плекса $\sum$ *с главной точкой типа* $\mu \neq n$ *не лежит ни в одной грани, то полярные гиперплоскости вершин* $\sum$ *образуют опять n-симплекс* $\sum^*$ *с той же самой главной точкой, с тем же самым главным* $(n - 1)$*-шаром, однако типа* $\mu^* = n - \mu - 1$ (и здесь $\mu^* \neq n$).

Summary

GEOMETRY OF THE SIMPLEX IN E_n (3rd part)

MIROSLAV FIEDLER, Praha.

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This is the third and final part of the paper, the first part of which was published in Čas. pro pěst. mat. *79* (1954), 297—320, the second in the same journal *80* (1955), 462—476. In the present part special types of simplexes in Euclidean spaces are studied.

First, a special type of rectangular n-simplexes is treated, the $(n - 1)$-dimensional faces of which may be numerated in such a way that exactly the interior angles φ_{12} (of the faces 1 a 2), $\varphi_{23}, \varphi_{34}, \ldots, \varphi_{n,n+1}$ are acute, all remaining interior angles right.

In the theorem 32 is shown that *the necessary and sufficient condition for an n-simplex to be rectangular of the type mentioned is that real numbers $c_1, c_2, \ldots, c_{n+1}$ (different from each other) exist so that the lengths d_{ij} of the edges of the simplex satisfy the relations*

$$d_{ij}^2 = |c_i - c_j| \quad (i \neq j, i, j = 1, 2, \ldots, n + 1) .$$

It follows (theorem 33) that *every m-dimensional face $(1 < m \leqq n - 1)$ of such an n-simplex is a rectangular m-simplex of this type*. Especially, every two-dimensional face of such an n-simplex is a rectangular triangle. Conversely, *an n-simplex, each two-dimensional face of which is rectangular, is rectangular of the type mentioned* (theorem 34). In the theorem 35 the following properties of such a rectangular n-simplex are proved:

(i) *the centre of the circumscribed hypersphere is in the middle of the (unique) longest edge,*

(ii) *in the E_n considered there exists a rectangular parallelepiped the vertices of which include all vertices of the simplex.*

In the further theorems 36—39 some properties of the *orthocentric n-simplex* and *orthocentric sets of $n + 2$ points in E_n* (i. e. $n + 1$ vertices and the orthocentre) are studied.

First, the idea of an *orthogonal n-hyperbola* is established as such a rational

algebraic curve of degree n in E_n, the n asymptotic directions of which are mutually rectangular. For the sake of brevity two of such n-hyperbolae in E_n are called *independent* if there is no k-dimensional $(0 \leqq k \leqq n - 1)$ linear improper subspace of E_n spanned by $k + 1$ asymptotic directions of one n-hyperbola containing more than k asymptotic directions of the other (i. e. if the asymptotic n-tuples of the both n-hyperbolae are *independent*).

Let a set of $n + 2$ points in E_n be of that kind that there exist two independent orthogonal n-hyperbolae both passing through all the points of the set. Then the set is orthocentric. Conversely, every rational algebraic curve of degree n passing through $n + 2$ points of an orthocentric set in E_n is an orthogonal n-hyperbola (theorem 36). The proof of this theorem is based upon this lemma: *if in the projective $(n - 1)$-space S_{n-1} two independent* (in the preceding sense) *sets of n points are given then there exists at most one regular hyperquadric with respect to which both n-tuples are autopolar.*

For an orthocentric n-simplex, the orthocentre of which not lying in any hyperplane of symmetry of an edge, there exists (theorem 37) *exactly one orthogonal n-hyperbola passing through all the vertices and the centre of gravity of the n-simplex.* This n-hyperbola being a generalized Kiepert's hyperbola of the triangle, has the following properties:

(i) *the asymptotic directions of it are the axes-directions* (uniquely determined) *of Steiner hyperellipsoids,*

(ii) *it contains the feet of all normals from the orthocentre to any regular hyperquadric of the Steiner system* (i. e. the pencil of hyperquadrics including the Steiner's circumscribed hyperellipsoid and the double improper hyperplane).

In the next theorem 38 a characteristic property of those orthocentric simplexes is formulated, the orthocentre of which is an interior point of the simplex (so called *positively orthocentric simplexes*). *An n-simplex is positively orthocentric if and only if there exists such an interior point P (the orthocentre) that for every selfadjoint point S (as far as $S \neq P$) of the reciprocal transform with respect to the simplex* (i. e. a transform of the type $x_i' = \dfrac{c_i}{x_i}$ for $c_i \neq 0$ real where x_i and x_i' are the barycentric homogeneous coordinates), *for which the centre of gravity and the point P correspond each other, holds that the line PS is orthogonal to the harmonic polar of S with respect to the simplex.*

Another characteristic property of an orthocentric n-simplex (theorem 39) is that *there exist real numbers $c_1, c_2, \ldots, c_{n+1}$, such that* ($\varphi_{ij}$ being interior angles)

$$\cos \varphi_{ij} = c_i c_j \text{ for } i \neq j .$$

Theorem 40 shows that *a characteristic property of equifacial simplexes* (with all the $(n - 1)$-dimensional faces of the same volume) *is that two* (and then three) *of the three points coincide: the centre of gravity, the centre of the inscribed* *hypersphere and the Lemoine point of the simplex* (i. e. the point with the property the sum of the squares of the distances between this point and the $(n-1)$-dimensional faces is minimum).

A necessary and sufficient condition (theorem 41) *for an n-simplex with the vertices* $O_1, \ldots, O_{n+1}$ *that its centre of gravity and the centre of the circumscribed hypersphere coincide is that for* $i = 1, \ldots, n+1$

$$\sum_{j=1}^{n+1} \overline{O_iO_j^2} = \text{const}\,.$$

Theorem 42 shows that *for every* $n > 2$ *there exist not equilateral n-simplexes with a unique distinguished point.*

In the further theorems other types of n-simplexes are treated. For these n-simplexes there exist real numbers $\alpha, \beta, t_1 \neq 0, \ldots, t_{n+1} \neq 0$, such that ($\sqrt{e_{ij}}$ is the length of the edge O_iO_j)

$$e_{ij} = \alpha(t_i^2 + t_j^2) + 2\beta t_i t_j\,, \quad i \neq j\,. \tag{*}$$

This class (*) of n-simplexes includes for special $\alpha : \beta$ some types of simplexes with certain generalized properties of the triangle.

Let Σ *be an n-simplex in* E_n *with vertices* $O_1, \ldots, O_{n+1}$. *A necessary and sufficient condition for the existence of such a point* $P \neq O_i$ *in* E_n *that the angles of the (not oriented) lines* PO_i, PO_j $[i \neq j]$ *be all equal is that* Σ *be of the type* (*) *with* $\alpha = n\beta$ [theorem 44; P is then the generalized point of Torricelli].

Let $P_1, \ldots, P_{n+1}$ *be the points in which a hypersphere inscribed (in the wider sense) in* Σ *touches the faces* $\omega_1, \ldots, \omega_{n+1}$ (ω_i opposite to O_i). *The lines* O_iP_i *pass through a point* Q *if and only if* Σ *is of the type* (*) *for* $\alpha = (n-1)\beta$ [theorem 45; Q is a generalized point of Gergonne].

Further, *a hypersphere touching all the lines* O_iO_j $(i \neq j)$ *exists if and only if* Σ *is of the type* (*) *for* $\alpha = \beta$ [theorem 46]. *Then a point* R *exists* [another generalization of the point of Gergonne] *such that all hyperplanes joining the point of contact in the line* O_iO_j *with the* $(n-2)$*-dimensional face opposite to* O_iO_j, *pass through* R.

If for Σ *all the hyperspheres* K_{ij}^k $(k \neq i \neq j \neq k)$ *containing the points* X *such that*

$$\overline{XO_i} : \overline{XO_j} = \overline{O_kO_i} : \overline{O_kO_j}$$

intersect (in two or one points called *isodynamic centres*) *then* Σ *is of the type* (*) *with* $\alpha = 0$. Conversely, *an n-simplex of the type* (*) *with* $\alpha = 0$ *is isodynamic*, i. e. the hyperspheres intersect in isodynamic centres (theorem 47).

The class (*) includes some more types of special simplexes, for example the positively orthocentric simplexes etc.

In the theorem 53 a geometrical characteristic of these simplexes is given. At first, *the principal point of an n-simplex* Σ ($n \geqq 2$) is defined (as far as it exists): *it is a point* H *not lying in any* ($n - 1$)*-dimensional face of* Σ, *of the manner that either* $H \neq T$ (T being the centre of gravity of Σ) *and the quadratic polar* Q *of* H *with respect to* Σ *is a hyperquadric of revolution with its axis passing through* H, *or* $H = T$ *and* Σ *is equilateral.* In the first case the axis mentioned is called *the principal axis of* Σ, the (unique) hypersphere in the pencil of hyperquadrics determined by Q and the double linear polar of H with respect to Σ *the principal hypersphere of* Σ. (A definition of the principal hypersphere in the second case is also added.) The principal point H is of *the type* μ if a certain crossratio in the pencil mentioned is $-\mu$. For such *n-simplexes with a principal point* the following theorem 53 holds:

An n-simplex Σ *is of the type* (*) *if and only if* Σ *is an n-simplex with a principal point* $\left(\textit{of the type } \mu = \frac{\alpha}{\beta}\right)$.

From (*) it follows immediately that *all faces of an n-simplex with a principal point of the type* μ *are also simplexes with a principal point of the type* μ (theorem 54).

Let Σ be an n-simplex with a principal point of the type μ. *If* μ *is an integer*, $0 < \mu \leqq n - 1$, *then* (theorem 55) *there exists a hypersphere* (the principal hypersphere) *touching all the* μ*-dimensional faces of* Σ. Besides, *all the linear spaces joining the point of contact in such a* μ*-dimensional face with the opposite* ($n - \mu - 1$)*-dimensional face of* Σ, *pass through a common point* (the principal point). *This property is characteristic for the n-simplexes with a principal point of the types* μ, $\mu = 1, \ldots, n - 1$.

Further it is shown that *every n-simplex with a principal point may be obtained from some positively orthocentric simplex by a dilatation in a certain direction* (theorem 57). It follows (theorem 58) that *all the altitudes of an n-simplex with a principal point intersect a line* (the principal axis). This last property is not yet characteristic.

Another *characteristic* property of an n-simplex with a principal point is the following: *there exists a set of* $n + 1$ *oriented hyperspheres in* E_n *with the centres in the vertices* O_i *intersecting each another under equal* (generalised) *angles.* From this, another way of constructing the simplexes with a principal point follows:

Let Σ' *be an equilateral n-simplex in* E_n *and let* $S'_1, \ldots, S'_{n+1}$ *be hyperspheres with centres in the vertices of* Σ' *and with equal radii* (real or purely imaginary). *By every hyperspherical inversion the* S'_i *are transformed in the hyperspheres* S_i, *the centres of which* (if not in a hyperplane) *are vertices of an n-simplex* Σ *with a principal point. Every n-simplex with a principal point may be constructed this way.*

Finally, in the theorem 62 it is shown that a kind of duality in the class of

n-simplexes with a principal point of the type $\mu \neq n$ may be defined. *Let Σ be an n-simplex with a principal point H of the type $\mu \neq n$. If the centre of the principal hypersphere S is not lying in any $(n - 1)$-dimensional face of Σ, the polar hyperplanes of the vertices of Σ with respect to S are faces of another n-simplex Σ^* with the same principal point H, with the same principal hypersphere S, but of the type $\mu^* = n - \mu - 1$.*