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GEOMETRY OF LAGRANGEAN STRUCTURES. 1.

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Abstract. This paper is the first one of the series intended as a self-contained, relatively complete exposition of differential geometry of Lagrangean structures. It develops the basic differentiation and integration theory of differential odd forms on smooth manifolds and differential odd base forms on smooth fibered manifolds. Except a few minor innovations it does not contain new results.

Key words. Differential form, differential odd form, differential odd base form, exterior product, interior product, pull-back, exterior derivative, integral of differential odd form.

MS Classification. 58 E 99; 58 C 20, 58 C 35.

A *Lagrangean structure* is a pair (Y, λ), where Y is a manifold endowed with the structure of a fibered manifold over an n-dimensional base manifold X, and λ is an odd base form on some r-jet prolongation J^rY of Y ($r \geqq 0$), horizontal with respect to the projection of J^rY onto X (*a lagrangian of order* r for Y). Let $J^r\gamma$ denote the r-jet prolongation of a section γ of Y, and let $J^r\gamma^*\lambda$ denote the pull-back of λ by $J^r\gamma$. Let Ω be a compact, n-dimensional submanifold of X with boundary. The *variational function*, or the *action function*, over Ω, associated with (Y, λ), is the real-valued function $\gamma \to \int_\Omega J^r\gamma^*\lambda$, defined on the set of sections of Y over Ω.

The main concern of the theory of Lagrangean structures is to study the variational functions, restricted to prescribed subsets of the set of sections; in particular, one is interested in their *critical points* and variational differential equations connected with them, *extrema*, and *symmetry properties*.

The purpose of the series beginning by this paper is to explain systematically the geometric foundations of the theory of Lagrangean structures, and of the integral variational problems in fibered spaces associated with them. Since the late 1960s, when the first papers on the geometric structure of this class of variational problems (of order 1) appeared, several branches of the subject have developed substantially. Our treatment reflects this development; on the other hand, we also introduce new concepts and ideas, and give original contributions to the theory.

Unless otherwise stated, all manifolds in this work will be real finite-dimensional, C^∞-smooth, Hausdorff manifolds with countable base, and all mappings of manifolds will be C^∞-smooth.

1. ODD BASE FORMS

This introductory part of the work contains the elementary calculus of odd forms on smooth manifolds (Sections 1.1 – 1.4) and the differentiation theory of odd base forms on smooth fibered manifolds (Sections 1.5 – 1.6). We emphasize those notions and theorems which will be utilized later in the theory of Lagrangean structures.

The theory of odd forms (covariant, antisymmetric pseudotensor fields) was initiated by de Rham [2] (see also [6]), and has been completed by Bourbaki [1]. The concept of an odd base form was introduced by the author [3] as a field of antisymmetric, covariant geometric objects on a fibered manifold which is "odd" with respect to the base of this fibered manifold only. Our exposition follows the work [4] where this concept is discussed in detail.

1.1. Odd scalars, odd forms. Let X be an n-dimensional manifold, FX the bundle of frames over X. Let us consider the set of real numbers R as a vector space endowed with the linear representation of the general linear group $GL_n(R) \times R \ni \ni (A, s) \to (\operatorname{sgn} \det A) \,.\, s \in R$, where sgn denotes the sign of a real number. The fiber bundle with base X and type fiber R, associated with the principal $GL_n(R)$-bundle FX by means of this linear representation, is called the *bundle of odd scalars* over X, and is denoted by $\mathring{R}X$. The fiber in $\mathring{R}X$ over a point $x \in X$ is denoted by $\mathring{R}_x X$. An equivalence class in $\mathring{R}_x X$ whose representative is a pair $(\xi, s) \in FX \times R$, is denoted by $[(\xi, s)]$, and is called an *odd scalar* at the point x. $\mathring{R}X$ is a vector bundle, and $\dim \mathring{R}X = 1 + \dim X$.

Let $p \geqq 1$ be any integer and let $\wedge^p T^*X$ denote the bundle of p-forms over X. The vector bundle $\mathring{R}X \otimes \wedge^p T^*X$ is called the *bundle odd p-form* over X. The fiber over a point $x \in X$ is the tensor product $\mathring{R}_x X \otimes \wedge^p T_x^* X$, where $T_x^* X = (T_x X)^*$ is the dual of tangent vector space $T_x X$ at x; the points of this fiber are called *odd p-forms* at the point x. The bundle $\mathring{R}X$ is also called the *bundle of odd 0-forms*, and an odd scalar at a point x is called an *odd 0-form* at x. A section of the bundle $\mathring{R}X \otimes \wedge^p T^*X$ defined on an open set $V \subset X$, is called a *(differential) odd p-form* on V; a section of $\mathring{R}X$ defined on V is called a *(differential) odd 0-form* on V, or a *field of odd scalars* on V.

Convention 1.1. For effective computation with differential forms and differential odd forms we establish the following summation convention. Let E be an m-dimensional vector space, (e_i) its basis, (e^i) the dual basis of the dual vector space E^*. Let $\omega \in \wedge^p T^*E$ be any element, $p \geqq 1$. ω is uniquely expressible in the form

$$\omega = \Sigma \omega_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}, \tag{1.1.1}$$

(summation over all sequences $(i_1, \dots, i_p)$ such that $1 \leqq i_1 < \dots < i_p \leqq m$), where $\omega_{i_1 \dots i_p} \in R$ are *components* of ω with respect to the basis $(e^{i_1} \wedge \dots \wedge e^{i_p})$.

$1 \leqq i_1 < \ldots < i_p \leqq m$, of the vector space $\wedge^p T^* E$. ω is also uniquely expressible in the form

$$\omega = \frac{1}{p!} \omega_{j_1 \ldots j_p} e^{j_1} \wedge \ldots \wedge e^{j_p} \tag{1.1.2}$$

(summation over *all* $j_1, \ldots, j_p = 1, 2, \ldots, m$), where the system of coefficients $\omega_{j_1 \ldots j_p}$, $1 \leqq j_1, \ldots, j_p \leqq m$, is antisymmetric in the subscripts; this system *extends* the system of components of ω defined by (1.1.1) to all sequences $(j_1, \ldots, j_p)$. We shall use both expressions (1.1.1) and (1.1.2) without explicit mentioning the range of summation. In general, when no sign of summation appears, the standard summation convention is applied to the repeated subscripts and superscripts.

Chart expressions. Let $x \in X$ be a point, (U, φ), $\varphi = (x^i)$, a chart at x. We put

$$\hat{\varphi}(x) = \left[\left(\left(\left(\frac{\partial}{\partial x^1}\right)_x, \ldots, \left(\frac{\partial}{\partial x^n}\right)_x\right), 1\right)\right]. \tag{1.1.3}$$

$\hat{\varphi}(x)$ is an element of $\hat{R}_x X$, called the *odd scalar* at x, *associated* with the chart (U, φ). Since $\dim \hat{R}_x X = 1$ and $\hat{\varphi}(x) \neq 0$, $\hat{\varphi}(x)$ can be taken as a basis of the vector space $\hat{R}_x X$. Thus any odd scalar $\hat{s} \in \hat{R}_x X$ has a unique expression of the form

$$\hat{s} = \sigma_\varphi \hat{\varphi}(x), \tag{1.1.4}$$

where $\sigma_\varphi \in R$ is the component of $\hat{s}$ with respect to (U, φ). The correspondence $x \to \hat{\varphi}(x)$ is a field of odd scalars on U; we call it the field of odd scalars *associated* with $(U, \hat{\varphi})$.

Let $\varrho \in \hat{R}_x X \otimes \wedge^p T_x^* X$ be any odd p-form at $x, p \geqq 1$. There exists a unique (ordinary) p-form $\varrho_\varphi \in \wedge^p T_x^* X$ such that

$$\varrho = \hat{\varphi}(x) \otimes \varrho_\varphi. \tag{1.1.5}$$

Writing $\varrho_\varphi = \Sigma \varrho_{\varphi, i_1 \ldots i_p} (\mathrm{d}x^{i_1})_x \wedge \ldots \wedge (\mathrm{d}x^{i_p})_x$ we obtain a unique expression of ϱ in the form

$$\varrho = \Sigma \varrho_{\varphi, i_1 \ldots i_p} \hat{\varphi}(x) \otimes (\mathrm{d}x^{i_1})_x \wedge \ldots \wedge (\mathrm{d}x^{i_p})_x, \tag{1.1.6}$$

where $\varrho_{\varphi, i_1 \ldots i_p} \in R$ are the components of ϱ with respect to (U, φ).

Let (V, ψ), $\psi = (y^j)$, be some other chart at x. We easily obtain the following *transformation formulas*:

$$\hat{\psi}(x) = (\operatorname{sgn} \det D\varphi\psi^{-1}(\psi(x)))\, \hat{\varphi}(x), \tag{1.1.7}$$

$$\sigma_\psi = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x)))\, \sigma_\varphi, \tag{1.1.8}$$

$$\varrho_\psi = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x)))\, \varrho_\varphi, \tag{1.1.9}$$

$$\varrho_{\psi, j_1 \ldots j_p} = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x))). \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{j_p}} \varrho_{\varphi, i_1 \ldots i_p}. \tag{1.1.10}$$

In these formulas Df denotes the derivative of a mapping f, and the derivatives $\partial x^i/\partial y^j$ on the right in (1.1.10) are considered at the point $\psi(x)$.

Let ω be an odd n-form on X ($n = \dim X$). We say that ω is *positive* at a point $x \in X$, if there exists a chart (U, φ), $\varphi = (x^i)$, at x such that the chart expression

$$\omega = F_\varphi \hat{\varphi} \otimes \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \tag{1.1.11}$$

satisfies $F_\varphi(x) > 0$. If ω is positive at x then for any other chart (V, ψ) at x $F_\varphi(x) > 0$; this follows from the transformation formula $F_\varphi(x) = \det D\varphi\psi^{-1}(\psi(x))$. An odd n-form on X, positive at each point, is called a *volume element* on X.

Theorem 1.1. *On each manifold there exists a volume element.*

Proof. For any chart (U, φ), $\varphi = (x^i)$, on X, $\hat{\varphi} \otimes \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ is a volume element on U. A volume element on X can be constructed with the help of such volume elements by means of a partition of unity.

An element $\hat{s} \in \hat{R}_x X$ is called a *unit odd scalar* at x if there exists a chart (U, φ) at x such that $\hat{s} = \hat{\varphi}(x)$. We shall now give conditions ensuring that the vector bundles $\wedge^p T^*X$ and $\hat{R}X \otimes \wedge^p T^*X$, $p \geqq 1$, be isomorphic.

Theorem 1.2. *Let X be an n-dimensional manifold. The following three conditions are equivalent*:

(1) *X is orientable.*

(2) *There exists a field of unit odd scalars defined on X.*

(3) *For each $p \geqq 1$ the vector bundles $\wedge^p T^*X$ and $\hat{R}X \otimes \wedge^p T^*X$ are isomorphic over id_X, and the vector bundle $\hat{R}X$ is isomorphic to $X \times R$ over id_X.*

Proof. 1. If X is orientable, then there exists an atlas on X, formed by charts (U_ι, φ_ι), $\varphi_\iota = (x_\iota^i)$, $\iota \in I$, such that for any $\iota, \varkappa \in I$, $\det D\varphi_\iota\varphi_\varkappa^{-1} > 0$. Then according to (1.1.7), $\hat{\varphi}_\iota = \hat{\varphi}_\varkappa$, and there exists a field of unit odd scalars δ, defined on X, such that the restriction of δ to U_ι is φ_ι.

2. Let δ be a field of unit odd scalars defined on X. For any element $\varrho \in \wedge^p T_x^* X$ we set $\nu(\varrho) = \delta(x) \otimes \varrho$. Then ν defines an isomorphism of the vector bundles $\wedge^p T^*X$ and $\hat{R}X \otimes \wedge^p T^*X$ over id_X. The same holds for $\varrho \in X \times R$.

3. Take $p = n$ and suppose that the vector bundles $\wedge^n T^*X$ and $\hat{R}X \otimes \wedge^p T^*X$ are isomorphic over id_X. Let $\nu : \wedge^p T^*X \to \hat{R}X \otimes \wedge^p T^*X$ be an isomorphism. Let ω be a volume element on X (Theorem 1.1). Then the mapping $x \to \nu^{-1}(\omega(x))$ is an everywhere non-zero (ordinary) n-form on X, and X must be orientable.

Let $\Omega^p(X)$ (resp. $\hat{\Omega}^p(X)$) denote the module of (ordinary) p-forms (resp. the module of odd p-forms) over the ring of functions. Suppose that X is orientable and choose an orientation of X, i.e. a maximal atlas (U_ι, φ_ι), $\iota \in I$, such that for any $\iota, \varkappa \in I$, $\det D\varphi_\iota\varphi_\varkappa^{-1} > 0$ on $U_\iota \cap U_\varkappa$. By the proof of Lemma 2, relation

$$\delta = \varphi_\iota \tag{1.1.12}$$

defines a field of unit odd scalars on X, which is said to be *associated* with the given orientation. The arising mapping $\Omega^p(X) \ni \varrho \to \delta \otimes \varrho \in \hat{\Omega}^p(X)$ is an isomorphism of modules, *associated* with the orientation.

1.2. Odd base scalars, odd base forms. Recall the definition of the pull-back of a vector bundle. Let E be a vector bundle with base X and projection $\pi : E \to X$, $f : Y \to X$ a mapping of manifolds. We set $f^*E = \{(y, z) \in Y \times E \mid f(y) = \pi(z)\}$. Let π_1 (resp. π_2) be the restriction of the canonical projection $Y \times E \to Y$ (resp. $Y \times E \to E$) to the set f^*E. On f^*E there exists precisely one structure of a vector bundle with base Y and projection π_1 such that $\pi_2 : f^*E \to E$ is a homomorphism of vector bundles over f. f^*E with this vector bundle structure is called the *pull-back* of the vector bundle E with respect to f. The homomorphism π_2 is called *canonical.*

Let (W, χ), $\chi = (x^i, z^\nu)$, be a vector bundle chart on E, (U, φ), $\varphi = (x^i)$, the associated chart on X, and (V, ψ), $\psi = (y^\sigma)$, a chart on Y. Suppose that $f(V) \subset U$. Writing for simplicity y^σ (resp. z^ν) instead of $y^\sigma \circ \pi_1$ (resp. $z^\nu \circ \pi_2$) we obtain a vector bundle chart $(\pi_1^{-1}(V), \varkappa)$, $\varkappa = (y^\sigma, z^\nu)$, on f^*E, which is called *associated* with the charts (V, ψ) and (W, χ).

Let $f : Y \to X$ be a fixed mapping of manifolds. The pull-back $f^*\mathring{R}X$ of the bundle of odd scalars $\mathring{R}X$ is called the *bundle of odd base scalars* over Y. An *odd base scalar* at a point $y \in Y$ is an element of the fiber in $f^*\mathring{R}X$ over y. A section of $f^*\mathring{R}X$, defined on an open subset $V \subset Y$, is called a *field of odd base scalars* on V, or a *(differential) odd base* 0*-form* on V.

Let $p \geqq 1$ be any integer. The vector bundle $f^*\mathring{R}X \otimes \wedge^p T^*Y$ is called the *bundle of odd base p-forms* over Y. An *odd base p-form* at a point $y \in Y$ is an element of the fiber in $f^*\mathring{R}X \otimes \wedge^p T^*Y$ over y. A section of $f^*\mathring{R}X \otimes \wedge^p T^*Y$, defined on an open subset $V \subset Y$, is called a *(differential) odd base p-form* on V.

Remark 1.1. If $Y = X$ and $f = id_X$ then the notion of an odd base p-form $(p \geqq 0)$ coincides with the notion of an odd p-form.

For any p, $0 \leqq p \leqq \dim Y$, odd base p-forms defined on an open set $V \subset Y$, form a module over the ring of functions; if the mapping f is fixed, this module is denoted by $\hat{\Omega}^p(Y)$.

Chart expressions. Let $f : Y \to X$ be a mapping, $x \in X$ a point, (U, φ) a chart at x, $\hat{\varphi}$ the field of odd scalars on U, associated with (U, φ). We set for each $y \in f^{-1}(U)$

$$f^*\hat{\varphi}(y) = (y, \hat{\varphi}(f(y))). \tag{1.2.1}$$

$f^*\hat{\varphi}(y)$ is an odd base scalar at y, called the odd base scalar, *associated* with (U, φ). Any odd base scalar δ at y has a unique expression of the form

$$\delta = \delta_\varphi f^*\hat{\varphi}(y), \tag{1.2.2}$$

where $\delta_\varphi \in R$ is the component of δ with respect to (U, φ). The correspondence $y \to f^*\hat{\varphi}(y)$ is a field of odd base scalars on the open set $f^{-1}(U) \subset Y$; we call it the field of odd base scalars *associated* with (U, φ).

Let $y \in f^{-1}(U)$ be any point, ϱ an odd base p-form at y, $p \geqq 1$. There exists a unique (ordinary) p-form $\varrho_\varphi \in \wedge^p T_y^* Y$ such that

(1.2.3) $$\varrho = f^*\hat{\varphi}(y) \otimes \varrho_\varphi.$$

Let (V, χ), $\chi = (y^\sigma)$, be a chart at y such that $f(V) \subset U$. Writing $\varrho_\varphi = \Sigma \varrho_{\varphi, \sigma_1 \ldots \sigma_y} (\mathrm{d}y^{\sigma_1})_y \wedge \ldots \wedge (\mathrm{d}y^{\sigma_p})_y$ we obtain a unique expression of ϱ in the form

(1.2.4) $$\varrho = \Sigma \varrho_{\varphi, \chi, \sigma_1 \ldots \sigma_p} f^*\hat{\varphi}(y) \otimes (\mathrm{d}y^{\sigma_1})_y \wedge \ldots \wedge (\mathrm{d}y^{\sigma_p})_y,$$

where $\varrho_{\varphi, \chi, \sigma_1 \ldots \sigma_p} \in R$ are the components of ϱ with respect to (U, φ) and (V, χ).

Let $x \in X$ be a point, (U, φ) and (U, ψ) two charts at x, $y \in f^{-1}(U)$ a point, and (V, χ), $\chi = (y^\sigma)$, (V, ζ), $\zeta = (\bar{y}^\sigma)$, two charts at y. Using the expressions (1.2.1) – (1.2.4) we easily obtain the following *transformation formulas*:

(1.2.5) $$f^*\hat{\psi}(y) = (\operatorname{sgn} \det D\varphi\psi^{-1}(\psi(x)))\, f^*\hat{\varphi}(y),$$

(1.2.6) $$\delta_\psi = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x)))\, \delta_\varphi,$$

(1.2.7) $$\varrho_\psi = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x)))\, \varrho_\varphi,$$

(1.2.8) $$\varrho_{\psi, \zeta, \nu_1 \ldots \nu_p} = (\operatorname{sgn} \det D\psi\varphi^{-1}(\varphi(x))) . \frac{\partial y^{\sigma_1}}{\partial \bar{y}^{\nu_1}} \cdots \frac{\partial y^{\sigma_p}}{\partial \bar{y}^{\nu_p}} .$$

In (1.2.8), the derivatives $\partial y^\sigma / \partial \bar{y}^\nu$ are considered at the point $\zeta(y)$.

1.3. Differentiation of odd forms and odd base forms. Let χ_1, χ_2 be two n-dimensional manifolds, $\alpha : X_1 \to X_2$ a local diffeomorphism. If $\zeta = (\zeta_1, \ldots, \zeta_n)$ is a frame at a point $x \in X_1$, then by definition of a local diffeomorphism, $T\alpha\zeta = (T\alpha\zeta_1, \ldots, \ldots, T\alpha\zeta_n)$ is a frame at $\alpha(x) \in X_2$. α induces a homomorphism of vector bundles $\mathring{R}\alpha : \mathring{R}X_1 \to \mathring{R}X_2$ over α by the formula

(1.3.1) $$\mathring{R}\alpha([(\zeta, s)]) = [(T\alpha\zeta, s)].$$

$\mathring{R}\alpha$ is obviously a linear isomorphism on each fiber in $\mathring{R}X_1$; its restriction to the fiber $\mathring{R}_x X_1$ is denoted by $\mathring{R}_x\alpha$. We have

(1.3.2) $$\mathring{R} id_X = id_{\mathring{R}X}, \qquad \mathring{R}(\beta \circ \alpha) = \mathring{R}\beta \circ \mathring{R}\alpha$$

for any n-dimensional manifold X, and for any two local diffeomorphisms of n-dimensional manifolds α, β such that $\beta \circ \alpha$ is defined.

Let δ be a field of odd scalars on X_2. We put for each $x \in X$

(1.3.3) $$\alpha^*\delta(x) = (\mathring{R}_x\alpha)^{-1}\, \delta(\alpha(x)).$$

$\alpha^*\delta$ is a field of odd scalars on X_1, called the *pull-back* of δ with respect to α. Analogously, let $p \geqq 1$, and let ϱ be an odd p-form on X_2. We put for each $x \in X$

and $\xi_1, \ldots, \xi_p \in T_x X_1$

$$(1.3.4) \qquad (\alpha^*\varrho)(x)(\xi_1, \ldots, \xi_p) = (\mathring{R}_x\alpha)^{-1}\, \varrho(\alpha(x))\,(T\alpha\xi_1, \ldots, T\alpha\xi_p).$$

$\alpha^*\varrho$ is an odd p-form on X_1, called the *pull-back* of ϱ with respect to α.

We shall now generalize the concept of the pull-back to odd base forms. Let Y_1 (resp. Y_2) be a fibered manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2). By a *homomorphism* of fibered manifolds Y_1, Y_2 we shall mean a mapping $\alpha : V \to Y_2$, where $V \subset Y_1$ is an open set, such that there exists a mapping $\alpha_0 : \pi_1(V) \to X_2$ satisfying

$$(1.3.5) \qquad \pi_2 \circ \alpha = \alpha_0 \circ \pi_1 .$$

Obviously, in this case $\pi_1(V) \subset X_1$ is open, and α_0 is unique; we call it the *projection* of α. Unless otherwise mentioned, we take for simplicity $V = Y_1$. Suppose, moreover, that α_0 is a local diffeomorphism. Then α induces a homomorphism of vector bundles, again denoted by $\mathring{R}\alpha : \pi_1^*\mathring{R}X_1 \to \pi_2^*\mathring{R}X_2$, by the formula

$$(1.3.6) \qquad \mathring{R}\alpha(y, \delta) = (\alpha(y), \mathring{R}\alpha_0(\delta)).$$

$\mathring{R}\alpha$ is a linear isomorphism on each fiber, and its projection is α; its restriction to the fiber over a point $y \in Y_1$ is denoted by $\mathring{R}_y\alpha$. We have

$$(1.3.7) \qquad \mathring{R}id_Y = id_{\pi^*\mathring{R}X}, \qquad \mathring{R}(\beta \circ \alpha) = \mathring{R}\beta \circ \mathring{R}\alpha$$

for any fibered manifold Y with base X and projection π, and for any two homomorphisms α, β of fibered manifolds whose projections are local diffeomorphisms, such that $\beta \circ \alpha$ is defined.

Let δ be a field of odd base scalars on Y_2. We set for each $y \in Y_1$

$$(1.3.8) \qquad \alpha^*\delta(y) = (\mathring{R}_y\alpha)^{-1}\, \delta(\alpha(y)).$$

$\alpha^*\delta$ is a field of odd base scalars on Y_1, called the *pull-back* of the field of odd base scalars δ with respect to the homomorphism α. Analogously, let $p \geqq 1$, and let ϱ be an odd base p-form on Y_2. We set for each $y \in Y_1$ and $\xi_1, \ldots, \xi_p \in T_y Y_1$

$$(1.3.9) \qquad \alpha^*\varrho(y)(\xi_1, \ldots, \xi_p) = (\mathring{R}_y\alpha)^{-1}\, \varrho(\alpha(y))\,(T\alpha\xi_1, \ldots, T\alpha\xi_p).$$

$\alpha^*\varrho$ is an odd base p-form on Y_1, called the *pull-back* of the odd base p-form ϱ with respect to α.

Remark 1.2. If $Y_1 = X_1$, $\pi_1 = id_{X_1}$, $Y_2 = X_2$, $\pi_2 = id_{X_2}$, then the pull-back of the corresponding odd base p-forms coincides with the pull-back of odd p-forms.

Let Y be a fibered manifold with base X and projection π, let $\gamma : X \to Y$ be its *section*, i.e., $\pi \circ \gamma = id_X$. γ can be viewed as a homomorphism of the fibered manifold X with base X and projection id_X into Y, whose projection is id_X; that is, the pull-back of an odd base p-form on Y with respect to γ has sense, and is an odd form on X. π can also be viewed as a homomorphism of fibered manifolds whose projection is id_X; in this case the pull-back of an odd p-form on X with respect to π is an odd base p-form on Y.

Remark 1.3. For $\alpha = \pi$ and $\delta = \hat{\varphi}$, definition (1.3.8) reduces to (1.2.1) (see Remark 1).

Chart expressions. Let $y \in Y_1$ be a point, $x = \pi_1(y)$, and let (U_1, φ_1) (resp. (U_2, φ_2)) be a chart at x (resp. $\alpha_0(x)$) such that $\alpha_0(U_1) \subset U_2$. Let δ be a field of odd base scalars on Y_2. Suppose that

$$\delta = \delta_0 \pi_2^* \hat{\varphi}_2 \tag{1.3.10}$$

with respect to (U_2, φ_2). $\alpha^*\delta(y)$ is a unique odd base scalar at y such that $\hat{R}_y \alpha\alpha^*\delta(y) = \delta(\alpha(y))$. Since for any odd base scalar $\sigma \in \pi_1^* \hat{R} X_1$ at y

$$\hat{R}_y \alpha\sigma = (\operatorname{sgn} \det D\varphi_2 \alpha_0 \varphi_1^{-1}(\varphi_1(x)))\, \sigma_0 \pi_2 \hat{\varphi}_2(\alpha(y)), \tag{1.3.11}$$

where

$$\sigma = \sigma_0 \pi_1^* \hat{\varphi}_1(y), \tag{1.3.12}$$

we have

$$\alpha^*\delta(y) = (\operatorname{sgn} \det D\varphi_1 \alpha_0^{-1} \varphi_2^{-1}(\varphi_2 \alpha_0(x)))\, \delta_0 \pi_1^* \hat{\varphi}_1(\alpha(y)). \tag{1.3.13}$$

Let $p \geqq 1$, and let ϱ be an odd base p-form on Y_2. Let

$$\varrho = \pi_2^* \hat{\varphi}_2 \otimes \varrho_{\varphi_2} \tag{1.3.14}$$

with respect to (U_2, φ_2). Then

$$\alpha^*\varrho = \alpha^* \pi_2^* \hat{\varphi}_2 \otimes \alpha^* \varrho_{\varphi_2}, \tag{1.3.15}$$

where $\alpha^*\varrho_{\varphi_2}$ is the pull-back of (ordinary) p-form.

The mapping $\hat{\Omega}^p(Y_2) \ni \varrho \to \alpha^*\varrho \in \hat{\Omega}^p(Y_1)$ has the following elementary properties. For any $\varrho_1, \varrho_2 \in \hat{\Omega}^p(Y_2)$ and any function $F : Y_2 \to R$,

$$\alpha^*(\varrho_1 + \varrho_2) = \alpha^*\varrho_1 + \alpha^*\varrho_2, \qquad \alpha^*(F\varrho_1) = (F \circ \alpha)\, \alpha^*\varrho_1. \tag{1.3.16}$$

Moreover, if $\beta : Y \to Y_3$ is a homomorphism of fibered manifolds whose projection is a local diffeomorphism, then for any $\varrho \in \hat{\Omega}^p(Y_3)$

$$\alpha^*\beta^*\varrho = (\beta \circ \alpha)^*\varrho. \tag{1.3.17}$$

Let Y be a fibered manifold with base X and projection π, $p \geqq 1$, ξ a vector field on Y. We put for each $y \in Y$ and $\xi_1, \ldots, \xi_{p-1} \in T_y Y$

$$(i_\xi \varrho)(y)(\xi_1, \ldots, \xi_{p-1}) = \varrho(y)(\xi(y), \xi_1, \ldots, \xi_{p-1}). \tag{1.3.18}$$

$i_\xi \varrho$ is an odd base $(p-1)$-form on Y, called the *inner product* of ϱ and ξ.

Chart expressions. If ϱ is expressed by

$$\varrho = \pi^* \hat{\varphi} \otimes \varrho_\varphi \tag{1.3.19}$$

with respect to a chart (U, φ) on X, then $i_\xi \varrho$ is expressed by

$$i_\xi \varrho = \pi^* \hat{\varphi} \otimes i_\xi \varrho_\varphi, \tag{1.3.20}$$

where $i_\xi \varrho_\varphi$ is the inner product of the (ordinary) p-form ϱ_φ and ξ.

For any $\varrho \in \hat{\Omega}^p(Y)$, any two vector fields ξ_1, ξ_2 and two functions f_1, f_2 on Y,

$$\begin{aligned} i_{f_1\xi_1 + f_2\xi_2}\varrho &= f_1 \cdot i_{\xi_1}\varrho + f_2 \cdot i_{\xi_2}\varrho, \\ i_{\xi_1} i_{\xi_2}\varrho &= -i_{\xi_2} i_{\xi_1}\varrho. \end{aligned} \tag{1.3.21}$$

If Y_1 (resp. Y_2) is a fibered manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2) and $\alpha : Y_1 \to Y_2$ is a homomorphism of fibered manifolds whose projection is a local diffeomorphism, then for any $\varrho \in \hat{\Omega}^p(Y_2)$ and any π-related vector fields ξ, ζ

$$\alpha^* i_\xi \varrho = i_\zeta \alpha^* \varrho. \tag{1.3.22}$$

Let $\varrho \in \hat{\Omega}^p(Y)$ be an odd base form. There exists a unique odd base form $d\varrho \in \Omega^{p+1}(Y)$ such that for each chart (U, φ) on X

$$d\varrho = \pi^* \hat{\varphi} \otimes d\varrho_\varphi, \tag{1.3.23}$$

where ϱ_φ is defined by the chart expression (1.3.19), and $d\varrho_\varphi$ is the exterior derivative of the (ordinary) p-form ϱ_φ. $d\varrho$ is called the *exterior derivative* of the odd base p-form ϱ.

The mapping $\varrho \to d\varrho$ is R-linear and by definition, for each ϱ (1.3.23), $d(d\varrho) = 0$. If $\alpha : Y_1 \to Y_2$ is a homomorphism of fibered manifolds whose projection is a local diffeomorphism, then for any $\varrho \in \hat{\Omega}^p(Y_2)$,

$$\alpha^* d\varrho = d\alpha^* \varrho. \tag{1.3.24}$$

Let Y be a fibered manifold. An odd base form $\varrho \in \hat{\Omega}^p(Y)$ is called *closed,* if $d\varrho = 0$. ϱ is called *exact* if there exists an odd base form $\eta \in \hat{\Omega}^{p-1}(Y)$ such that $\varrho = d\eta$. Each exact odd base form is closed; as in the case of ordinary forms, the converse is also valid locally (the *Poincaré lemma*).

Theorem 1.3. *Let $p \geqq 1$ be an integer, $\varrho \in \hat{\Omega}^p(Y)$ a closed odd base form. Then each point $y \in Y$ has a neighbourhood V such that there exists an odd base form $\eta \in \hat{\Omega}^{p-1}(V)$ for which $\varrho = d\eta$.*

Proof. This follows from the Poincaré lemma for (ordinary) forms.

Let Y be a fibered manifold with base X and projection π. A vector field Ξ on Y is called *π-projectable,* if there exists a vector field ξ on X such that

$$T\pi\Xi = \xi \circ \pi. \tag{1.3.25}$$

If ξ exists, it is unique, and is called the *π-projection* of Ξ. Ξ is called *π-vertical,* if it is π-projectable and its π-projection is the zero vector field.

Let Ξ be a π-projectable vector field on Y, ξ its π-projection, α_t^{Ξ} (resp. α_t^{ξ}) the local one-parameter group of Ξ (resp. ξ). Then for any $t \in R$,

$$\pi \circ \alpha_t^{\Xi} = \alpha_t^{\xi} \circ \pi, \tag{1.3.26}$$

on the domain of definition of α_t^{Ξ}; α_t^{Ξ} is therefore a homomorphism of fibered manifolds.

Let $\varrho \in \hat{\Omega}^p(Y)$, let $y \in Y$ be a point. There exists a neighbourhood V of y and $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$, α_t^{Ξ} is defined on V. Thus $\alpha_t^{\Xi *}\varrho$ is defined, and is an odd base p-form on V. The curve $t \to (\alpha_t^{\Xi *}\varrho(y))$ lies in the fiber over y in $\pi^* \mathring{R}X \otimes \wedge^p T^*X$; hence the derivative of this curve at a point belongs to the same fiber. We set

$$\partial_{\Xi}\varrho(y) = \left\{\frac{d}{dt}\alpha_t^{\Xi *}\varrho(y)\right\}_0, \tag{1.3.27}$$

(the derivative considered at $t = 0$). The mapping $y \to \partial_{\Xi}\varrho(y)$ is an odd base p-form on Y, called the *Lie derivative* of the odd base p-form ϱ with respect to the π-projectable vector field Ξ.

Remark 1.4. The Lie derivative of an odd base p-form with respect to a vector field which is *not* π-projectable, is not defined.

Chart expressions. Let (U, φ) be a chart on X such that α_t^{ξ} is defined on U for all sufficiently small t. Then $\operatorname{sgn}\det D\varphi\alpha_t\varphi^{-1} = 1$ and by (1.1.3) and (1.3.15), $\alpha_t^{\Xi *}\varrho = \pi^*\hat{\varphi} \otimes \alpha_t^{\Xi *}\varrho_{\varphi}$. This shows that

$$\partial_{\Xi}\varrho = \pi^*\hat{\varphi} \otimes \partial_{\Xi}\varrho_{\varphi}, \tag{1.3.28}$$

where $\partial_{\Xi}\varrho_{\varphi}$ is the Lie derivative of (ordinary) p-form ϱ_{φ} with respect to Ξ.

Let $\varrho \in \hat{\Omega}^p(Y)$, $a, b \in R$, and let Ξ and Θ be two π-projectable vector fields on Y. Then the following formulas easily follow from the analogous ones for (ordinary) forms:

$$\partial_{\Xi}\varrho = i_{\Xi}\, \mathrm{d}\varrho + \mathrm{d}i_{\Xi}\varrho, \tag{1.3.29}$$

$$\partial_{\Xi}\, \mathrm{d}\varrho = \mathrm{d}\, \partial_{\Xi}\varrho, \tag{1.3.30}$$

$$\partial_{a\Xi + b\Theta}\varrho = a\, \partial_{\Xi}\varrho + b\, \partial_{\Theta}\varrho. \tag{1.3.31}$$

Obviously, the mapping $\varrho \to \partial_{\Xi}\varrho$ is R-linear.

Let $\omega \in \hat{\Omega}^p(Y)$, $\varrho \in \hat{\Omega}^q(Y)$. For each $y \in Y$ and $\xi_1, \dots, \xi_{p+q} \in T_y Y$ we put

$$\omega \wedge \varrho(y)(\xi_1, \dots, \xi_{p+q}) = \tag{1.3.32}$$
$$= \Sigma \frac{1}{p!\,q!} \operatorname{sgn}\sigma.\ \omega(y)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\, \varrho(y)(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}),$$

$$\varrho \wedge \omega(y)(\xi_1, \dots, \xi_{p+q}) = \tag{1.3.33}$$
$$= \Sigma \frac{1}{p!\,q!} \operatorname{sgn}\sigma.\ \omega(y)(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})\, \varrho(y)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(q)})$$

(summation over all permutations σ of the set $\{1, 2, \ldots, p+q\}$). $\omega \wedge \varrho$ (resp. $\varrho \wedge \omega$) is an odd base $(p+q)$-form on Y, called the *exterior product* of the p-form ω and odd base q-form ϱ (resp. odd base q-form ϱ and p-form ω).

Chart expressions. If (U, φ) is a chart on X and ϱ is expressed by (1.3.19), then

$$\omega \wedge \varrho = \pi^*\hat{\varphi} \otimes (\omega \wedge \varrho_\varphi), \qquad \varrho \wedge \omega = \pi^*\hat{\varphi} \otimes (\varrho_\varphi \wedge \omega) \tag{1.3.34}$$

with respect to (U, φ).

The mapping $(\omega, \varrho) \to \omega \wedge \varrho$ is bilinear over the ring of functions. Moreover,

$$\omega \wedge \varrho = (-1)^{pq}\, \varrho \wedge \omega, \tag{1.3.35}$$

$$(\eta \wedge \omega) \wedge \varrho = \eta \wedge (\omega \wedge \varrho), \tag{1.3.36}$$

where $\eta \in \Omega^r(Y)$ is any element. If Ξ is a π-projectable vector field on Y, we have

$$i_\Xi(\omega \wedge \varrho) = i_\Xi\omega \wedge \varrho + (-1)^p\, \omega \wedge i_\Xi\varrho, \tag{1.3.37}$$

$$\partial_\Xi(\omega \wedge \varrho) = \partial_\Xi\omega \wedge \varrho + \omega \wedge \partial_\Xi\varrho. \tag{1.3.38}$$

Finally,

$$\alpha^*(\omega \wedge \varrho) = \alpha^*\omega \wedge \alpha^*\varrho \tag{1.3.39}$$

for any homomorphism of fibered manifolds $\alpha : Y' \to Y$ whose projection is a local diffeomorphism.

1.4. Integration of odd forms. In this section we develop the integration theory of continuous odd n-forms on compact n-dimensional manifolds with boundary; within this theory, the integration domains need not be orientable.

Let X be a compact n-dimensional manifold with boundary ∂X, ϱ a continuous odd n-form on X. Suppose that there exists a chart (U, φ), $\varphi = (x^i)$, on X such that the support of ϱ satisfies $\operatorname{supp} \varrho \subset U$. Let ϱ be expressed by

$$\varrho = f \cdot \hat{\varphi} \otimes dx^1 \wedge \ldots \wedge dx^n \tag{1.4.1}$$

with respect to (U, φ). We define the *integral* of ϱ on X by

$$\int_X \varrho = \int f\varphi^{-1}, \tag{1.4.2}$$

where the integral on the right is the standard Lebesgue integral on R^n. Using the change of variables rule and the transformation formula for the components of an odd n-form one can easily verify that the number (1.4.2) is independent of the choice of (U, φ). Let now ϱ be an arbitrary continuous odd n-form on X, (U_i, φ_i), $i = 1, 2, \ldots, N$, a finite system of charts such that $X = \cup U_i$, and (χ_i) a partition of unity, subordinate to the covering (U_i) of X. We define the *integral* of ϱ on X by

$$\int_X \varrho = \sum_i \int_X \chi_i\varrho, \tag{1.4.3}$$

where each of the summands on the right is given by (1.4.2).

Theorem 1.4. *Let $\alpha : X \to Y$ be a diffeomorphism of compact n-dimensional manifolds with boundary, ϱ a continuous odd n-form on Y. Then*

$$\int_Y \varrho = \int_X \alpha^*\varrho. \tag{1.4.4}$$

Proof. 1. Suppose first that $\operatorname{supp} \varrho \subset V$, where (V, ψ), $\psi = (y^j)$, is a chart on Y. Then (U, φ), $\varphi = (x^j)$, where $\varphi = \psi\alpha$ and $U = \psi^{-1}(V)$, is a chart on X. If ϱ has an expression $\varrho = f \,.\, \hat{\psi} \otimes dy^1 \wedge \ldots \wedge dy^n$, then $\alpha^*\varrho = (f \circ \alpha) \,.\, \hat{\varphi} \otimes$ $\otimes\, dx^1 \wedge \ldots dx^n$, and (1.4.2) gives (1.4.4).

2. Let (V_i, ψ_i) be a finite system of charts on Y such that $\cup V_i = Y$, and let (χ_i) be a partition of unity, subordinate to the covering (V_i) of Y. Then (U_i, φ_i), where $U_i = \alpha^{-1}(V_i)$, $\varphi_i = \psi_i\alpha$, is a system of charts on X such that (U_i) is a covering of X, and $(\chi_i\alpha)$ is a partition of unity subordinate to this covering. Since $(\chi_i\alpha) \,.\, \alpha^*\varrho = \alpha^*(\chi_i\varrho)$, we get from the definition

$$\int_X \alpha^*\varrho = \sum_i \int_X \alpha^*(\chi_i\varrho), \tag{1.4.5}$$

and apply the first part of the proof to each summand on the right.

Let $I \subset R$ be an open interval. A one-parameter system (ϱ_t), $t \in I$, of odd n-forms, defined on an n-dimensional manifold with boundary X, is called *differentiable*, if there exists a volume element ω on X (see Sec. 1.1) such that the function $(t, x) \to f(t, x)$, defined by the formula

$$\varrho_t(x) = f(t, x) \,.\, \omega(x), \tag{1.4.6}$$

is differentiable. If (ϱ_t) is differentiable, we set

$$\frac{d}{dt}\varrho_t = \frac{\partial f}{\partial t} \,.\, \omega. \tag{1.4.7}$$

$(d\varrho_t/dt)$ is a one-parameter system of odd n-forms on X, called the *derivative* of (ϱ_t) (with respect to the parameter).

Theorem 1.5. *Let (ϱ_t) be a differentiable system of odd n-forms on a compact n-dimensional manifold X. Then the function $t \to \int_X \varrho_t$ is differentiable, and*

$$\frac{d}{dt}\int_X \varrho_t = \int_X \frac{d}{dt}\varrho_t. \tag{1.4.8}$$

Proof. Let us apply the definition (1.4.3) to any element ϱ_t of the system (ϱ_t) We get

$$\int_X \varrho_t = \sum_i \int_X \chi_i\varrho_t. \tag{1.4.9}$$

Write $\varrho_t = f_t \,.\, F_i \,.\, \hat{\varphi}_i \otimes dx_i^1 \wedge \ldots \wedge dx_i^n$ with respect to (U_i, φ_i), $\varphi_i = (x_i^k)$, where $\omega = F_i \,.\, \hat{\varphi}_i \otimes dx_i^1 \wedge \ldots \wedge dx_i^n$ is some volume element. Then

(1.4.10) $$\int_X \chi_i \varrho_t = \int \chi_i \varphi_i^{-1} f_t \varphi_i^{-1} F_i \varphi_i^{-1}.$$

Since the mapping $(t, x') \to \chi_i\varphi_i^{-1}(x') \,.\, f_t\varphi_i^{-1}(x') \,.\, F_i\varphi_i^{-1}(x')$ is differentiable, the function $t \to \int_X \chi_i \varrho_t$ is also differentiable, and by the classical Leibniz rule

(1.4.11) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_X \chi_i\varrho_t = \int \chi_i\varphi_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t} f_t\varphi_i^{-1}F_i\varphi_i^{-1} = \int_X \chi_i \frac{d}{dt}\varrho_t.$$

By (1.4.9), $t \to \int_X \varrho_t$ is differentiable, and we get (1.4.8).

Let ϱ be an odd $(n-1)$-form on X, $x_0 \in \partial X$ a point, and (U, φ), $\varphi = (x^i)$, a chart at x_0. That is, the set $\varphi(U)$ is open in $R^n_{(-)} = \{y \in R^n \mid y^1(y) \leqq 0\}$, where $y^1, \ldots, y^n$ are the canonical coordinates on R^n, and the set $\varphi(\partial X \cap U)$ is given by the equation $x^1(x) = 0$. Denote for each i

(1.4.12) $$\omega_i = (-1)^{i-1}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \ldots \wedge \mathrm{d}x^n.$$

Then

(1.4.13) $$\varrho = \hat{\varphi} \otimes \varrho_\varphi, \qquad \varrho_\varphi = \Sigma f^p \omega_p$$

with respect to (U, φ). Denote by $(U_{\partial X}, \varphi_{\partial X})$ the chart on ∂X induced by (U, φ). We define

(1.4.14) $$\varrho\,|_{\partial X} = \hat{\varphi}\,|_{\partial X} \otimes \varrho_\varphi\,|_{\partial X} = f^1\,|_{\partial X} \cdot \hat{\varphi}_{\partial X} \otimes \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n,$$

where $\varrho_\varphi\,|_{\partial X}$ means the restriction of the (ordinary) $(n-1)$-form ϱ_φ to $\partial X \cap \cap\, U$. $\varrho\,|_{\partial X}$ is an odd $(n-1)$-form on $\partial X \cap U$. It is easily seen that there exists a unique odd $(n-1)$-form $\varrho\,|_{\partial X}$ on ∂X whose restriction to $\partial X \cap U$ is given by (1.4.14), for any (U, φ). Let (V, ψ), $\psi = (y^j)$, be another chart at x_0, and write $\varrho = \hat{\psi} \otimes \Sigma g^q \eta_q$, where $\eta_q = (-1)^{q-1} \,.\, \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^{q-1} \wedge \mathrm{d}y^{q+1} \wedge \ldots \wedge \mathrm{d}y^n$.Then $f^i = |\det D\psi\varphi^{-1}| \,.\, (\partial x^i/\partial y^j) \,.\, g^j$. Since by definition, $x^1 = 0 = x^1(0, y^2, \ldots, y^n)$ on $\partial X \cap U \cap V$ and the function $y^1 \to x^1(y^1, y_0^2, \ldots, y_0^n)$, where $(y_0^1, \ldots, y_0^n) = \psi(x_0)$, is increasing, we have $|\partial y^1/\partial x^1| \,.\, (\partial x^1/\partial y^1) = sgn(\partial y^1/\partial x^1) = 1$, and

(1.4.5) $$f^1\,|_{\partial X} = |\det D\psi_{\partial X}\varphi_{\partial X}^{-1}| \cdot g^1\,|_{\partial X}$$

on $\partial X \cap U \cap V$. This formula assures us the existence of $\varrho\,|_{\partial X}$. We call $\varrho\,|_{\partial X}$ the *restriction* of ϱ to the boundary ∂X of X, and denote it simply by ϱ.

Remark 1.5. Analogous construction of the restriction can be given for any *orientable* $(n-1)$-dimensional submanifold of X and (ordinary) forms. This construction fails, however, for non-orientable submanifolds.

The following is the *Stokes' theorem* on integration of exact odd forms on compact manifolds with boundary.

Theorem 1.6. *Let X be a compact n-dimensional manifold with boundary, and ϱ a differentiable odd $(n-1)$-form on X. Then*

(1.4.16)
$$\int_X d\varrho = \int_{\partial X} \varrho.$$

Proof. Let (U_i, φ_i), $\varphi_i = (x_i^k)$, be a finite system of charts on X such that $X = \cup U_i$, (χ_i) a partition of unity subordinate to the covering (U_i) of X. It is sufficient to show that for each i,

(1.4.17)
$$\int_X d(\chi_i\varrho) = \int_{\partial X} \chi_i\varrho.$$

We distinguish two cases.

(a) $U_i \cap X = \emptyset$. Then $\int_{\partial X} \chi_i\varrho = 0$. Writing $\chi_i\varrho$ in the form

(1.4.18)
$$\chi_i\varrho = \hat{\varphi}_i \otimes \Sigma f_i^p \omega_{i,p},$$

we get $d(\chi_i\varrho) = \hat{\varphi}_i \otimes \Sigma(\partial f_i^p/\partial x_i^p) \,.\, dx_i^1 \wedge \ldots \wedge dx_i^n$. Hence by the Fubini theorem,

(1.4.19)
$$\int_X d(\chi_i\varrho) = \Sigma \int \frac{\partial f_i^p}{\partial x_i^p} = 0,$$

since each of the functions f_i^p has a compact support.

(b) $U_i \cap X \neq \emptyset$. We get as above

(1.4.20)
$$\int_X d(\chi_i\varrho) = \Sigma \int \frac{\partial f_i^p}{\partial x_i^p} = \int \frac{\partial f_i^1}{\partial x_i^1},$$

since each of the functions f_i^p, $p \neq 1$, has a compact support, and we integrate over $(-\infty, \infty)$. We get for the remaining integral in (1.4.20)

(1.4.21)
$$\int \frac{\partial f_i^1}{\partial x_i^1} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_i^1}{\partial x_i^1} =$$
$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_i^1(0, x_i^2, \ldots, x_i^n)\, dx_i^2 \ldots dx_i^n = \int_{\partial X} \chi_i\varrho$$

as required.

Remark 1.6. Let X be a compact *orientable* n-dimensional manifold with boundary, δ a field of unit odd scalars on X. Let ω be a continuous (ordinary) n-form on X. We define the *integral* of ω on X by

(1.4.22)
$$\int_X \omega = \int_X \delta \otimes \omega.$$

This integral obviously depends on the orientation of X. By means of (1.4.22), Theorems 1.4–1.6 are easily reformulated for this case.

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