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ON REPRESENTATIONS OF FUZZY QUANTUM POSETS

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1. Introduction

Recently, there have appeared some models of fuzzy set theory [7, 3, 6, 4] wich can describe an axiomatic structure of, for example, quantum mechanics. These models are based on the one-to-one correspondence between subsets and their characteristic functions. If a quantum mechanical event, say a, is defined only vaguely, then by a fuzzy event we mean a function a defined on a crisp set Ω with the values in [0, 1].

The present paper will deal with fuzzy quantum posets as they have been defined in [4], and which generalize $q - \sigma$-algebras, we give the representations by an orthomodular, σ-orthoposet, $\bar{M}$, as well as by an appropriate $q - \sigma$-algebra, which is an analogue of the Loomis-Sikorski representation of $\bar{M}$.

2. Congruences on fuzzy quantum posets

We recall that according to [4], a fuzzy quantum poset is a couple (Ω, M) where Ω is a nonempty set, and M is a family of fuzzy sets from Ω, ie, $M \subseteq [0, 1]^{\Omega}$, such that

(i) If $1(\omega) = 1$ for any $w \in \Omega$, then $1 \in M$;

(ii) $a \in M$ implies $a^{\perp} := 1 - a \in M$;

(iii) if $1/2(w) = 1/2$ for any $w \in \Omega$, then $1/2 \notin M$;

(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$ whenever $\min(a_i, a_j) \leq 1/2$

In particular, if Q is a $q - \sigma$-algebra, i.e. Q is a nonvoid family of subsets of Ω, which is closed with respect to complements and unions of countably many mutually disjoint sets of Q, then (Ω, M), where $M = \{I_A : A \in Q\}$, is a fuzzy quantum poset.

Operations $\cup$, $\perp$, and $\cap$, which is defined $\bigcap_i a_i = \inf_i a_i$, are Zadeh's fuzzy union, fuzzy complement and fuzzy intersection, correspondingly, of fuzzy sets. We recall that $\cup$, $\cap$, $\perp$ may be defined for all $\{a_i\} \subset [0, 1]^\Omega$, if necessary. Two fuzzy sets a and b are fuzzy orthogonal and we write $a \perp_F b$ iff $a \cap b \leq 1/2$. We note that Mesiar [5] in the same way defined the F-disjointness of two fuzzy sets. The structure M has been suggested in the paper [1]. In the models [7, 3, 6], the orthogonality of a and b, $a \perp b$, is defined via $a \perp b$ iff $a \leq b^\perp$. It is clear that if $a \perp b$ then $a \perp_F b$, but the converse does not hold in general. If $a \cup a^\perp = b \cup b^\perp$, then $a \perp_F b$ iff $a \perp b$. Moreover, according to the natural ordering $\leq$, M is a poset with an orthogonality $\perp : a \mapsto a^\perp$, for which $a \cup a^\perp \leq 1$ for any $a \in M$. We denote by $W_0(M)$ $(W_1(M))$ the set of a alall $a \in M$ such that $a \leq 1/2$, $(a \geq 1/2)$. Moreover, for any $a \in M$, $a \cap a^\perp \in W_0(M)$, $a \cup a^\perp \in W_1(M)$, and $W_0(M)$ and $W_1(M)$ consist only of elements of those forms.

A relation $R \subseteq M \times M$ is said to be a congruence relation on M if (i) R is an equivalence relation on M; (ii) if aRb then $a^\perp R b^\perp$, for any a, $b \in M$; (iii) if $a_i \perp_F a_j$, $b_i \perp_F b_j$ for $i \neq j$, $a_i R b_i$, $i \geq 1$ then $\bigcup_i a_i R \bigcup_i b_i$.

Now we define a relation $\sim\ \subseteq M \times M$ via

$$a \sim b \text{ iff } a \perp_F b^\perp, \qquad a^\perp \perp_F b. \tag{2.1}$$

It is clear that (i) $a \sim a$ for any $a \in M$; (ii) if $a \sim b$, then $a^\perp \sim b^\perp$; (iii) if $a \sim b$, then $b \sim a$. On the other hand, $\sim$ is not transitive, in general, as we may verify on simple examples. Let $\approx$ be the transitive closure of $\sim$, i.e., the smallest equivalence relation on M containing $\sim$. It is obvious that

$$\begin{gathered} a \approx b \text{ iff there are } a_1, \ldots, a_n \in M \text{ such that} \\ a \sim a_1,\ a_1 \sim a_2,\ \ldots,\ a_n \sim b \end{gathered} \tag{2.2}$$

We recall that if $\{c_i\} \subset W_1(M)$, then [4] $c = \bigcap_{i=1}^\infty c_i \in W_1(M)$ and $\bigcup_{i=1}^\infty \{c_i = 1/2\} \neq$ $\neq \Omega$, where $\{c_i = 1/2\} = \{w \in \Omega : c_i(w) = 1/2\}$. Indeed, we have $\bigcup_{i=1}^\infty \{c_i = 1/2\} \subseteq$ $\subseteq \{c = 1/2\} \neq \Omega$.

Let us define relations θ_0, θ_f and θ_∞ on M as follows:

(i) $a \theta_0 b$ iff there is an $c \in W_1(M)$ such that

$$a \cap b^\perp \cap c,\ a^\perp \cap b \cap c \leq 1/2; \tag{2.3}$$

(ii) $a\theta_f b$ iff there are $c_1, \ldots, c_n \in W_1(M)$ such that

$$\{a \cap b^\perp > 1/2\} \cup \left\{a^\perp \cap b > \frac{1}{2}\right\} \subseteq \bigcup_{i=1}^{n} \{c_i = 1/2\}; \tag{2.4}$$

(iii) $a\theta b$ iff there are $\{c_n\} \subset W_1(M)$ such that

$$\{a \cap b^\perp > 1/2\} \cup \{a^\perp \cap b > 1/2\} \subseteq \bigcup_{n=1}^{\infty} \{c_n = 1/2\} \tag{2.5}$$

The following result generalizes that in [3].

Lemma 2.1. The transitive closure $\approx$ is a proper congruence relation on M and, moreover, $\approx\, = \theta_0 = \theta_f = \theta_\infty$.

Proof: Since (2.3) is equivalent to the assertion $\{a \cap b^\perp > 1/2\} \cup \left\{a^\perp \cap b > \frac{1}{2}\right\} \subseteq \{c = 1/2\}$, we conclude that $\theta_0 \subset \theta_f \subset \theta_\infty$. Suppose $a\theta_\infty b$, then we define a sequence $\{c_n\} \subset W_1(M)$ with (2.5). Let us put $c = \bigcap_{n=1}^{\infty} c_n \in W_1(M)$, then $\bigcup_{n=1}^{\infty} \{c_n = 1/2\} \subseteq \{c = 1/2\}$ which yields $\theta_\infty \subseteq \theta_0$. Moreover, θ_0 is a congruence relation on M. It suffices to verify the transitivity of θ_0. Let $a\theta_0 b$, $b\theta_0 c$. We can find $c_1, c_2 \in W_1(M)$ such that $a \cap b^\perp \cap c_1$, $a^\perp \cap b \cap c_1$, $b \cap c^\perp \cap c_2$, $b^\perp \cap c \cap c_2 \leq \leq 1/2$. It is obvious that.

$$a \cap c^\perp \cap (c_1 \cap c_2), \qquad a^\perp \cap c \cap (c_1 \cap c_2) \leq 1/2,$$

which entails $a\theta_0 c$.

Therefore, $\approx\, \subseteq \theta_0$. On the other hand, let $a\theta_0 b$. Then, for some $c \in W_1(M)$, (2.3) holds, and it is evident that $a \sim a \cap c \sim b \cap c \sim b$ which gives us $a \approx b$ and $\theta_0 = \,\approx$.

To finish our proof, assume that $a_i \theta_0 b_i$ for $a_i \perp_F a_j$, $b_i \perp_F b_j$ whenever $i \neq j$. There exists a sequence $\{c_i\} \subset W_1(M)$ such that $a_i \cap b_i^\perp \cap c_i$, $a_i^\perp \cap b_i \cap c_i \leq 1/2$. If we put $c = \bigcap_{i=1}^{\infty} c_i$, then $\bigcup_i a_i \cap \left(\bigcup_i b_i\right)^\perp \cap c$, $\left(\bigcup_i a_i\right)^\perp \cap \left(\bigcup_i b_i\right) \cap c \leq 1/2$, so that $\approx$ is a congruence on M. The fact that $\approx$ is proper follows from obvious fact $0 \not\approx 1$. Q.E.D.

3. Quotient of a fuzzy quantum poset

A nonvoid subset $I_0 \subseteq M$ is said to be an $F-\sigma$-ideal of (Ω, M) if

(i) $a \cap a^\perp \in I$ for any $a \in M$;

(ii) if $a \leq b$, $a \in M$, $b \in I$, then $a \in I$;

(iii) if $a_i \perp_F a_j$ for $i \neq j$, $\{a_i\} \subset I$, then $\bigcup_i^{\infty} a_i \in I$

(iv) if $a \cap c \in I$ for a $c \in W_1(M)$, then $a \in I$.

By an F-state on (Ω, M) we mean any function $m: M \to [0, 1]$ such that

(i) $m\left(a \bigcup_{\infty} a^{\perp}\right) = 1$ for any $a \in M$;

(ii) $m\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$, if $a_i \perp_F a_j$ for $i \neq j$.

Then $I_m := \{a \in M: m(a) = 0\}$ is a proper $F - \sigma$-ideal of (Ω, M), (see [4]). Moreover, M is an $F - \sigma$-ideal, too.

Proposition 3.1. Put

$$I_0 = \{a \in M: \text{there is a } c \in W_1(M),\ a \cap c \leq 1/2\} \tag{3.1}$$

Then I_0 is a proper $F - \sigma$-ideal of (Ω, M) containing $W_0(M)$ such that $I_0 = \{a \in M: a\,\theta_0\,0\}$. Moreover, if I is any $F - \sigma$-ideal of (Ω, M), then $I_0 \subseteq I$.

Proof. It follows from Lemma 2.1 and from the definition of $F - \sigma$-ideals. Q.E.D.

Lemma 3.2: For any $a \in M$, we put

$$\bar{a} = \{b \in M: b\,\theta_0\,a\} \tag{3.2}$$

and

$$\bar{M} = M/I_0 = \{\bar{a}: a \in M\} \tag{3.3}$$

If in $\bar{M}$ we define a relation $\leq$ via

$$\bar{a} \leq \bar{b} \text{ iff there is a } c \in W_1(M) \text{ with } a \cap b^{\perp} \cap c \leq 1/2, \tag{3.4}$$

then $\leq$ is a well-defined partial ordering on $\bar{M}$. In addition, if $a \cup b \in M$, then $\bar{a} \leq \bar{b}$ if $\overline{a \cup b} = \bar{b}$.

Proof. To show that the relation $\leq$ defined via (3.4) is correct, it is sufficient to prove that if $\bar{a} \leq \bar{b}$ then $\bar{a}_1 \leq \bar{b}_1$ whenever $a\,\theta_0\,a_1$, $b\,\theta_0\,b_1$. Supposing this, we find $c, c_1, c_2 \in W_1(M)$ such that $a \cap b^{\perp} \cap c$, $a \cap a_1^{\perp} \cap c_1$, $a^{\perp} \cap a_1 \cap c_1$, $b \cap b_1^{\perp} \cap c_2$, $b^{\perp} \cap b_1 \cap c_2 \leq 1/2$. Hence $a_1 \cap b_1^{\perp} \cap (c \cap c_1 \cap c_2 \cap (a \cup a^{\perp}) \cap (b \cup b^{\perp})) \leq 1/2$. It is simple that $\bar{a} \leq \bar{a}$, and $\bar{a} \leq \bar{b}$, $\bar{b} \leq \bar{a}$ entail $\bar{a} = \bar{b}$. The transitivity of $\leq$ follows from the following. Let $\bar{a} \leq \bar{b}$, $\bar{b} \leq \bar{c}$, we can find c_1, $c_2 \in W_1(M)$ such that $a \cap b^{\perp} \cap c_1$; $b \cap c^{\perp} \cap c_2 \leq 1/2$. Hence $a \cap c^{\perp} \cap (c_1 \cap c_2 \cap (b \cup b^{\perp})) \leq 1/2$.

The last property may be proved in the same way. Q.E.D.

Remark. With respect to the partial ordering $\leq$, we define in the poset $\bar{M}$ the join V and the meet $\wedge$, if they exist in it.

Lemma 2.1 proves that the mapping $\perp: \bar{M} \to \bar{M}$ defined via

$$\bar{a} \mapsto \bar{a}^{\perp}, \qquad a \in M, \tag{3.5}$$

is defined well. In accordance with quantum logic theory, two elements $\bar{a}$, $\bar{b}$ of $\bar{M}$ are orthogonal, and we write $\bar{a} \perp \bar{b}$, iff $\bar{a} \leq \bar{b}^{\perp}$.

Theorem 3.3. Let (Ω, M) be a fuzzy quantum poset. Then the quotient poset $\bar{M} = M/I_0$ is, with respect to the partial ordering $\leq$ and $\perp$ which are defined via (3.4) and (3.5), respectively, an orthomodular σ-orthoposet with the minimal and maximal elements $\bar{0}$ and $\bar{1}$, correspondingly. Moreover, the canonical mapping φ: $a \mapsto \bar{a}$, $a \in M$, is a surjective σ-homomorphism from Monto $\bar{M}$, ie., it preserves the maximal elements, fuzzy orthogonal elements and joins of mutually orthogonal sequences.

Proof. It is evident that, for any $a \in M$, $\bar{0} \leq \bar{a} \leq \bar{1}$. If $a \perp_F b$ then $\bar{a} \perp \bar{b}$, indeed, $a \cap b \leq 1/2$ entails $a \cap b \cap 1 \leq 1/2$, so that $\bar{a} \leq \bar{b}^{\perp}$.

Let $\{\bar{a}_i\}$ be any sequence of mutually orthogonal elements of $\bar{M}$. Then we can find a sequence $\{a_i'\}$ of mutually fuzzy orthogonal elements from M such that $\bar{a}_i' = \bar{a}_i$ for any i. For this, it suffices to find a sequence $\{c_{ij}\} \subseteq W_1(M)$ such that $a_i \cap a_j \cap c_{ij} \leq 1/2$. Putting $c = \bigcup_{i,j} c_{ij} \in W_1(M)$ and $a_i' = a_i \cap c$, $i \geq 1$, we obtain the elements in question. Moreover, we assert $\bigvee_i \bar{a}_i = \bigvee_i \bar{a}_i' = \bar{a}$ where $a = \bigcup_i a_i'$. It is evident that $\bar{a}_i' \leq \bar{a}$ for any $i \geq 1$. If $\bar{a}_i' \leq \bar{b}$, $i \geq 1$, for some $\bar{b} \in \bar{M}$, then there exists an $c_0 \in W_1(M)$ such that $a_i' \cap b^{\perp} \cap c_0 \leq 1/2$ for any $i \geq 1$. This yields $(\cup a_i') \cap b^{\perp} \cap c_0 \leq 1/2$, $a \cap b^{\perp} \cap c_0 \leq 1/2$, and $\bar{a} \leq \bar{b}$.

The orthogonality $\perp$: $\bar{a} \mapsto \bar{a}^{\perp}$, $a \in M$, has the properties:

(i) $(\bar{a}^{\perp})^{\perp} = \bar{a}$ for any $\bar{a} \in \bar{M}$;

(ii) if $\bar{a} \leq \bar{b}$, then $\bar{b}^{\perp} \leq \bar{a}^{\perp}$;

(iii) $\bar{a} \vee \bar{a}^{\perp} = \bar{1}$ for any $\bar{a} \in \bar{M}$;

(iv) if $\bar{a} \leq \bar{b}$, then there is a $\bar{c} \in \bar{M}$ such that $\bar{a} \perp_0 \bar{c}$ and $\bar{a} \vee \bar{c} = \bar{b}$.

The first three properties are simple. To prove the fourth, suppose $\bar{a} \leq \bar{b}$. Then there is a $c_1 \in W_1(M)$ such that $a \cap b^{\perp} \cap c_1$, $a^{\perp} \cap b \cap c_1 \leq 1/2$. Put $a_1 = a \cap c_1 \in M$, then $a_1 \perp_F b^{\perp}$ and $c := b \cap a_1^{\perp}$; $c \perp_F a_1$ calculate $\bar{a} \vee \bar{c} = \bar{a}_1 \vee \vee \bar{c} = \overline{a_1 \cup c} = \bar{b}$. The last equality follows from the observations: $(a_1 \cup c) \cap \cap b^{\perp} = a_1 \cap b^{\perp} \cup c \cap b^{\perp} = a \cap c_1 \cap b^{\perp} \cup b \cap a_1^{\perp} \cap b^{\perp} \leq 1/2$ and $(a_1^{\perp} \cap c^{\perp}) \cap b = = (a^{\perp} \cup c_1^{\perp}) \cap c^{\perp} \cap b \leq 1/2$.

The properties of the canonical mapping are now obvious.

4. The Loomis-Sikorski representation

From Theorem 3.3 we conclude that the quotient M/I_0 is a quantum logic [9] which is not necessary a lattice. If M is closed with respect to the fuzzy union of any sequence of fuzzy sets of M, then M is a σ-lattice; consequently, M/I_0 is a Boolean σ-algebra. In this case, due to the famous Loomis-Sikorski theorem [8], we find a measurable space $(X, \mathscr{S})$ and a σ-homomorphism h from $\mathscr{S}$ onto M/I_0.

In this section we show that for any fuzzy quantum poset (Ω, M) we can find a $q-\sigma$-algebra Q of some $X \neq \emptyset$ which can be surjectively embedded onto M/I_0, moreover $X = \Omega$.

According to [4], we introduce the following system of subsets of Ω:

$$K(M) = \left\{A \subseteq \Omega\text{: there is an } a \in M \text{ such that } \left\{a > \frac{1}{2}\right\} \subseteq A \subseteq \{a \geq 1/2\}\right\} \tag{4.1}$$

then $K(M)$ is a $q-\sigma$-algebra as it has been proved in [4]. The $K(M)$ has the following simple properties:

(i) if for $\{a_i\} \subset M$, $\cap a_i \in M$, then $\bigcap_i A_i \in K(M)$, where $\{a_i > 1/2\} \subseteq A_i \subseteq \{a_i \geq 1/2\}$;

(ii) $\{a > 1/2\} \subseteq A_t \subseteq \{a \geq 1/2\}$, then $\bigcup_t A_t \in K(M)$.

Theorem 4.1. Let (Ω, M) be any fuzzy quantum poset. Then there is a surjective σ-homomorphism h from $K(M)$ onto M/I_0 which preserves maximal elements, complements and transmits unions of countably many mutually disjoint subsets to joins of mutually orthogonal elements.

Proof. We define a mapping h: $K(M) \to \tilde{M}$ via $h(A) = \tilde{a}$ iff $\{a > 1/2\} \subseteq \subseteq A \subseteq \{a \geq 1/2\}$. We show that h is defined well. If $\{b > 1/2\} \subseteq A \subseteq \{b \geq 1/2\}$, then $\{b^\perp > 1/2\} \subseteq A^c \subseteq \{b^\perp \geq 1/2\}$ which gives $\{a > 1/2\} \cap \{b^\perp > 1/2\} \subseteq A \cap \cap A^c = \emptyset$, so that $a \cap b^\perp \leq 1/2$, similarly $a^\perp \cap b \leq 1/2$, which proves $\tilde{a} = \tilde{b}$. Therefore, $h(\Omega) = \tilde{1}$, $h(A^c) = h(A)^\perp$ for any $A \in K(M)$; and $h\left(\bigcup_i A_i\right) = \bigvee_i h(A_i)$ if $A_i \cap A_j = \emptyset$, $i \neq j$, $\{A_i\} \subset K(M)$. Q.E.D.

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SÚHRN

O REPREZENTÁCIÁCH FUZZY KVANTOVÝCH POSETOV

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V práci sú predstavené dve reprezentácie fuzzy kvantových posetov pomocou kvantovej logiky a $q - \sigma$-algebry, ktorá je Loomisov-Sikorského analóg danej logiky.

РЕЗЮМЕ

ОБ ПРЕДСТАВЛЕНИЯХ НЕЧЕТНИХ КВАНТОВЫХ ЧАСТИЛНО УПОРЯДОЧЕННЫХ ПРОСТРАНСТВ

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В работе представлены две представления нечетких квантовых частилно упорядоченных пространств с помощью квантовой логики и q-σ-алгебры, которая является аналогом Лумиса-Сикорского данной логики.