CENTRALLY DETERMINED STATES ON VON NEUMANN ALGEBRAS

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(Received October 19, 1990)

Summary. It is shown that every von Neumann algebra whose centre determines the state space is already abelian.

Keywords: State space of operator algebras, noncommutative Radon-Nikodym theorem.

AMS classification: 46L30, 46L50

The following question was posed in [4]: Is every von Neumann algebra with centrally determined state space abelian? The aim of this note is to establish a positive answer to this question.

Let $\mathcal{A}$ be an arbitrary von Neumann algebra and let Z be its centre. Let $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(Z)$ stand for the orthomodular lattices of all projections in $\mathcal{A}$ and Z, respectively (see [6]). Let us call a mapping $\eta\colon \mathcal{A} \to Z$ a *centre state* if it is positive, $\eta(C) = C$ and $\eta(CA) = C\eta(A)$ for every $C \in Z$ and $A \in \mathcal{A}$ (see [1]). Further, let us call a mapping $s\colon \mathcal{P}(\mathcal{A}) \to \langle 0,1\rangle$ a *state* if $s(I) = 1$ (I is an identity in A) and $s\Big(\sum_{n\in N} P_n\Big) = \sum_{n\in N} s(P_n)$, whenever (P_n) is sequence of mutually orthogonal elements of $\mathcal{P}(\mathcal{A})$. Finally, let us say that $\mathcal{A}$ has a *centrally determined state space* (see [2, 3]) if states s_1 and s_2 on $\mathcal{P}(\mathcal{A})$ coincide whenever they agree on $\mathcal{P}(Z)$.

Theorem. *A von Neumann algebra $\mathcal{A}$ has a centrally determined state space if and only if it is abelian.*

Proof. The sufficiency is obvious. Let us take up the necessity. Suppose that $\mathcal{A}$ is not abelian. Looking for a contradiction let us assume that $\mathcal{A}$ has centrally determined state space. Let us choose $A \in \mathcal{A} \setminus Z$. According to [1, Lemma 8.2.3, p. 512] $\mathcal{A}$ admits an ultraweakly continuous centre state $\eta\colon \mathcal{A} \to Z$, that is, $\mathcal{A}$ admits