Пусть A — симметрическая матрица такая, что $c(A) > 0$, $\sigma = \frac{\sqrt{Q^*(A)}}{c(A)} \leqq \xi$. Тогда существуют матрицы $\varphi^k(A)$ для $k = 1, 2, 3, \ldots$ и справедливы утверждения:

1° *последовательность матриц $\varphi^k(A)$ сходится к диагональной матрице L;*

2° $Q^*(\varphi^k(A)) \leqq Q^*(A)\, \varrho^k \mu^{2^k-1}$ *где $\mu = \sigma/\xi$;*

3° *если $\varphi^k(A) = U_k \varphi^{k-1}(A)\, U_k^*$, существует (унитарная) матрица $V = U_1^* U_2^* U_3^* \ldots$ и $L = V^* A V$.*

Также можно дать оценку сходимости бесконечного произведения $U_1^* U_2^* U_3^* \ldots$

Метод может быть применен и в более общем случае, когда некоторые из диагональных элементов матрицы A довольно близки друг другу. Матрицу A в таком случае разобьем подходящим способом на блоки, причем имеет место теорема, аналогичная выше приведенной.

Summary

AN ITERATIVE METHOD OF COMPUTING THE EIGENVALUES AND EIGENVECTORS OF A SYMMETRIC MATRIX

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An iterative method is described for bringing a symmetric matrix to the diagonal form. The method is suitable for matrices sufficienty near the diagonal form and may be compared to the Newton method for finding roots of polynomials.

Let A be the given complex symmetric matrix and let D be the diagonal matrix consisting of the diagonal elements of A. Let us consider the case where the diagonal elements of A are all different from each other. Then there exists exactly one antisymmetric matrix S such that $DS - SD = A - D$. Under certain conditions (if all proper values of S are less than one in modulus) the matrix $E + S^2$ is positive definite. If W is the positive definite square root of $E + S^2$, the matrix $U = S + W$ is a unitary one. The matrix $\varphi(A) = UAU^*$ is then the next approximation of the diagonal matrix similar to A. In a similar manner, the matrices $\varphi^2(A) = \varphi(\varphi(A))$, $\varphi^3(A) = \varphi(\varphi^2(A))$, are constructed.

We introduce the following notation: if A is a matrix with elements a_{ik}, we put

$$c(A) = \min_{i,j,i\neq j} |a_{ii} - a_{jj}|\,, \quad Q^*(A) = \sum_{i,j,i\neq j} |a_{ij}|^2\,.$$

The following theorem is proved. *There exist two numbers* $\xi \doteq 0{,}47172$ and $\varrho \doteq 0{,}24051$ with the following properties:

Let A *be a symmetric metrix such that* $c(A) > 0$, $\sigma = \dfrac{\sqrt{Q^*(A)}}{c(A)} \leqq \xi$. *Under these conditions, the iterated matrix* $\varphi^k(A)$ *has a meaning for each* k *and*

1° *the sequence* $\varphi^k(A)$ *converges to a diagonal matrix* L;

2° $Q^*(\varphi^k(A)) \leqq Q^*(A)\, \varrho^k\, \mu^{2^k-1}$ *where* $\mu = \dfrac{\sigma}{\xi}$;

3° *if* $\varphi^k(A) = U_k\, \varphi^{k-1}(A)\, U_k^*$, *the (unitary) matrix* $V = U_1^* U_2^* U_3^* \ldots$ *exists and fulfills* $L = V^* A V$.

An estimate of the convergence of the infinite product $U_1^* U_2^* U_3^* \ldots$ may also be given.

The method may be applied even in the more general case when some of the diagonal elements are nearly equal; the matrix is then divided into blocks in a suitable manner and a theorem similar to the preceding result may be obtained.