(t,x,y,z) = f\bigl(t,\alpha(t,x),\beta(t,y),z\bigr) - y + \beta(t,y),$$

$$\alpha(t,x) = \begin{cases} \min\{\sigma_1(t),\sigma_2(t)\} & \text{for } x < \min\{\sigma_1(t),\sigma_2(t)\},\\ x & \text{for } \min\{\sigma_1(t),\sigma_2(t)\} \leqslant x \leqslant \max\{\sigma_1(t),\sigma_2(t)\},\\ \max\{\sigma_1(t),\sigma_2(t)\} & \text{for } x > \max\{\sigma_1(t),\sigma_2(t)\}, \end{cases}$$

$$\beta(t,y) = \begin{cases} \sigma_1'(t) & \text{for } y' < \sigma_1'(t),\\ y & \text{for } \sigma_1'(t) \leqslant y \leqslant \sigma_2'(t),\\ \sigma_2'(t) & \text{for } y' > \sigma_2'(t). \end{cases}$$

The BVP

$$(9)_s \qquad u''' + g(t,u,u',u'') = s, \ (2)$$

can be written in the form of an operator equation

$$Lu + G_s u = 0 \quad \text{in } \operatorname{dom} L,$$

where $G_s\colon X \to C^0(0,1)$, $G_s u = g(t,u,u',u'') - s$.

In $\overline{\Omega}$ the BVP $(1)_s$, (2) is equivalent to the BVP $(9)_s$, (2), the operator equation $Lu + N_s u = 0$ is equivalent to the operator equation $Lu + G_s u = 0$ and

$$d_L(L + G_s, \Omega) = d_L(L + N_s, \Omega).$$

We define $\Omega_1 = \{x \in \operatorname{dom} L\colon \|x'\| < r^*, \ \|x''\| < k\}$, where $r^* > \max\{\|\sigma_1\|, \|\sigma_2\|\}$. We shall prove that for $\lambda \in [0,1]$ every solution of the equation

$$(10) \qquad Lu - (1-\lambda)Iu + \lambda G_s u = 0,$$

where $Iu = u'$, satisfies $u \notin \overline{\Omega}_1$. If $\|u'\| \geqslant r^*$, then there exists $t_0 \in (0,1)$ such that

$$\begin{aligned} &u'(t_0) \geqslant r^* \quad (\text{or } u'(t_0) \leqslant -r^*),\\ &u''(t_0) = 0,\\ &u'''(t_0) \leqslant 0 \quad (u'''(t_0) \geqslant 0). \end{aligned}$$

If r^* is large enough, then

$$\begin{aligned} &f\big(t, \alpha(t,x), \sigma_1', 0\big) - s + r^* + \sigma_1' > 0 \quad \text{and}\\ &f\big(t, \alpha(t,x), \sigma_2', 0\big) - s - r^* + \sigma_2' < 0 \quad \text{for } x \in \mathbb{R}, \ t \in [0,1]. \end{aligned}$$

For $u'(t_0) \leqslant -r^*$ we obtain

$$u'''(t_0) - (1-\lambda)u'(t_0) + \lambda\Big(f\big(t_0, \alpha(t_0, u(t_0)), \sigma_1'(t_0), 0\big) - s - u'(t_0) + \sigma_1'(t_0)\big)\Big) = 0.$$

It follows from the last equality that $u'''(t_0) < 0$ which contradicts $u'''(t_0) \geqslant 0$. A similar contradiction can be obtained if we suppose that $u'(t_0) \geqslant r^*$. We have proved that $\|u'\| < r^*$. Since (3) is valid we get the inequality

$$\Big| - (1-\lambda)y - \lambda\Big(f\big(t, \alpha(t,x), \beta(t,y), z\big) - s - y + \beta(t,y)\big)\Big| \leqslant h_R(|z|) + 2r^* + |s|$$

for $y < r^*$, and

$$\int_0^\infty \frac{s\,ds}{h_R(s) + 2r^* + |s|} \geqslant \frac{1}{1 + \frac{2r^*+|s|}{a_R}} \int_0^\infty \frac{s\,ds}{h_R(s)} = \infty.$$

The last inequality implies that we can use Lemma 1 and for k large enough also $\|u''\| < k$ is satisfied.

For $\lambda = 0$ the equation (10) has only the trivial solution and $d_L(L - I, \Omega_1) = \pm 1 \pmod 2$. By the property of invariance under a homotopy we obatain $d_L(L + G_s, \Omega_1) = \pm 1 \pmod 2$. Next we prove that every solution u of the equation $Lu + G_s u = 0$ satisfies $u \in \Omega \subset \Omega_1$. If $u'(t_1) > \sigma_2'(t_1)$ for some $t_1 \in (0,1)$ then there exists an interval $(a,b) \subset (0,1)$, $t_1 \in (a,b)$, $u'(t) > \sigma_2'(t)$ for $t \in (a,b)$ and $u'(a) = \sigma_2'(a)$, $u'(b) = \sigma_2'(b)$. This implies that there exists $t_2 \in (a,b)$ such that

$$\begin{aligned} u'(t_2) &> \sigma_2'(t_2), \\ u''(t_2) &= \sigma_2''(t_2), \\ u'''(t_2) &\leqslant \sigma_2'''(t_2). \end{aligned}$$

Since u is a solution of (9) and σ_2 is a strict upper solution of $(1)_s$, (2), it follows that

$$u'''(t_2) + f\Big(t, \alpha\big(t_2, u(t_2), \sigma_2'(t_2), \sigma_2''(t_2)\big)\Big) - s - u'(t_2) + \sigma_2'(t_2) = 0,$$
$$u'''(t_2) > \sigma_2'''(t_2).$$

This contradicts the inequality $u'''(t_2) \leqslant \sigma_2'''(t_2)$. If $u'(t) \leqslant \sigma_2'(t)$ for $t \in (0,1)$ and there exists $t_3 \in (0,1)$ such that $u'(t_3) = \sigma_2'(t_3)$ then $u''(t_3) = \sigma_2''(t_3)$ and $u'''(t_3) \leqslant \sigma_2'''(t_3)$. This implies that

$$u'''(t_3) + f\Big(t_3, \alpha\big(t_3, u(t_3), \sigma_2'(t_3), \sigma_2''(t_3)\big)\Big) - s = 0$$

and since σ_2 is a strict upper solution of (9) we obtain $u'''(t_3) > \sigma_2'''(t_3)$. This contradicts $u'''(t_3) \leqslant \sigma_2'''(t_3)$.

It is possible to prove in a similar way that $u'(t) > \sigma_1'(t)$ for every possible solution u of the equation $Lu + G_s u = 0$ and for every $t \in [0,1]$.

By using the excision property of the degree we obtain

$$d_L(L + G_s, \Omega) = \pm 1 \pmod 2$$

and, finally,

$$d_L(L + N_s, \Omega) = \pm 1 \pmod 2.$$

Lemma 4 is proved. □

Theorem 5. *Let us suppose that the assumptions of Theorem 3 are fulfilled. Moreover, suppose that there exists $M(s_1) \in \mathbb{R}$ such that for $s \leqslant s_1$ any solution of the BVP $(1)_s$, (2) satisfies the inequality*

$$u'(t) \leqslant M(s_1) \quad \text{for } t \in [0,1] \tag{11}$$

and that there exists $\alpha \in \mathbb{R}$ such that

$$f(t,x,y,z) \geqslant \alpha \tag{12}$$

for $t \in [0,1]$, $x \in \bigl[\min\{-R_1(t-\eta), M(s_1)(t-\eta)\}, \max\{-R_1(t-\eta), M(s_1)(t-\eta)\}\bigr]$, $y \in [-R_1, M(s_1)]$, $z \in \mathbb{R}$. Then the number s_0 provided by Theorem 3 is finite and

for $s < s_0$ the BVP $(1)_s$, (2) has no solution,

for $s = s_0$ the BVP $(1)_s$, (2) has at least one solution,

for $s \in (s_0, s_1]$ the BVP $(1)_s$, (2) has at least two solutions.

Proof. First we prove that s_0 is finite. Let u be a solution of $(1)_s$, (2). From $(1)_s$ it follows that $u''' \leqslant s - \alpha$. From (2) it follows that

$$u''(t) \geqslant \frac{1}{4}(\alpha - s) \quad \text{for } t \in [0, \tfrac{1}{4}] \quad \text{or}$$
$$u''(t) \leqslant \frac{1}{4}(s - \alpha) \quad \text{for } t \in [\tfrac{3}{4}, 1].$$

If we take s such that $\frac{\alpha - s}{16} > M(s_1)$ we obtain a contradiction to (10).

Let $\mathrm{s} \in (s_0, s_1)$ and let u be a solution of the BVP $(1)_s$, (2) for $s = \mathrm{s}$. We can assume that $R_1 \leqslant |M(s_1)|$.

Let $\Omega_1 = \{x \in X : \|x(t)\| < |M(s_1)|,\ \|x'(t)\| < |M(s_1)|,\ \|x''(t)\| < \varrho\}$, where ϱ is taken sufficiently large. Since the BVP $(1)_s$, (2) has no solution for $s_{-1} < s_0$, it is a consequence of the basic properties of the degree that

$$d_L(L + N_{s_{-1}}, \Omega_1) = 0. \tag{13}$$

On the other hand, for $s \leqslant s_1$ all solutions of $(1)_s$, (2) satisfy the inequality $\|u'\| < |M(s_1)|$. If ϱ is large enough and $s \in [s_{-1}, s_1]$ then we have $\|u''\| < \varrho$ for all solutions of $(1)_s$, (2) (the bound given by Lemma 1 can be taken independent of s for $s \in [s_{-1}, s_1]$). From the properties of the degree and from (13) it follows that $d_L(L + N_s, \Omega_1) = 0$ for $s \in [s_{-1}, s_1] \supset (s_0, s_1]$.

Let $\Omega_\varepsilon = \{x \in X : \|x(t)\| < |M(s_1)|,\ -|M(s_1)| < x'(t) < \underline{u}'(t) + \varepsilon \text{ for } t \in [0,1],\ \|x''(t)\| < \varrho\}$, where $u(t)$ is a solution of $(1)_s$, (2) for $s = \mathrm{s} \in (s_0, s_1)$ and $\underline{u}(t) = u(t) + \varepsilon(t - \eta)$. For $s \in (\mathrm{s}, s_1]$ it is possible (because f is continuous) to take ε such that $\|\underline{u}'\| < |M(s_1)|$ and $\underline{u}(t)$ is a strict upper solution of $(1)_s$, (2). $-|M(s_1)|(t-\eta)$ is a strict lower solution of $(1)_s$, (2). According to Lemma 5 for $s \in (\mathrm{s}, s_1]$ we have

$$d_L(L+N_s,\Omega_\varepsilon) = \pm 1 \pmod 2. \tag{14}$$

From the additivity property of the degree it follows that

$$d_L(L+N_s,\Omega_1-\overline{\Omega}_\varepsilon) = \pm 1 \pmod 2 \tag{15}$$

for $s \in (\mathrm{s}, s_1]$. Relations (14), (15) imply the existence of a solution of the BVP $(1)_s$, (2) in Ω_ε and in $\Omega_1 - \overline{\Omega}_\varepsilon$. Since s is arbitrary in (s_0, s_1), the BVP $(1)_s$, (2) has at least two solutions for $s \in (s_0, s_1]$.

Now we prove that $(1)_s$, (2) has a solution for $s = s_0$. Let us take a sequence $\{s_n\}_{n=1}^{\infty}$, where $s_n \in (s_0, s_1]$, $n \in N$, $\lim\limits_{n\to\infty} s_n = s_0$. We know that for any s_n $(1)_s$, (2) has a solution u_n satisfying $\|u_n\| < |M(s_1)|$, $\|u_n'\| < |M(s_1)|$, and according to Lemma 1 we get $\|u_n''\| < \varrho$ for ϱ large enough. Since u_n is a solution of $(1)_{s_n}$, (2) the sequence $\{u_n'''\}_{n=1}^{\infty}$ is bounded in $C^0(0,1)$. By the Arzela-Ascoli lemma we can suppose that $\{u_n\}_{n=1}^{\infty}$ converges in $C^2(0,1)$ to a solution of $(1)_s$, (2). Theorem 5 is proved. □

References

[1] *A. Ambrosetti and G. Prodi*: On the inversion of some differentiable mappings with singularities between Banach spaces. Ann. Mat. Pura Appl. *93(4)* (1972), 231–247.

[2] *S. H. Ding and J. Mawhin*: A multiplicity result for periodic solutions of higher order ordinary differential equations. Differential and Integral Equations *1(1)*.

[3] *C. Fabry, J. Mawhin and M. Nkashama*: A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations. Bull. London Math. Soc. *18* (1986), 173–180.

[4] *J. Mawhin*: Topological degree methods in nonlinear boundary value problems. CBMS Regional Confer. Ser. Math. No. 40. Amer. Math. Soc., Providence, 1979.

[5] *J. Mawhin*: First order ordinary differential equations with several solutions. Z. Angew. Math. Phys. *38* (1987), 257–265.

[6] *M. Šenkyřík*: Method of lower and upper solutions for a third-order three-point regular boundary value problem. Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. XXXI (1992), 60–70.

Author's address: *Martin Šenkyřík*, Přírodovědecká fakulta UP, Tomkova 38, 771 46 Olomouc, Czech Republic;

Present address: Department of Mathematics, Oregon State University, Kidden Hall 368, Corvallis, OR 97331, U.S.A.