Если $G \in \mathfrak{G}_1$, то для каждой ограниченной непрерывной на $H(G)$ функции f существует точно одно ограниченное решение F задачи Дирихле; имеем $F = D(f)$.

Пусть φ — изометрическое отображение пространства E_m в E_m. Если $G \in \mathfrak{G}$ (соотв. $G \in \mathfrak{G}_0$), то и $\varphi(G) \in \mathfrak{G}$ (соотв. $\varphi(G) \in \mathfrak{G}_0$).

Если $G_1 \in \mathfrak{G}_0$, $G_2 \in \mathfrak{G}$, $G_1 \subset G_2$, то $G_2 \in \mathfrak{G}_0$.

Предположим, что $U \in \mathfrak{G}^m$ и что для всякого $b \in H(U)$ существует гиперплоскость R и замкнутая сфера K с центром в R так, что $b \in R \cap K$ и что $R \cap K \cap G = \emptyset$. Тогда $U \times E_1 \in \mathfrak{G}^{m+1}$. Если, далее, $U \in \mathfrak{G}_1^m$, $A = E_m - U$,) $G \in \mathfrak{G}^{m+1}$, $G \cap (A \times \langle 0, \infty)) = \emptyset$, то $G \in \mathfrak{G}_1^{m+1}$.*

Теорема 37 имеет следующий наглядный смысл:

Предположим, что $G \in \mathfrak{G}$ и что $f \in \mathfrak{D}(G)$. Пусть K — большая открытая сфера; пусть g — непрерывная функция на $H(K \cap G)$, которая равна нулю на $H(K)$ и приблизительно равна f на $H(G)$. Тогда функция $D(K \cap G, g)$ приблизительно равна $D(G, f)$ на множестве $\overline{K \cap G}$.

Теорема 39 утверждает, что зависимость $D(f)$ от f непрерывна:

Если $G \in \mathfrak{G}$, f, f_0, $f_1, \ldots \in \mathfrak{D}(G)$ и если $f_n(x) \to f_0(x)$, $|f_n(x)| \leqq f(x)$ $(n = 1, 2, \ldots)$ для всякого $x \in H(G)$, то $D(f_n, x) \to D(f_0, x)$ для всякого $x \in \overline{G}$.

Отдел 40 содержит определение системы $\mathfrak{M}$, элементы которой — функции на множестве $\overline{G}$ ($G \in \mathfrak{G}$), обладающей следующим свойством: *Для каждого $f \in \mathfrak{D}(G)$ существует точно одно решение задачи Дирихле F такое, что $F \in \mathfrak{M}$; имеем $F = D(f)$.*

Summary

THE DIRICHLET PROBLEM

JAN MAŘÍK, Praha.

(Received May 23, 1956.)

Let G be an open subset of the m-dimensional Euclidean space E_m, $\emptyset \neq G \neq E_m$; let f be a (finite real) continuous function on $H(G)$. ($H(G)$ is the boundary, $\overline{G}$ is the closure of G.) If a function F is continuous on $\overline{G}$, harmonic on G and equal to f on $H(G)$, we say that F is a solution of the Dirichlet problem corresponding to the function f and the set G. If G is bounded, then there

*) Если, напр., $m = 2$ и если A — сегмент, то множество $U = E_2 - A$ удовлетворяет всем предположениям.

exists at most one such function F. Let $\mathfrak{g}$ be the system of all non-empty bounded open sets $G \subset E_m$ such that for each continuous function on $H(G)$ there exists a solution of the Dirichlet problem; further, let $\mathfrak{G}^m = \mathfrak{G}$ be the system of all $G \subset E_m$ $(G \neq E_m)$ such that $G \cap K \in \mathfrak{g}$ for each sufficiently large open sphere K with center 0. If $G \in \mathfrak{G}$, let $\mathfrak{D}(G)$ be the family of all functions f which are continuous on $H(G)$ and which have the following property: There exists a non-negative function F, which is continuous on $\bar{G}$, harmonic on G and which fulfils the relation $F(x) \geqq |f(x)|$ for each $x \in H(G)$. The following assertion is an easy consequence of Theorem 13:

For each non-negative function $f \in \mathfrak{D}(G)$ $(G \in \mathfrak{G})$ there exists a smallest non-negative solution of the Dirichlet problem.

We denote this solution by $D(G, f)$ and for an arbitrary $f \in \mathfrak{D}(G)$ put $D(G, f) = D(f) = D(f_+) - D(f_-)$, where $f_+(x) = \max(f(x), 0)$, $f_-(x) = \max(-f(x), 0)$. The function $D(f)$ is evidently a solution of the corresponding Dirichlet problem; the values of $D(f)$ will be denoted by $D(f, x)$. — Theorem 23 asserts:

Let G be open in E_m, $\emptyset \neq G \neq E_m$. Suppose that there exists, for each $b \in H(G)$, a hyperplane R and a closed sphere K with center in R such that $b \in R \cap K$, $R \cap K \cap G = \emptyset$. Then $G \in \mathfrak{G}$.

Now let $\mathfrak{G}_1^m = \mathfrak{G}_1$ be the system of all $G \in \mathfrak{G}$ such that $D(G, 1, x) = 1$ $(x \in G)$; further put $\mathfrak{G}_0^m = \mathfrak{G}_0 = \mathfrak{G} - \mathfrak{G}_1$. In accordance with Exercise 12 we have $\mathfrak{G}_1^2 = \mathfrak{G}^2$; each bounded set $G \in \mathfrak{G}^m$ (m arbitrary) evidently belongs to $\mathfrak{G}_1^m$. If $m > 2$, then, according to Theorem 35, each set $G \in \mathfrak{G}^m$ with bounded complement belongs to $\mathfrak{G}_0^m$. — According to Sections 17—19 and 24—31, the following theorems hold:

If $G \in \mathfrak{G}_0$, then $\inf_{x \in G} D(f, x) = 0$ for each non-negative function $f \in \mathfrak{D}(G)$.

If $G \in \mathfrak{G}_1$, then there exists, for each bounded continuous function f on $H(G)$, a unique bounded solution F of the corresponding Dirichlet problem, namely $F = D(f)$.

Let φ be an isometrical mapping of E_m into E_m. If $G \in \mathfrak{G}$ (resp. $G \in \mathfrak{G}_0$), then $\varphi(G) \in \mathfrak{G}$ (resp. $\varphi(G) \in \mathfrak{G}_0$).

If $G_1 \in \mathfrak{G}_0$, $G_2 \in \mathfrak{G}$, $G_1 \subset G_2$, then $G_2 \in \mathfrak{G}_0$.

Suppose that $U \in \mathfrak{G}^m$ and that there exists, for each $b \in H(U)$, a hyperplane R and a closed sphere K with center in R such that $b \in R \cap K$, $R \cap K \cap U = \emptyset$. Then $U \times E_1 \in \mathfrak{G}^{m+1}$. If, moreover, $U \in \mathfrak{G}_1^m$, $A = E_m - U$,) $G \in \mathfrak{G}^{m+1}$, $G \cap (A \times \langle 0, \infty)) = \emptyset$, then $G \in \mathfrak{G}_1^{m+1}$.*

Theorem 37 has the following intuitive sense:

Suppose that $G \in \mathfrak{G}$ and that $f \in \mathfrak{D}(G)$. Let K be a large open sphere; let g be a continuous function on $H(K \cap G)$, which vanishes on $H(K)$ and approximately

*) If, for example, $m = 2$ and if A is a segment, then the set $U = E_2 - A$ fulfils all the conditions.

equals f on H(G). Then the function $D(K \cap G, g)$ *is approximately equal to* $D(G, f)$ *on the set* $\overline{K \cap G}$.

Theorem 39 asserts that $D(f)$ depends continuously on f:

If $G \in \mathfrak{G}$, $f, f_0, f_1, \ldots \in \mathfrak{D}(G)$ *and if* $f_n(x) \to f_0(x)$, $|f_n(x)| \leqq f(x)$ $(n = 1, 2, \ldots)$ *for each* $x \in H(G)$, *then* $D(f_n, x) \to D(f_0, x)$ *for each* $x \in \overline{G}$.

Section 40 contains the definition of a system $\mathfrak{M}$, the elements of which are functions on $\overline{G}$ $(G \in \mathfrak{G})$ and which has the following property: *For each* $f \in \mathfrak{D}(G)$ *there exists a unique solution F of the corresponding Dirichlet problem such that* $F \in \mathfrak{M}$, *namely* $F = D(f)$.