всякий другой подбор α, γ может вести к менее выгодному результату. Если дополнительно предположить существование ограниченной $M'''(x)$ в некоторой окрестности Θ, то лучшим является подбор $\alpha = 1$, $\gamma = \frac{1}{6}$, дающий $\mathsf{E}[(x_n - \Theta)^2] = O(n^{-\frac{2}{3}})$; если в некоторой окрестности Θ функция $M(x)$ — аналитическая и симметричная относительно Θ, то $\mathsf{E}[(x_n - \Theta)^2] = O(n^{-(1-\varepsilon)})$ для $\alpha = 1$, $\gamma = \frac{1}{2}\varepsilon$ и для произвольного $\varepsilon > 0$. При дополнительном условии, что $\sigma^2(x) = \int_{-\infty}^{\infty} (y - M(x))^2 \, \mathrm{d}H(y|x)$ непрерывна и положительна в некоторой окрестности Θ, и при ограничении значений α, γ доказано, что случайные величины $n^{\frac{1}{2}(\alpha - 2\gamma)}$ $(x_n - \Theta)$ асимптотически нормальны.

В варианте, рассматриваемом в § 2-ом, на функции распределения $H(y|x)$ наложены только два условия, а именно (2.2) и выполнение неравенства $K_0|x - \Theta| \leqq |M'(x)| \leqq K_1|x - \Theta|$ для всех $x \in (-\infty, +\infty)$. С небольшими изменениями в дополнительных предположениях справедливы и в этом случае все выше упомянутые теоремы.

§ 1-ый является реферирующим.

Примечание. В статье C. Derman, An application of Chung's lemma to the Kiefer-Wolfowitz stochastic approximation procedure, Annals Math. Stat. *27* (1956), 532—536 доказаны две теоремы, похожие на теоремы 5 и 7 (случай 3°) нашей статьи.

Summary

ON THE KIEFER-WOLFOWITZ APPROXIMATION METHOD

VÁCLAV DUPAČ, Praha.

(Received January 20, 1956.)

Asymptotic properties are established for the Kiefer-Wolfowitz [9] procedure of finding the value $x = \Theta$, for which the regression function $M(x) = \int_{-\infty}^{\infty} y \, \mathrm{d}H(y|x)$ achieves its maximum. The original assumptions are modified in such a way that it is possible to use the mathematical tools due to Chung [7]. Two different cases are treated. In the case considered in Sec. 3, all assumptions from [9] are accepted, some of them being strengthened: instead of (2.8) — from [9] — it is supposed that the inequality

$$K_0 \, |x - \Theta| \leqq |M'(x)| \leqq K_1 \, |x - \Theta| \, , \quad K_0 > 0, \quad K_1 > 0$$

holds in some neighbourhood of Θ; instead of (2.2) it is assumed, that a finite interval (A, B), containing Θ, is known, together with some lower estimates of $M(A)$, $M(B)$, and that $\int_{M(x)-C}^{M(x)+C} \mathrm{d}H(y|x) = 1$ holds for all $x \in \langle A, B \rangle$ (C being a constant).

The sequences $\{a_n\}$, $\{c_n\}$, occuring in the approximation scheme, are supposed to be of the type $a_n = \frac{a}{n^\alpha}$, $c_n = \frac{c}{n^\gamma}$; then the choice $\alpha = 1$, $\gamma = \frac{1}{4}$ (c being positive and a greater than a specific constant) is proved to be optimal. This choice ensures that $\mathsf{E}[(x_n - \Theta)^2] = O(n^{-\frac{1}{2}})$ — every other choice can actually lead to a worse result. If in addition the existence of bounded $M'''(x)$ is supposed in some neighbourhood of Θ, then the best choice is $\alpha = 1$, $\gamma = \frac{1}{6}$, giving $\mathsf{E}[x_n - \Theta)^2] = O(n^{-\frac{2}{3}})$; if in some neighbourhood of Θ the function $M(x)$ is analytical and symmetrical about Θ, then for an arbitrary $\varepsilon > 0$ we can reach that $\mathsf{E}[(x_n - \Theta)^2] = O(n^{-(1-\varepsilon)})$ by a suitable choice of α, γ. Under the additional hypothesis that $\sigma^2(x) = \int_{-\infty}^{\infty} (y - M(x))^2 \, \mathrm{d}H(y|x)$ is continuous and positive in a neighbourhood of Θ and with a restriction on the range of α, γ, the asymptotic normality of the random variables $n^{\frac{1}{2}(\alpha-2\gamma)}(x_n - \Theta)$ is proved.

In the case considered in Sec. 2, only two conditions are set upon the distribution functions $H(y|x)$, namely, the (2.2) and the inequality $K_0|x - \Theta| \leqq |M'(x)| \leqq K_1|x - \Theta|$, holding for all $x \in (-\infty, +\infty)$. With some changes in additional conditions, all the theorems mentioned in the above case hold.

The Sec. 1 is an expository one.

Added in proof. C. Derman, An application of Chung's lemma to the Kiefer-Wolfowitz stochastic approximation procedure, AMS *27* (1956), pp. 532—536, has proved two theorems, which are similar to our Theorem 5 and Theorem 7 (case 3°).