x с разложением (2), $0 < x \leqq 1$, k-ю частичную сумму (формального) ряда

$$c^1_{m_1} u_1 + \ldots + c^r_{m_r} u_r + 0 + 0 + 0 + \ldots + 0 + \ldots$$

и $\varphi_k(0) = 0$. Тогда справедлива

Теорема 2. *Пусть* $\sum_{k=1}^{\infty} u_k$ — *ряд с действительными членами, пусть последовательность* $\{M_k\}_{k=1}^{\infty}$ *нормальна относительно ряда* $\sum_{k=1}^{\infty} u_k$. *Тогда для всех* $x \in \langle 0, 1 \rangle$, *за исключением точек множества первой категории,*

$$\limsup_{k \to \infty} \varphi_k(x) = + \infty , \quad \liminf_{k \to \infty} \varphi_k(x) = - \infty .$$

Summary

ON AN APPLICATION OF THE CONTINUED FRACTIONS IN THE THEORY OF THE INFINITE SERIES

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In the paper [1] J. D. Hill deals with some interesting properties of the subseries. In this paper proves the author analogical results for the series of more general structure (than are the subseries), using some fundamental properties of the continued fractions.

Let $\sum_{k=1}^{\infty} u_k$ be a series with real numbers and let for every $k = 1, 2, 3, \ldots$, be $M_k = (c^k_1, c^k_2, \ldots, c^k_n, \ldots)$ a sequence of real numbers. Let us put

$$A_k = \sup_{n=1,2,3,\ldots} |c^k_n| \quad (k = 1, 2, 3, \ldots)$$

and let $\sum_{k=1}^{\infty} A_k |u_k| < \infty$. Let

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots + \cfrac{1}{n_l + \ddots}}} \tag{1}$$

be the infinite expansion of the irrational number $x \in (0, 1)$ into the continued fraction. Let us define the function φ in the following way: $\varphi(0) = 0$, $\varphi(x) =$

$= \sum_{l=1}^{\infty} c_{n_l}^l u_l$, if x is irrational number with the expansion (1). If x is rational, $0 < x \leqq 1$ and it's expansion into the continued fraction is

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cdots + \cfrac{1}{m_r}}} \tag{2}$$

then we put $\varphi(x) = \sum_{l=1}^{r} c_{m_l}^l u_l$.

According to this definition is the set of all values of the function φ on the set of all irrational number of the interval (0, 1) identic with the set of all the sums of the series of the form $\sum_1^{\infty} \varepsilon_k u_k$, where ε_k is the member of the sequence M_k $(k = 1, 2, 3, \ldots)$.

In the paper this theorem is proved:

Theorem 1. *The function φ is integrable on $\langle 0, 1\rangle$ in the Riemann sense.*

From the proof of the theorem follows, that the points of discontinuity of the function φ may be only the rational numbers of the interval $\langle 0, 1\rangle$.

Let

$$\sum_{k=1}^{\infty} u_k\,, \quad M_k\ (k = 1, 2, 3, \ldots,) \tag{3}$$

have the previous meaning. The sequence $\{M_k\}_{k=1}^{\infty}$ will be called normal with respect to the series (3), if there exist sequences $\{\varepsilon_k\}_1^{\infty}, \{\varepsilon_k'\}_1^{\infty}$; $\varepsilon_k, \varepsilon_k'$ being the members of the sequence M_k $(k = 1, 2, 3, \ldots)$, such that the relations

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \varepsilon_k u_k = +\infty\,, \quad \liminf_{n\to\infty} \sum_{k=1}^{n} \varepsilon_k' u_k = -\infty$$

are valid.

Let for every natural k means $\varphi_k(x)$ for x irrational (1) the k-th partial sum of the series $\sum_{l=1}^{\infty} c_{n_l}^l u_l$, for x rational (2), $0 < x \leqq 1$, the k-th partial sum of the (formal) series

$$c_{m_1}^1 u_1 + \ldots + c_{m_r}^r u_r + 0 + 0 + \ldots + 0 + \ldots$$

and $\varphi_k(0) = 0$. Then is in force

Theorem 2. *Let $\sum_{k=1}^{\infty} u_k$ be a series with real numbers, let the sequence $\{M_k\}_{k=1}^{\infty}$ is normal with respect to the series $\sum_{k=1}^{\infty} u_k$. Then for every $x \in \langle 0,1\rangle$ except the points of a set of the first cathegory*

$$\limsup_{k\to\infty} \varphi_k(x) = +\infty\,, \quad \liminf_{k\to\infty} \varphi_k(x) = -\infty\,.$$