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ON THE CONVERGENCE IN BANACH SPACES

by M. MARJANOVIĆ, BEOGRAD

1. In this paper will be given a theorem concerning the interdependence of two topologies for a Banach space. It will be also shown how same well-known theorems can be obtained as easy consequences. To do it, we must introduce a new topological notion; that of sequencially weaker topology.

2. Our terminology and notations essentially follow Dunford and Schwartz [1]. We recall only some definitions.

Let $(\mathfrak{X}, \tau)$ be a topological space. The sets in τ are called the open sets of $(\mathfrak{X}, \tau)$, and a *neighborhood* of a point $x \varepsilon \mathfrak{X}$ is any subset of $\mathfrak{X}$ containing an open set to wich x belongs.

If $A \subseteq \mathfrak{X}$, then a sequence $\{x_n\}$, $x_n \varepsilon \mathfrak{X}$ is *eventually* in A if there is an integer N such that if $n \geqslant N$ then $x_n \varepsilon A$.

A sequence $\{x_n\}$ in $\mathfrak{X}$ *converges* τ to a point x if $\{x_n\}$ is eventually in every neighborhood of x. We also use the notations $x_n \to_\tau x$ or $\tau - \lim x_n = x$ to mean $\{x_n\}$ converges τ to x.

A subset A of a topological space is *sequentially compact*, if every sequence in A has a subsequence converging to a point of $\mathfrak{X}$.

3. Let $(\mathfrak{Y}, \tau_y)$ and $(\mathfrak{X}, \tau_x)$, $\mathfrak{X} \subseteq \mathfrak{Y}$ be two Hausdorff spaces. If every $\tau_x -$convergent sequence $\{x_n\}$, $x_n \varepsilon \mathfrak{X}$ converging to a point x, has a $\tau_y -$convergent subsequence to the same point x, then we shall say that the $\tau_x -$topology is *sequencially weaker* than the relative $\tau_y -$topology of $\mathfrak{X}$ as a subset of $\mathfrak{Y}$.

The main result of this paper is the following.

Theorem. *Let $(\mathfrak{Y}, \tau)$ be a Hausdorff space and $\mathfrak{X} \subseteq \mathfrak{Y}$ a Banach space such that the metric topology of $\mathfrak{X}$ is sequencially weaker then the relative $\tau -$topology of $\mathfrak{X}$ as a subset of $\mathfrak{Y}$. Then, a $\tau -$convergent sequence $\{x_n\}$, $x_n \varepsilon \mathfrak{X}$ converges in the metric topology of $\mathfrak{X}$ if and only if it is sequencially compact.*

Proof. Let $\{x_n\}$ be convergent in the τ – topology and sequencially compact in the metric topology. Then, there is a subsequence $\{x_{n_k}\}$ converging in $\mathfrak{X}$, i. e.

$$\lim_k x_{n_k} = x.$$

On the other hand, we have

$$\tau - \lim_n x_n = \bar{x} \in \mathfrak{Y}.$$

Since there is a τ - convergent subsequence $\{x_{n_k'}\}$ of $\{x_{n_k}\}$, it follows

$$\tau - \lim_k x_{n_k'} = x$$

and we also have

$$\tau - \lim_k x_{n_k'} = x,$$

what means that $\bar{x} = x$. To prove that

(1)
$$\lim_n x_n = x,$$

we may make an indirect proof by supposing that

$$\lim_n (\sup_{m>n} | x - x_m |) = \lim_n \sup | x - x_n | = \varepsilon > 0.$$

Now, there exists a subsequence $\{x - x_{n_k}\}$ of the sequence $\{x - x_n\}$ which does not converge to the zero in $\mathfrak{X}$. Hence,

$$\lim_k (x - x_{n_k}) = x' \neq 0$$

i. e.

$$\lim_k x_{n_k} = x - x'.$$

On the other hand, there is a subsequence $\{x_{n_k'}\}$ of the sequence $\{x_{n_k}\}$ such that

$$\tau - \lim_k x_{n_k'} = x - x'$$

and

$$\tau - \lim_k x_{n_k'} = \tau - \lim_n x_n = x,$$

what is not possible if $x' \neq 0$. Therefore, (1) is proved.

Conversely, if $\{x_n\}$ converges to x in the metric topology then every subsequence of $\{x_n\}$ does it, and $\{x_n\}$ is sequencially compact. It remains to show that

(2)
$$\tau - \lim_n x_n = x.$$

If $\{x_n\}$ does not converge τ to x, then a subsequence $\{x_{n_k}\}$ may be selected in a such way that every of its subsequence fails to converge to x in τ – topology (Kelley [3]). But this contradicts the fact that

$$x_{n_k} \to x,$$

and then the existence of a subsequence of $\{x_{n_k}\}$, τ – converging to x, too. Thus the theorem is completely proved.

4. We give below three examples choosing special spaces with the topologies satisfying the condition stated in the proved theorem.

Example 1. Let $\mathfrak{X}$ be a Banach space and let the τ – topology be the weak topology of $\mathfrak{X}$. Then, the theorem proved above can be formulated in a such way

A $\mathfrak{X}^*$ - convergent sequence $\{x_n\}$ is convergent in the metric topology of $\mathfrak{X}$ if and only if it is sequencially compact.

This result has been given by Gagaev [2].

Example 2. Let $(\mathfrak{Y}, \tau)$ be the space of all real functions $x(t)$ defined over the interval [0, 1] and topologized relative to pointwise convergence and $C_{[0,1]} \subseteq \mathfrak{Y}$ be the space of all real continuons functions on [0, 1], where by the norm we mean

$$|x| = \sup_{t \in [0,1]} |x(t)|.$$

If a convergent sequence (in the τ – topology) $\{x_n(t)\}$ of functions belonging to $C_{[0,1]}$ is sequencially compact, then in accordance with the theorem,

$$\lim_{n \to \infty} x_n(t) \in C_{[0,1]}.$$

It is easily seen that the pointwise convergence together with the sequencially compactness is equivalent to the generalized uniform convergence. Hence, we have:

The limit of a generalized uniform convergent sequence of continuous functions is a continuous function.

This statement presents the well-known Dini's theorem.

Example 3. Let (S, Σ, μ) be a finite measure space, where μ is a countably additive complex or extended real valued set function. Let $1 \leqslant p < \infty$. Then, $L_p(S, \Sigma, \mu)$ will denote the set of all μ measurable functions f on S to a Banach space $\mathfrak{X}$ such that the function $|f(\cdot)|^p$ is μ — integrable. By the norm we mean the quantity

$$|f|_p = \left\{ \int_S |f(s)|^p \, v(\mu, ds) \right\}^{1/p}.$$

Since a convergent sequence of functions in $L_p(S, \Sigma, \mu)$ converges in measure and thereby has a subsequence wich converges almost everywhere, the τ-topology for the space of all functions on S to $\mathfrak{X}$, as the topology given by the convergence almost everywhere, is sequencially weaker than the metric topology of $L_p(S, \Sigma, \mu)$. Consequently, we have this consequence of the proved theorem.

Let $1 \leqslant p < \infty$, let (S, Σ, μ) be a finite measure space and let $\{f_n\}$ be a sequence of functions in $L_p(S, \Sigma, \mu)$ converging almost everywhere to a function f. Then f is in $L_p(S, \Sigma, \mu)$ and $|f_n - f|_p$ converges to zero if and only if $\{f_n\}$ is sequencially compact.

This statement is known as Vitali's convergence theorem where the condition.

(I) $$\lim_{v(\mu, E) \to 0} \int_E |f_n(s)|^p v(\mu, ds) = 0,$$

uniformly in n, is replaced with

(I') the sequence $\{f_n\}$ is sequencially compact in $L_p(S, \Sigma, \mu)$.

Let us prove that the conditions (I) and (I') are equivalent. We may suppose without lose of generality that $\{f_n\}$ converges almost everywhere to zero.

From (I), it folows that $\{f_n\}$ converges in $L_p(S, \Sigma, \mu)$ and therefore, $\{f_n\}$ is sequencially compact.

Conversely, (I'), implies (I). Indeed, let $\{f_n\}$ be sequencially compact. Let us form the convergent subsequences

$$\{fn_k^l\}, \quad (k = 1, 2, \dots; \ l = 1, 2, \dots)$$

which exhaust $\{f_n\}$. From

$$fn_k^l(s) \to 0$$

it follows

$$|fn_k^l(s)|_p \to 0, \text{ as } k \to \infty.$$

Then, for an arbitrary $\varepsilon > 0$,

$$|fn_k^l| < \varepsilon^{1,p}, \text{ if } n_k^l > N_l(\varepsilon).$$

We can now choose an $\eta_l(\varepsilon) > 0$, such that

$$\int_E |fn_k^l(s)|^p v(\mu, ds) < \varepsilon \quad \text{for } v(\mu, E) < \eta_l(\varepsilon).$$