*Nechť* $P, H \in \mathbf{E}(X)$. *Nechť*

1° $\mathbf{R}(P) \subset \tilde{X}$,

2° $\mathbf{R}(E - PH) \supset \tilde{X}$,

3° $\mathbf{N}(E - PH) \cap \tilde{X} = \{0\}$.

*Potom* $E - PH$, *uvažován jako operátor na* $\tilde{X}$, *má inversní operátor* $W \in \mathbf{E}(\tilde{X})$. *Označme* $V = WP$, *takže* $V \in \mathbf{E}(X)$. *Potom operátory* $E - PH$ *a* $E - HP$ *mají inversní a platí*

$$(E - PH)^{-1} = E + VH\,, \quad (E - HP)^{-1} = E + HV\,.$$

*Jestliže dále* $P^2 = P$ *a* $\tilde{X} = \mathbf{R}(P)$, *má také operátor* $E - PHP$ *inversní operátor a platí*

$$(E - PHP)^{-1} = E + VHP = E + V - P\,.$$

Má-li se tedy řešit například $(E - PH)\,x = y$, najdeme bod $\tilde{x} \in \tilde{X}$ tak, že $(E - PH)\,\tilde{x} = PHy$ a položíme $x = y + \tilde{x}$. Podobně, vztah $(E - HP)\,x_2 = y$ je ekvivalentní vztahu $x_2 = y + Hz_2$, kde $z_2 \in \tilde{X}$, $(E - PH)\,z_2 = Py$. Jestliže dále $P^2 = P$ a $\tilde{X} = \mathbf{R}(P)$, vztah $(E - PHP)\,x_3 = y$ je ekvivalentní vztahu $x_3 = y - Py + z_2$.

Tyto výsledky tvoří algebraickou basi dalších vyšetřování. Nechť nyní $P$ je projekce na konečně dimensionální podprostor $\tilde{X}$. Dá se očekávati, že, bude-li $\tilde{X}$ vhodně volen, operátor $E - PHP$ bude aproximovati operátor $E - H$. Studium $E - PH$ na $\tilde{X}$ se potom redukuje na studium konečné matice. Dejme tomu, že tato matice je regulární; máme potom existenci inversního operátoru $(E - PHP)^{-1}$. Stupeň aproximace závisí na „vzdálenosti $H$ od $\tilde{X}$“, která je definována jako

$$\varrho(H, \tilde{X}) = \sup_{|x| \leq 1} \inf_{\tilde{x} \in \tilde{X}} |Hx - \tilde{x}|\,.$$

Zřejmě $\varrho(H, \tilde{X}) \leq |H - PH|$. Je-li toto číslo dostatečně malé, můžeme dokázati existenci inversního operátoru k $E - H$. Řešení rovnice $(E - H)\,x = y$ může býti potom aproximováno řešením $(E - PHP)\,\tilde{x} = y$ a může být podán odhad pro $|x - \tilde{x}|$.

## Summary

## ON APPROXIMATE SOLUTIONS OF LINEAR EQUATIONS IN BANACH SPACES

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In the present paper, we attempt to use methods of Functional Analysis to the study of approximate solutions of Linear Equations in Banach space. The main idea is the use of projections on finite dimensional subspaces.

Let $X$ be a Banach (i. e. a complete normed) space. Let $\tilde{X}$ be a closed subspace of $X$. The Banach algebra of all bounded linear operators on $X$ will be denoted by $\mathbf{E}(X)$. The range and kernel of an operator $A \in \mathbf{E}(X)$ will be denoted by $\mathbf{R}(A)$ and $\mathbf{N}(A)$.

We prove the following theorem:

*Let $P, H \in \mathbf{E}(X)$. Suppose that the following inclusions are fulfilled:*

$$1^\circ \quad \mathbf{R}(P) \subset \tilde{X} ,$$

$$2^\circ \quad \mathbf{R}(E - PH) \supset \tilde{X} ,$$

$$3^\circ \quad \mathbf{N}(E - PH) \cap \tilde{X} = \{0\} .$$

*Then $E - PH$ considered as an operator on $\tilde{X}$, has an inverse operator $W \in \mathbf{E}(\tilde{X})$. Let us denote by $V$ the product $WP$, so that $V \in \mathbf{E}(X)$. The operators $E - PH$ and $E - HP$ have inverses (on the whole of $X$) and we have*

$$(E - PH)^{-1} = E + VH , \quad (E - HP)^{-1} = E + HV .$$

*If further, $P^2 = P$ and $\tilde{X} = \mathbf{R}(P)$, the operator $E - PHP$ has an inverse as well and*

$$(E - PHP)^{-1} = E + VHP = E + V - P .$$

Thus, e. g., if $(E - PH)\, x = y$ is to be solved, we find a point $\tilde{x} \in \tilde{X}$ such that $(E - PH)\, \tilde{x} = PHy$ and put $x = y + \tilde{x}$. Similarly, the relation $(E - HP)\, x_2 = y$ is equivalent to $x_2 = y + Hz_2$, where $z_2 \in \tilde{X}$, $(E - PH)\, z_2 = Py$. If, further, $P^2 = P$ and $\tilde{X} = \mathbf{R}(P)$, the relation $(E - PHP)\, x_3 = y$ is equivalent to $x_3 = y - Py + z_2$.

These results form the algebraic basis of the further considerations. Let us choose now $P$ as a projection on a finite dimensional subspace $\tilde{X}$. It is to be expected that, if $\tilde{X}$ and $P$ are suitably chosen, the operator $E - PHP$ will approximate the operator $E - H$. The study of $E - PH$ on $\tilde{X}$ reduces to the study of a finite matrix. Suppose that the regularity of this matrix is assured. We have, then, the existence of an inverse to $E - PHP$. The degree of the approximation is measured by the "distance of $H$ from $\tilde{X}$" defined as

$$\varrho(H, \tilde{X}) = \sup_{|x| \leqq 1} \inf_{\tilde{x} \in \tilde{X}} |Hx - \tilde{x}| .$$

Clearly $\varrho(H, \tilde{X}) \leqq |H - PH|$. If this number is small enough, we can prove the existence of an inverse to $E - H$. The solution of $(E - H)\, x = y$ may then by approximated by that of $(E - PHP)\, \tilde{x} = y$ and an estimate of the difference $|x - \tilde{x}|$ may be given.