ON THE EXISTENCE OF SOLUTIONS OF THE n-TH ORDER NON-LINEAR DIFFERENTIAL EQUATION WITH DELAY

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In paper [2], the existence theorem for a non-linear differential equation of the fourth order with delay is proved by means of Schauder-Tychonoff fixed point theorem.

In this paper several assertions from [3] are generalized to the differential equation (1). The method from [2] is used to prove Theorem 1.

Consider a differential equation of the n-th order with delay of the form

$$(1) \qquad y^{(n)}(t) + \sum_{k=0}^{n-1} r_k(t)\, y^{(k)}(t) = f(t, y(t), \ldots, y^{(n-1)}(t), y[h(t)], \ldots, y^{(n-1)}[h(t)]) ,$$

where $n \geqq 2$ is a natural number. Let the following conditions be fulfilled:

(a) $r_k \in C(J \equiv [t_0, \infty), R)$, $k = 0, 1, \ldots, n - 1$,
(b) $h \in C(J, R)$, $h(t) \leqq t$,
(c) $f(t, v_1, \ldots, v_n, u_1, \ldots, u_n) \in C(D \equiv J \times R^{2n})$.

Let $\Phi(t) = \{\Phi_0(t), \Phi_1(t), \ldots, \Phi_{n-1}(t)\}$ be a vector-function defined and continuous on the initial set

$$E_{t_0} = (\inf_{t \in J} h(t), t_0] .$$

If $\inf h(t) = \min h(t)$, $t \in J$, then $E_{t_0} = [\inf_{t \in J} h(t), t_0]$.

Initial Problem. Find a solution $y(t)$ of the differential equation (1) on the interval J which fulfils the initial conditions

$$(2) \qquad y^{(k)}(t_0+) = \Phi_k(t_0) = y_0^{(k)}, \quad y^{(k)}[h(t)] \equiv \Phi_k[h(t)], \quad h(t) < t_0 ,$$
$$k = 0, 1, \ldots, n - 1 .$$

Let $x_j(t)$, $j = 0, 1, \ldots, n - 1$ be the solutions on J of the differential equation

$$(3) \qquad x^{(n)}(t) + \sum_{k=0}^{n-1} r_k(t)\, x^{(k)}(t) = 0$$

which fulfil the initial conditions

$$(4)\qquad x_j^{(k)}(t_0) = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases} \quad j, k = 0, 1, \dots, n-1.$$

Then every solution $x(t) = \sum_{j=0}^{n-1} C_j x_j(t)$ of (3) where C_j are real numbers satisfies

$$x^{(k)}(t_0) = C_k, \quad k = 0, 1, \dots, n-1.$$

Remark 1. The Wronskian $W(t)$ of solutions $x_j(t)$, $j = 0, 1, \dots, n-1$ satisfies

$$W(t) = \exp\left\{-\int_{t_0}^{t} r_{n-1}(s)\,\mathrm{d}s\right\}.$$

For the sake of brevity we shall further write $W(t)$ only.

Denote

$$(5)\qquad W_k(t, s) = \begin{vmatrix} x_0(s), & x_1(s), & \dots, x_{n-1}(s) \\ x_0'(s), & x_1'(s), & \dots, x_{n-1}'(s) \\ \vdots & \vdots & \vdots \\ x_0^{(n-2)}(s), & x_1^{(n-2)}(s), & \dots, x_{n-1}^{(n-2)}(s) \\ x_0^{(k)}(t), & x_1^{(k)}(t), & \dots, x_{n-1}^{(k)}(t) \end{vmatrix}, \quad k = 0, 1, \dots, n-1.$$

Evidently $W_k(t, s) = \partial^k W_0(t, s)/\partial t^k$ for every $t, s \in J$, $s \leqq t$, $k = 1, 2, \dots, n-1$

We define

$$(6)\qquad D(s) = \max\{|W_{k0}(s)|, |W_{k1}(s)|, \dots, |W_{kn-1}(s)|\}, \quad s \in J,$$

$k = 0, 1, \dots, n-1$, where $K_{ki}(s)$, $i = 0, 1, \dots, n-1$ are determinants obtained from $W_k(t, s)$ by omitting the i-th column and the n-th row.

We define further

$$C = \sum_{j=0}^{n-1} |C_j|$$

and

$$(7)\qquad \alpha_k(t) = \max\{|x_0^{(k)}(t)|, |x_1^{(k)}(t)|, \dots, |x_{n-1}^{(k)}(t)|\}, \quad t \in J,$$

where $x_j(t)$, $j = 0, 1, \dots, n-1$ are the solutions of (3) fulfilling the conditions (4).

From (6) and (7) it is evident that the functions $\alpha_k(t)$, $k = 0, 1, \dots, n-1$ and $D(t)$ are continuous on J.

Because $\alpha_k(t_0) = 1$, we put $\alpha_k(t) \equiv 1$ for $t \in E_{t_0}$, $k = 0, 1, \dots, n-1$.

Denote

$$(8)\qquad \beta_k(t) = \begin{cases} \max\{\alpha_k(t), \alpha_k[h(t)]\}, & t \in J, \\ \alpha_k(t) \equiv 1, & t \in E_{t_0}, \end{cases} \quad k = 0, 1, \dots, n-1.$$

Remark 2. *If the functions $\alpha_k(t)$ are nondecreasing, then $\beta_k(t) = \alpha_k(t)$.*

Theorem 1. *Let the conditions* (a)–(c) *be fulfilled and let there exists a constant* $\lambda > 0$ *such that*

$$(9)\qquad |\Phi_k(t)| \leqq \lambda\,, \quad k = 0, 1, \ldots, n-1\,, \quad t \in E_{t_0}\,.$$

Further suppose that there exists a function $\omega(t, r_1, \ldots, r_n, z_1, \ldots, z_n)$ *defined and continuous for* $t \in J$ *and* $0 \leqq r_1, \ldots, r_n, z_1, \ldots, z_n < \infty$, *which fulfils the following conditions*:

(i) *for every* $t \in J$ $\omega(t, r_1, \ldots, r_n, z_1, \ldots, z_n)$ *is non-negative and non-decreasing in all the other arguments;*

(ii) $|f(t, v_1, \ldots, v_n, u_1, \ldots, u_n)| \leqq \omega(t, |v_1|, \ldots, |v_n|, |u_1|, \ldots, |u_n|)$ *on* D;

(iii)

$$(10)\qquad \int_{t_0}^{\infty} \frac{\prod_{k=0}^{n-2} \alpha_k(t)}{W(t)}\, \omega(t, \beta_0(t)\,\lambda, \ldots, \beta_{n-1}(t)\,\lambda, \beta_0(t)\,\lambda, \ldots, \beta_{n-1}(t)\,\lambda)\, \mathrm{d}t < \frac{\lambda - C}{n!}\,.$$

Then every solution $y(t)$ *of the initial problem* (1), (2) *which fulfils the conditions*

$$(11)\qquad \sum_{k=0}^{n-1} |y_0^{(k)}| = \sum_{k=0}^{n-1} |C_k| = C < \lambda$$

exists on J *and satisfies*

$$(12)\qquad |y^{(k)}(t) - x^{(k)}(t)| < \beta_k(t)\,(\lambda - C)\,, \quad k = 0, 1, \ldots, n-1\,,$$

where $x(t) = \sum_{j=0}^{n-1} C_j\, x_j(t)$ *is the solution of* (3) *with* $C_j = y_0^{(j)}$ (*cf.* (2) *and* (11)).

Proof. Let Y_{n-1} be the space of functions $y(t)$ which have $n - 1$ continuous derivatives on $E_{t_0} \cup J$. Let $\{I_l\}_{l=1}^{\infty}$ be a sequence of compact intervals such that $\bigcup_{l=1}^{\infty} I_l = J$, where $I_l = [t_0, t_l]$ and $I_l \subset I_{l+1} \subset J$ for every l.

Define in the space Y_{n-1} a system of seminorms

$$R_l(y) = \max_{k=0,1,\ldots,n-1} \{ \sup_{t \in E_{t_0} \cup I_l} |y^{(k)}(t)| \}\,.$$

This system of seminorms induces a local by convex topology on Y_{n-1} and therefore the space Y_{n-1} is local by convex.

Consider a subset $F \subset Y_{n-1}$ defined as follows:

$$F = \{ y \in Y_{n-1},\ |y^{(k)}(t)| \leqq \lambda \beta_k(t),\ k = 0, 1, \ldots, n-1,\ t \in E_{t_0} \cup J \}\,,$$

where $\beta_k(t)$ are defined in (8).

Define for $y \in F$ an operator T:

$$(13)\qquad (Ty)^{(k)}(t) = \Phi_k(t)\,, \quad t \in E_{t_0}\,, \quad k = 0, 1, \ldots, n-1\,,$$

$$(Ty)^{(k)}(t) = x^{(k)}(t) + \int_{t_0}^{t} \frac{W_k(t, s)}{W(s)} f(s, y(s), \ldots, y^{(n-1)}(s), y[h(s)], \ldots, y^{(n-1)}[h(s)])\, ds\,,$$

$$k = 0, 1, \ldots, n - 1\,, \quad t \in J\,,$$

where $x(t)$ is a solution of (3).

a) It is obvious that F is a convex closed set.

b) We show that $TF \subset F$.

For $t \in E_{t_0}$ we obtain with regard to (9)

$$|(Ty)^{(k)}(t)| = |\Phi_k(t)| \leqq \lambda = \lambda \beta_k(t)\,, \quad k = 0, 1, \ldots, n - 1\,.$$

Since (5) implies the estimate

$$|W_k(t, s)| \leqq n!\, \alpha_k(t) \prod_{l=0}^{n-2} \alpha_l(s)\,,$$

we obtain for $t \in J$ from (13)

$$|(Ty)^{(k)}(t)| \leqq |x^{(k)}(t)| + \int_{t_0}^{t} \frac{|W_k(t, s)|}{W(s)} |f(s, y(s), \ldots, y^{(n-1)}(s),$$

$$y[h(s)], \ldots, y^{(n-1)}[h(s)])|\, ds \leqq$$

$$\leqq \alpha_k(t) \left[C + n! \int_{t_0}^{\infty} \frac{\prod_{l=0}^{n-2} \alpha_l(t)}{W(t)} \omega(t, \beta_0(t)\lambda, \ldots, \beta_{n-1}(t)\lambda, \beta_0(t)\lambda, \ldots, \beta_{n-1}(t)\lambda)\, dt \right] \leqq$$

$$\leqq \alpha_k(t) \left[C + n! \frac{(\lambda - C)}{n!} \right] \leqq \alpha_k(t)\lambda \leqq \beta_k(t)\lambda\,.$$

c) We show that T is continuous.

Let $\{y_j^{(k)}\}_{j=1}^{\infty}$, $k = 0, 1, \ldots, n - 1$, $y_j \in F$ be a sequence which converges to $y^{(k)}$, $k = 0, 1, \ldots, n - 1$, $y \in F$ uniformly on every compact subinterval of J.

Let $I_l = [t_0, t_l]$ be an arbitrary compact interval from J and let $\varepsilon > 0$ be given. We show that $(Ty_j)^{(k)}(t) \rightrightarrows (Ty)^{(k)}(t)$, $k = 0, 1, \ldots, n - 1$ provided $t \in I_l$.

Denote

$$A_k = \max_{t \in [t_0, t_l]} \alpha_k(t)\,, \quad k = 0, 1, \ldots, n - 1\,.$$

As the function f is continuous and $y_j^{(k)} \rightrightarrows y^{(k)}$, $k = 0, 1, \ldots, n - 1$ holds on every compact interval I_l, there exists such $M > 0$ that for $j \geqq M$

(14) $$\frac{\prod_{k=0}^{n-2} \alpha_k(t)}{W(t)} |f(t, y_j(t), \ldots, y_j^{(n-1)}(t), y_j[h(t)], \ldots, y_j^{(n-1)}[h(t)]) -$$

$$- f(t, y(t), \dots, y^{(n-1)}(t), y[h(t)], \dots, y^{(n-1)}[h(t)])| <$$

$$< \frac{\varepsilon}{A_k(t_l - t_0)\, n!}, \quad k = 0, 1, \dots, n-1, \quad t \in I_l .$$

From (13) with regard to (14) we obtain for $t \in I_l$ and $j \geqq M$

$$|(Ty_j)^{(k)}(t) - (Ty)^{(k)}(t)| \leqq \alpha_k(t)\, n! \int_{t_0}^{t} \frac{\prod_{l=0}^{n-2} \alpha_l(s)}{W(s)} |f(s, y_j(s), \dots$$

$$\dots, y_j^{(n-1)}(s), y_j[h(s)], \dots, y_j^{(n-1)}[h(s)]) - f(s, y(s), \dots$$

$$\dots, y^{(n-1)}(s), y[h(s)], \dots, y^{(n-1)}[h(s)])|\, \mathrm{d}s < \frac{A_k n!\, \varepsilon}{A_k(t_l - t_0)\, n!} \int_{t_0}^{t} \mathrm{d}s \leqq$$

$$\leqq \frac{\varepsilon(t - t_0)}{(t_l - t_0)} \leqq \frac{\varepsilon(t_l - t_0)}{(t_l - t_0)} = \varepsilon .$$

d) We show that $\overline{TF}$ is a compact set. The assertion a) implies

$$|(Ty)^{(k)}(t)| \leqq \beta_k(t)\, \lambda, \quad k = 0, 1, \dots, n-1, \quad t \in E_{t_0} \cup J .$$

If we choose $k = n - 1$ in (13) and differentiate, we obtain

$$(Ty)^{(n)}(t) = x^{(n)}(t) + \int_{t_0}^{t} \frac{W_n(t, s)}{W(s)} f(s, y(s), \dots, y^{(n-1)}(s), y[h(s)], \dots$$

$$\dots, y^{(n-1)}[h(s)])\, \mathrm{d}s + f(t, y(t), \dots, y^{(n-1)}(t), y[h(t)], \dots, y^{(n-1)}[h(t)]),$$

where

$$W_n(t, s) = \begin{vmatrix} x_0(s), & x_1(s), & \dots, x_{n-1}(s) \\ x_0'(s), & x_1'(s), & \dots, x_{n-1}'(s) \\ \vdots \;\dots\dots & \vdots \;\dots\dots\dots & \vdots \;\dots\dots \\ x_0^{(n-2)}(s), & x_1^{(n-2)}(s), & \dots, x_{n-1}^{(n-2)}(s) \\ x_0^{(n)}(t), & x_1^{(n)}(t), & \dots, x_{n-1}^{(n)}(t) \end{vmatrix} .$$

The last equality yields for $t \in J$ the estimate

$$|(Ty)^{(n)}(t)| \leqq |x^{(n)}(t)| + \int_{t_0}^{t} \frac{|W_n(t, s)|}{W(s)} \omega(s, \beta_0(s)\, \lambda, \dots, \beta_{n-1}(s)\, \lambda, \beta_0(s)\, \lambda, \dots$$

$$\dots, \beta_{n-1}(s)\, \lambda)\, \mathrm{d}s + \omega(t, \beta_0(t)\, \lambda, \dots, \beta_{n-1}(t)\, \lambda, \beta_0(t)\, \lambda, \dots, \beta_{n-1}(t)\, \lambda),$$

which implies that $(Ty)^{(n)}(t)$ is bounded on I_l. Thus have obtained the uniform boundedness of $(Ty)^{(k)}(t)$, $k = 0, 1, \ldots, n$ on $E_{t_0} \cup I_l$, hence the equicontinuity of $(Ty)^{(k)}(t)$, $k = 0, 1, \ldots, n - 1$ on $E_{t_0} \cup I_l$. Therefore $\overline{TF}$ is a compact set.

With regard to the Schauder-Tychonoff fixed point theorem, the operator T has at least one fixed point in F satisfying

$$(15) \qquad (Ty)^{(k)}(t) = y^{(k)}(t), \quad k = 0, 1, \ldots, n - 1 .$$

The assertion (12) follows now from (13) by virtue of (15) and (10). The proof of Theorem 1 is complete.

Theorem 2. *Let the assumptions from Theorem 1 hold with the condition (10) replaced by*

$$\int_{t_0}^{\infty} \frac{D(t)}{W(t)} \omega(t, \beta_0(t)\lambda, \ldots, \beta_{n-1}(t)\lambda, \beta_0(t)\lambda, \ldots, \beta_{n-1}(t)\lambda)\,\mathrm{d}t < \frac{\lambda - C}{n} .$$

Then every solution $y(t)$ of the initial problem (1), (2) which fulfils (11) exists on J and satisfies (12) with $x(t)$ from Theorem 1.

Proof proceeds as that of Theorem 1, only we use (6) to estimate $W_k(t, s)$.

Lemma 1. *Let (a)–(c) hold. Let $[t_0, T)$ be the maximal interval of a solution $y(t)$ of the initial problem (1), (2) and let the functions $y^{(k)}(t)$, $k = 0, 1, \ldots, n - 1$ be bounded on $[t_0, T)$. Let moreover $\Phi(t)$ be bounded on E_{t_0}. Then $T = \infty$.*

The proof can be found in [3].

Lemma 2. *Let $\gamma(t)$, $a(t)$, $F(t)$, $q(t)$ be functions belonging to the class $C([t_0, b), [0, \infty))$ and let a function $\omega(z) \in C([0, \infty), (0, \infty))$ be non-decreasing.*

Denote

$$(16) \qquad \Omega(z) = \int_{z_0}^{z} \frac{1}{\omega(s)}\,\mathrm{d}s, \quad z_0 > 0, \quad z \geqq 0 .$$

Let $z(t) \in C([t_0, b), [0, \infty))$ satisfy the relation

$$(17) \qquad z(t) \leqq \gamma(t) + a(t) \int_{t_0}^{t} F(s)\, q(s)\, \omega[z(s)]\,\mathrm{d}s, \quad t_0 \leqq t < b .$$

Then we have for every $t \in [t_0, b)$

$$(18) \qquad z(t) \leqq \Omega^{-1} \left\{ \Omega[\Gamma(t)] + A(t) \int_{t_0}^{t} F(s)\, q(s)\,\mathrm{d}s \right\},$$

where Ω^{-1} is the inverse function to (16), $\Gamma(t) = \max_{t_0 \leqq s \leqq t} \gamma(s)$ and $A(t) = \max_{t_0 \leqq s \leqq t} a(s)$, $t \in [t_0, b)$.

Proof. Define a function $Z(t)$ on the interval $[t_0, b)$ by the relation $Z(t) = \max_{t_0 \leqq s \leqq t} z(s)$. It is evident that $Z(t)$ is a continuous, non-negative and non-decreasing function. With respect to the properties of $\omega(z)$, we obtain from (17) that

$$z(t) \leqq \Gamma(t) + A(t) \int_{t_0}^{t} F(s)\, q(s)\, \omega[Z(s)]\, ds .$$

Let $\bar{t} \in [t_0, t]$ be a point at which $z(t)$ assumes its maximum on $[t_0, t]$. Then

$$Z(t) = z(\bar{t}) \leqq \Gamma(\bar{t}) + A(\bar{t}) \int_{t_0}^{\bar{t}} F(s)\, q(s)\, \omega[Z(s)]\, ds \leqq$$

$$\leqq \Gamma(t) + A(t) \int_{t_0}^{t} F(s)\, q(s)\, \omega[Z(s)]\, ds .$$

If we apply the Bihari lemma (see [1]) to the last inequality, we conclude

$$Z(t) \leqq \Omega^{-1} \left\{ \Omega[\Gamma(t)] + A(t) \int_{t_0}^{t} F(s)\, q(s)\, ds \right\} .$$

Since $z(t) \leqq Z(t)$, (18) holds.

Theorem 3. *Let the assumptions* (a)–(c) *be fulfilled. Moreover, let*

(i) $\psi(t) \in C(J, [0, \infty))$;

(ii) *the function* $\omega(z) \in C([0, \infty), (0, \infty))$ *be non-decreasing and*

$$\int_{t_0}^{\infty} \frac{ds}{\omega(s)} = \infty ;$$

(iii) $$|f(t, v_1, \ldots, v_n, u_1, \ldots, u_n)| \leqq \psi(t)\, \omega(|v_1|) ,$$

for every point $(t, v_1, \ldots, v_n, u_1, \ldots, u_n) \in D$.

Then every solution $y(t)$ *of the initial problem* (1), (2) *exists on* J *and fulfils the inequality*

(19) $$|y(t)| \leqq \Omega^{-1} \left\{ \Omega[\Gamma(t)] + A(t) \int_{t_0}^{t} \frac{D(s)}{W(s)}\, \psi(s)\, ds \right\} ,$$

where Ω, Ω^{-1} *have the meaning from Lemma* 2, $\Gamma(t) = \max_{t_0 \leqq s \leqq t} |x(s)|$, $A(t) = \max_{t_0 \leqq s \leqq t} \alpha_0(s)$, $x(t) = \sum_{j=0}^{n-1} C_j\, x_j(t)$ *is the solution of* (3) *with* $C_j = y_0^{(j)}$ (*cf.* (2) *and* (11)), $\alpha_0(s)$ *is defined in* (7) *and* $D(s)$ *in* (6).