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APPROXIMATION RELATIVE TO AN ULTRA FUNCTION

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Abstract. Let X be a non-empty set. A symmetric function $f. X \times X \to R$ is called an ultra function on X if $f(x, y) \leqq \max\{f(x, z), f(z, y)\}$ for all $x, y, z \in X$. If G is a subset of a set X with an ultra function f then an element $g_0 \in G$ is said to be (i) an f-best approximation to $x \in X$ if $f(x, g_0) \leqq f(x, g)$ for all $g \in G$ and (ii) an f-best co-approximation to x if $f(g_0, g) \leqq f(x, g)$ for all $g \in G$. In this paper we extend some of the known results on best approximation and best co-approximation in non-archimedean normed linear spaces to approximation relative to an ultra function which is defined either on an arbitrary set X or on a Hausdorff topological vector space X over a non-archimedean valued field F.

Key words. f-best approximation, f-best co-approximation, symmetric function and ultra function.

The main aim of the present study is to extend some known results on approximation in non-archimedean normed linear spaces to approximation relative to an ultra function which is defined either on an arbitrary set or on a Hausdorff topological vector space over a non-archimedean valued field.

1. Introduction

The notion of f-best approximation in a vector space X was given by Breckner and Brosowski [1] and in a Hausdorff topological space X by the author in [5]. Taking X to be a Hausdorff locally convex topological vector space and f to be a continuous sublinear functional on X, certain results on best approximation relative to the functional f were proved in [1], [2] and [8]. We shall discuss f-best approximation, f-best coapproximation and f-orthogonality in Hausdorff topological vector spaces over non-archimedean valued fields relative to an ultra function f in section 2, and in section 3 we shall discuss f-best approximation and f-best co-approximation for an ultra function f defined on an arbitrary set X. When X is a non-archimedean normed linear space and $f = \|\,.\,\|$, the norm on X, we get some of the results of [3], [4] and [6].

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2. f-Approximation in Topological Vector Spaces

In this section we discuss f-best approximation, f-best co-approximation and f-orthogonality in Hausdorff topological vector spaces over non-archimedean valued fields relative to an ultra function f.

Let X be a Hausdorff topological vector space over a non-archimedean (n.a.) valued field F and f a symmetric (i.e. $f(-x) = f(x)$ for all $x \in X$) real-valued ultra function (i.e. $f(x + y) \leqq \max\{f(x), f(y)\}$ for all $x, y \in X$) on X. Let K be a non-empty closed subset of X and $x \in X$.

An element $k_0 \in K$ is said to be an *f-best approximation* to x in K if

$$f(x - k_0) = f_K(x) \equiv \inf\{f(x - k) : k \in K\}.$$

We denote by $P_{K,f}(x)$ the collection of all such $k_0 \in K$. The set K is said to be *f-proximinal* if $P_{K,f}(x)$ is non-empty for each $x \in X$, *f-semi-Chebyshev* if $P_{K,f}(x)$ is atmost singleton for any $x \in X$ and *f-Chebyshev* if $P_{K,f}(x)$ is exactly singleton for each $x \in X$.

The set K is said to be *f-infimum compact* if for every $x \in X$ and every minimizing net $\{k_\alpha\}$ in K (i.e. $f(x - k_\alpha) \to f_K(x)$) has an f-convergent subset in K.

An element $g_0 \in K$ is said to be an *f-best coapproximation* of an element $x \in X$ if

$$f(g_0 - g) \leqq f(x - g)$$

for all $g \in K$. The set of all such $g_0 \in K$ is denoted by $R_{K,f}(x)$.

For $x, y \in X$, x is said to be *f-orthogonal* to y, $x \perp_f y$, if

$$f(x) \leqq f(x + \alpha y)$$

for every scalar α.

x is said to be f-orthogonal to K, $x \perp_f K$, if $x \perp_f y$ for all $y \in K$.

The following theorem gives existence of f-best approximation for a non-negative f.

Theorem 1. *Let K be a non-empty f-infimum compact subset of X. Then K is f-proximinal.*

Proof. Let $x \in X$ then by the definition of $f_K(x)$, there exists a net $\{k_\alpha\}$ in K such that

$$f(x - k_\alpha) \to f_K(x).$$

Since $\{k_\alpha\}$ is a minimizing net in K and K is f-infimum compact, there exists a subnet $\{k_\beta\}$ of $\{k_\alpha\}$ and $k_0 \in K$ such that $\lim_\beta f(k_\beta - k_0) = 0$. Consider

$$f(x - k_0) \leqq \max\{f(x - k_\beta), f(k_\beta - k_0)\}.$$

In the limiting case this gives

$$f(x - k_0) \leqq f_K(x),$$
$$\leqq f(x - k_0),$$

i.e. $f(x - k_0) = f_K(x)$ and so K is f-proximinal.

Remark. It will be interesting to study conditions under which K is f-semi-Chebyshev and f-Chebyshev. One such conditions under which K is f-Chebyshev is given in section 3, Theorem 1.

The following theorem characterizes elements of f-best approximation.

Theorem 2. *For a linear subspace G of X, $g_0 \in P_{G,f}(x)$ if and only if $(x - g_0) \perp_f G$*

Proof. $(x - g_0) \perp_f G \Leftrightarrow f(x - g_0 + \alpha g) \geqq f(x - g_0)$ for all $g \in G$, $\alpha \in F$
$$\Leftrightarrow g_0 \in P_{G,f}(x).$$

Corollary. *For a linear subspace G, $P_{G,f}(x)$ is empty for every $x \in X \mid G$ if there exist no $y \in X \mid \{o\}$ such that $y \perp_f G$.*

Proof. Suppose $P_{G,f}(x) \neq \varphi$ for some $x \in X \mid G$. Let $g_0 \in P_{G,f}(x)$. Then $(x - g_0) \perp_f G$. Take $y = x - g_0$. Then $y \in X \mid \{0\}$ and $y \perp_f G$, a contradiction.

The following theorem characterizes elements of f-best coapproximation when f is sublinear (a symmetric sublinear functional is homoneous i.e. $f(\alpha x) = |\alpha| f(x)$).

Theorem 3. *For a linear subspace G, $g_0 \in R_{G,f}(x)$ if and only if $G \perp_f (x - g_0)$.*

Proof. $G \perp_f (x - g_0) \Leftrightarrow f[g + \alpha(x - g_0)] \geqq f(g)$ for all $g \in G$, $\alpha \in F$,
$$\Leftrightarrow f(x - g_0 + \alpha^{-1}g) \geqq f(\alpha^{-1}g) \quad \text{for all } g \in G,\ \alpha \in F,\ \alpha \neq 0,$$
$$\Leftrightarrow f(x - g_0 + g') \geqq f(g') \quad \text{for all } g' \in G,$$
$$\Leftrightarrow f(x - g'') \geqq f(g_0 - g'') \quad \text{for all } g'' \in G,$$
$$\Leftrightarrow g_0 \in R_{G,f}(x).$$

Corollary. *For a linear subspace G, $R_{G,f}(x)$ is empty for every $x \in X \mid G$ if there exist no $y \in X \mid \{0\}$ such that $G \perp_f y$ when f is sublinear.*

Proof. It is similar to Corollary to Theorem 2.

The following result shows that for a sublinear f the f-orthogonality is symmetric in X.

Theorem 4. *For a sublinear f, the f-orthogonality is symmetric.*

Proof. Let $x \perp_f y$. Then

(1) $$f(x + \alpha y) \geqq f(x) \quad \text{for every scalar } \alpha,$$

we are to show that $y \perp_f x$ i.e.

$$f(y + \beta x) \geqq f(y) \quad \text{for every scalar } \beta.$$

Suppose that for some $\beta \neq 0 \in F$,

$$f(y + \beta x) < f(y).$$

This implies

(2) $$f(x + \beta^{-1}y) < f(\beta^{-1}y),$$

as f is homogeneous. Then

$$f(x) = f(x + \beta^{-1}y - \beta^{-1}y) = \max\{f(x + \beta^{-1}y), f(\beta^{-1}y)\},$$

as f is symmetric (if $f(x) < f(y)$

$$\text{then } f(x + y) = \max\{f(x), f(y)\}) = f(\beta^{-1}y).$$

Then (2) gives

$$f(x + \beta^{-1}y) < f(x),$$

a contradiction to (1). Hence $y \perp_f x$.

The following theorem shows that for a subspace G, elements of f-best approximation and f-best coapproximation coincide and so there is no need to study, f-best co-approximation separately for a sublinear f.

Theorem 5. *Let G be a subspace of X and $x \in X$. Then an element of f-best approximation to x in G is an element of f-best coapproximation and vice-versa i.e.* $P_{f,G}(x) = R_{f,G}(x)$.

Proof. The proof follows from Theorems 2, 3 and 4.

3. f-Approximation in Arbitrary Sets

In this section we discuss f-best approximation and f-best co-approximation where f is an ultra function defined on an arbitrary set X.

To start with we restate a few definitions of section 2 in the context of an ultra function defined on an arbitrary set.

Let X be any set. A symmetric function $f: X \times X \to \mathbf{R}$ is called an ultra function on X [7] if

$$f(x, y) \leqq \max\{f(x, z), f(z, y)\}$$

for all $x, y, z \in X$.

Let G be a subset of a set X with an ultra function f.

An element $g_0 \in G$ is said to be *f-best approximation* to $x \in X$ if

$$f(x, g_0) \leqq f(x, g)$$

for all $g \in G$.

An element $k_0 \in G$ is said to be *f-best co-approximation* of x if

$$f(k_0, g) \leqq f(x, g)$$

for all $g \in G$.

Regarding the uniqueness of best approximation the following result was proved in [3]:

For a linear subspace G of a n.a. normed linear space X, best approximation of $x \in X$, $x \notin G$ in G when it exists is never uniquely determined unless $G = \{0\}$.

The following example shows that in our case, f-best approximation may be unique.

Let $X = N$, the set of natural numbers,

$$f: N \times N \to R,$$

defined by

$$f(m, n) = \max\left\{\frac{1}{m}, \frac{1}{n}\right\},$$

$$G = \{1, 2, 3, \ldots, n : n > 1\},$$

and $n_0 \in X$, $n_0 \notin G$. Then it is easy to see that n is f-best approximation for n_0 and is unique.

It is interesting to note that every element of X which is not in G has n as f-best approximation in G.

The following theorem characterizes the uniqueness of f-best approximation:

Theorem 1. *Let E be a subset of a set X with an ultra function f and $x \in X$. An f-best approximation $z \in E$ to x is unique if and only if there exist no $t \in E$ such that $f(t, z) \leqq f(x, z)$.*

Proof. Firstly, suppose there exist $t \in E$ such that

$$f(t, z) \leqq f(x, z).$$

Then

$$f(x, t) \leqq \max\{f(x, z), f(z, t)\} = f(x, z),$$

implies that t is also an f-best approximation to x, a contradiction.

Conversely, suppose there exist no such t. Then z is unique f-best approximation to x. For, let if possible, there exist $\Theta \in E$, $\Theta \neq z$ such that Θ is also an f-best approximation to x. Then

$$f(x, \Theta) = f(x, z) = \inf_{y \in E} f(x, y).$$

Therefore

$$f(\Theta, z) \leqq \max\{f(\Theta, x), f(x, z)\}$$

gives

$$f(\Theta, z) \leqq f(x, z),$$

a contradiction.

The following result shows that as in section 2, there is no need to study best co-approximation separately in this case too.

Theorem 2. *Let G be a subset of X and $x \in X$. Then an element of f-best approximation to x in G is an element of f-best co-approximation and vice-versa.*

Proof. Let $g_0 \in G$ be an f-best approximation to x. Then

$$f(x, g_0) \leqq f(x, g)$$

for all $g \in G$. Consider

$$f(g_0, g) \leqq \max\{f(g_0, x), f(x, g)\} = f(x, g).$$

Thus $g_0 \in G$ is f-best co-approximation to x.

Conversely, suppose $g_0 \in G$ is f-best co-approximation to x. Then

$$f(g_0, g) \leqq f(x, g)$$

for all $g \in G$. Consider

$$f(x, g_0) \leqq \max\{f(x, g), f(g, g_0)\} = f(x, g).$$

Thus $g_0 \in G$ is f-best approximation to x.

Remark 1. When $f = d$, the metric on X, we get: In an ultra metric space elements of best approximation and best co-approximation coincide.

Remark 2. The notions of ε-approximation, best simultaneous approximation, proximal points of pairs of sets, strong approximation, strong co-approximation, farthest points and strong farthest points, available in literature can be discussed relative to an ultra function defined on an arbitrary set.

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