

Werk

Label: Article

Jahr: 1986

PURL: https://resolver.sub.uni-goettingen.de/purl?311067255_0022|log12

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COMPARISON THEOREMS FOR STURM–LIOUVILLE EQUATIONS

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(Received September 19, 1984)

Abstract. Concerning the differential equations $-(P(x)u')' + Q(x)u = 0$ and $-(p(x)u')' + q(x)u = 0$, $a \leq x \leq b$, Sturm-type comparison theorems are proved where the conditions on the coefficients in question are, for instance, $p \leq P$ and mean value conditions for q and Q on certain subintervals of $[a, b]$. The results are closely related to well-known theorems of Levin and Fink.

Key words. Sturm-Liouville equation, comparison of solutions.
MS Classification. 34 C 10.

Consider the differential equations

$$(1) \quad L[u] \equiv -(P(x)u')' + Q(x)u = 0, \quad P > 0, P \in C^1, Q \in C, \quad \dots \\ -\infty < a \leq x \leq b < \infty,$$

and

$$(2) \quad l[u] \equiv -(p(x)u')' + q(x)u = 0, \quad p > 0, p \in C^1, q \in C.$$

In the special case $P \equiv p \equiv 1$ a well-known comparison theorem of Levin [2] states the following (see [5]).

Theorem 1 (Levin): *Let $P \equiv p \equiv 1$ be fulfilled and suppose that there exists a nontrivial solution u of (1) with $u(a) = u(b) = u'(c) = 0$, $a < c < b$. If the inequality*

$$(3) \quad \int_{x_1}^{x_2} q(x) dx \leq - \left| \int_{x_1}^{x_2} Q(x) dx \right|$$

holds for all pairs of numbers x_1, x_2 with $a \leq x_1 \leq c \leq x_2 \leq b$, then every solution of (2) has at least one zero on $[a, b]$.

Condition (3) implies that all mean values of $q(x)$ on intervals $[x_1, x_2]$, $x_1 \leq c \leq x_2$, are non-positive. In the following we shall prove a corresponding comparison theorem where mean values of $q(x)$ can also be positive. We give the following preparation.

Let $u(x)$ be a nontrivial solution of the boundary problem

$$L[u] = 0, \quad u(a) = 0 = u(b),$$

with fixed sign on (a, b) ; assume that u is positive on (a, b) . Choose a positive function f belonging to $C^2[a, b]$. Then because of

$$\lim_{x \downarrow a} \frac{u'}{u} = \infty, \quad \lim_{x \uparrow b} \frac{u'}{u} = -\infty,$$

it is easily seen that there exist points t_1, t_2 with $a < t_1 \leq t_2 < b$ such that

$$(4) \quad \frac{f'(t_i)}{f(t_i)} = \frac{u'(t_i)}{u(t_i)}, \quad i = 1, 2, \quad \frac{f'(x)}{f(x)} \leq \frac{u'(x)}{u(x)}, \quad a < x \leq t_1, \\ \frac{f'(x)}{f(x)} \geq \frac{u'(x)}{u(x)}, \quad t_2 \leq x < b.$$

Note that there can exist several points t_1 or t_2 with the properties (4), respectively.

Set

$$(5) \quad c_i = f(t_i) u^{-1}(t_i), \quad i = 1, 2,$$

and define the function

$$(6) \quad v(x) = \begin{cases} c_1 u(x), & a \leq x < t_1, \\ f(x), & t_1 \leq x \leq t_2, \\ c_2 u(x), & t_2 < x \leq b. \end{cases}$$

It follows from (4) and (5) that $v(x)$ is a continuously differentiable function on $[a, b]$. Setting

$$v(x) = \mu(x) f(x), \quad a \leq x \leq b,$$

we have

$$\mu' \mu^{-1} = v' v^{-1} - f' f^{-1}$$

and (4) implies that

$$(7) \quad \mu \in C^1[a, b]; \quad \mu'(x) \geq 0, \quad a \leq x \leq t_1; \\ \mu(x) = 1, \quad t_1 \leq x \leq t_2; \quad \mu'(x) \leq 0, \quad t_2 \leq x \leq b.$$

v will be used as a test function to estimate the quadratic form of equation (2).

Supposing

$$(8) \quad p(x) \leq P(x), \quad a \leq x \leq b,$$

we have

$$\int_a^b [p(v')^2 + qv^2] dx = \int_a^b [(p - P)(v')^2 + (q - Q)v^2] dx + \int_a^b [P(v')^2 + Qv^2] dx \leq \\ (9) \quad \leq \int_a^b (q - Q)v^2 dx + \int_a^b [P(v')^2 + Qv^2] dx.$$

(7) shows that the function $\mu^2(x)$ is monotone increasing on $[a, t_1]$ from $\mu^2(a) = 0$ to $\mu^2(t_1) = 1$ and monotone decreasing on $[t_2, b]$ from $\mu^2(t_2) = 1$ to $\mu^2(b) = 0$.

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Therefore, by a mean value theorem of integral calculus, there exist points τ_1 , $a \leq \tau_1 \leq t_1$, and τ_2 , $t_2 \leq \tau_2 \leq b$, such that

$$(10) \quad \int_a^b (q - Q)v^2 dx = \int_{\tau_1}^{\tau_2} (q - Q)f^2 dx, \quad a \leq \tau_1 \leq t_1, \quad t_2 \leq \tau_2 \leq b.$$

The second integral on the right-hand side of (9) is handled by integration by parts as follows.

$$(11) \quad \begin{aligned} \int_a^b [P(v')^2 + Qv^2] dx &= c_1^2 \int_a^{t_1} [P(u')^2 + Qu^2] dx + \int_{t_1}^{t_2} [P(f')^2 + Qf^2] dx + \\ &+ c_2^2 \int_{t_2}^b [P(u')^2 + Qu^2] dx = c_1^2 P(t_1) u'(t_1) u(t_1) + P(t_2) f'(t_2) f(t_2) - \\ &- P(t_1) f'(t_1) f(t_1) + \int_{t_1}^{t_2} L[f] f dx - c_2^2 P(t_2) u'(t_2) u(t_2) = \int_{t_1}^{t_2} L[f] f dx. \end{aligned}$$

Thus, we obtain

$$(12) \quad \int_a^b [p(v')^2 + qv^2] dx \leq \int_{\tau_1}^{\tau_2} (q - Q) f^2 dx + \int_{t_1}^{t_2} L[f] f dx, \\ a \leq \tau_1 \leq t_1 \leq t_2 \leq \tau_2 \leq b,$$

where the numbers t_1 and t_2 are defined by (4).

Theorem 2: Let u be a nontrivial solution of equation (1) with fixed sign on (a, b) and $u(a) = 0 = u(b)$ and let f be a positive function belonging to $C^2[a, b]$. If (8) is fulfilled and the inequality

$$(13) \quad \int_{x_1}^{x_2} (q - Q) f^2 dx + \int_{t_1}^{t_2} L[f] f dx \leq 0$$

holds for all pairs of numbers x_1, x_2 with $a \leq x_1 \leq t_1$ and $t_2 \leq x_2 \leq b$ where t_1 and t_2 are defined by

$$(14) \quad \frac{f'(t_i)}{f(t_i)} = \frac{u'(t_i)}{u(t_i)}, \quad i = 1, 2, t_1 \leq t_2, \\ \frac{f'(x)}{f(x)} \leq \frac{u'(x)}{u(x)}, \quad a < x \leq t_1, \quad \frac{f'(x)}{f(x)} \geq \frac{u'(x)}{u(x)}, \quad t_2 \leq x < b,$$

then every solution v of equation (2) has a zero in (a, b) or v has the properties

- i) v is a constant multiple of u on $[a, t_1]$,
- ii) v is a constant multiple of f on $[t_1, t_2]$,
- iii) v is a constant multiple of u on $[t_2, b]$.

Proof: In view of (13) it follows from (12) that

$$(15) \quad \int_a^b [p(v')^2 + qv^2] dx \leq 0,$$

where v is the test function (6). v belongs to the domain of the closure of the form

$$l(\varphi, \psi) = \int_a^b (p\varphi'\psi' + q\varphi\psi) dx, \quad \varphi, \psi \in C_0^\infty(a, b),$$

of equation (2). Because of (15) two cases are possible,

$$\inf_{\varphi \in C_0^\infty, \|\varphi\|=1} l(\varphi, \varphi) < 0 \quad \text{or} \quad \inf_{\varphi \in C_0^\infty, \|\varphi\|=1} l(\varphi, \varphi) = 0,$$

where $\|\varphi\|$ denotes the norm of φ in the Hilbert space $L_2(a, b)$. In the first case equation (2) has a nontrivial solution with at least two zeros in (a, b) (cp. [3]). Then by Sturm's comparison theorem every solution of (2) has a zero in (a, b) . In the second case the infimum of the form is realized by the (normalized) function v . Consequently, this function v is an eigenfunction of the Friedrichs extension A of the operator A_0 ,

$$A_0\varphi = l[\varphi], \quad \varphi \in C_0^\infty(a, b),$$

in the Hilbert space $L_2(a, b)$. The corresponding eigenvalue is zero. Now it is easily seen that v belongs to $C^2[a, b]$. v is a classical solution of (2). This proves Theorem 2.

Corollary 1: Let u be a nontrivial solution of (1) with fixed sign on (a, b) and $u(a) = 0 = u(b)$ and let f be a positive function belonging to $C^2[a, b]$. Assume that there exists a point c , $a < c < b$, such that

$$\frac{f'(c)}{f(c)} = \frac{u'(c)}{u(c)}, \quad \frac{f'(x)}{f(x)} \leq \frac{u'(x)}{u(x)}, \quad a < x \leq c, \quad \frac{f'(x)}{f(x)} \geq \frac{u'(x)}{u(x)}, \quad c \leq x < b.$$

If (8) is fulfilled and the inequality

$$\int_{x_1}^{x_2} qf^2 dx \leq \int_{x_1}^{x_2} Qf^2 dx$$

holds for all pairs x_1, x_2 with $a \leq x_1 \leq c \leq x_2 \leq b$, then every solution v of equation (2) has a zero in (a, b) , or v is a constant multiple of u .

Proof: Set $t_1 = t_2 = c$ in Theorem 2.

Corollary 2: Let $P \equiv p \equiv 1$ and assume that u is a nontrivial solution of equation (1) with fixed sign on (a, b) and $u(a) = 0 = u(b)$. If the inequality

$$(16) \quad \int_{[x_1, x_2]} q(x - x_0)^2 dx \leq \int_{[x_1, x_2] \setminus [t_1, t_2]} Q(x - x_0)^2 dx$$

holds for a point $x_0 \notin [a, b]$ and all pairs x_1, x_2 with $a \leq x_1 \leq t_1 \leq t_2 \leq x_2 \leq b$ where t_1 and t_2 are defined by

$$(17) \quad \frac{1}{t_i - x_0} = \frac{u'(t_i)}{u(t_i)}, \quad i = 1, 2,$$

$$\frac{1}{x - x_0} \leq \frac{u'(x)}{u(x)}, \quad a < x \leq t_1, \quad \frac{1}{x - x_0} \geq \frac{u'(x)}{u(x)}, \quad t_2 \leq x < b,$$

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then every solution v of equation (2) has a zero in (a, b) , or v has the following properties:

- i) v is a constant multiple of u on $[a, t_1]$,
- ii) v is a constant multiple of $x - x_0$ on $[t_1, t_2]$,
- iii) v is a constant multiple of u on $[t_2, b]$.

Proof: By choosing $f(x) = x - x_0$ in Theorem 2 it follows that

$$\int_{t_1}^{t_2} L[f] f \, dx = \int_{t_1}^{t_2} Q(x - x_0)^2 \, dx.$$

Thus, (16) implies (13), and Corollary 2 follows from Theorem 2. The geometrical meaning of (17) is that there exist tangents $y_i(x) = \lambda_i(x - x_0)$, $i = 1, 2$, touching the curve of u at t_i , respectively.

The special case $f \equiv 1$ leads to the following corollaries.

Corollary 3: Let u be a nontrivial solution of (1) with fixed sign on (a, b) and $u(a) = 0 = u(b)$ and assume that

$$u'(t_1) = 0 = u'(t_2), \quad a < t_1 \leq t_2 < b; \quad u'(x) \geq 0, \quad a \leq x \leq t_1; \quad u'(x) \leq 0, \\ t_2 \leq x \leq b.$$

If (8) is fulfilled and the inequality

$$(18) \quad \int_{[x_1, x_2]} q \, dx \leq \int_{[x_1, x_2] \setminus [t_1, t_2]} Q \, dx$$

holds for all pairs of numbers x_1, x_2 with $a \leq x_1 \leq t_1 \leq t_2 \leq x_2 \leq b$, then v has a zero in (a, b) or v has the following properties:

- i) v is a constant multiple of u on $[a, t_1]$,
- ii) $v = \text{const}$ on $[t_1, t_2]$,
- iii) v is a constant multiple of u on $[t_2, b]$.

Proof: Set $f \equiv 1$ in Theorem 2.

A special case of Corollary 3 is the case $t_1 = t_2 = c$, $a < c < b$. Then inequality (18) has the form

$$\int_{x_1}^{x_2} q \, dx \leq \int_{x_1}^{x_2} Q \, dx, \quad a \leq x_1 \leq c \leq x_2 \leq b.$$

In this special case Corollary 3 is closely related to a result of Fink [1] concerning the smallest positive eigenvalues λ_1 and λ_2 of the problems

$$(p(x) u')' + \lambda_1 q_1(x) u = 0, \quad u(a) = 0 = u(b),$$

and

$$(p(x) u')' + \lambda_2 q_2(x) u = 0, \quad u(a) = 0 = u(b).$$

Concerning the importance of the quantity of these eigenvalues for oscillation or disconjugacy of the corresponding equations compare [4, p. 53].

In the following the restriction $x_0 \notin [a, b]$ supposed in Corollary 2 is to be omitted. Assume that there exist points $x_0 \in (a, b)$ and t with $x_0 < t < b$ such that

$$\frac{1}{t - x_0} = \frac{u'(t)}{u(t)} \quad \text{and} \quad \frac{1}{x - x_0} \geq \frac{u'(x)}{u(x)}, \quad t \leq x < b,$$

where u is the solution of equation (1) from above. Then, the function

$$(19) \quad v(x) = \begin{cases} 0, & a \leq x < x_0, \\ x - x_0, & x_0 \leq x \leq t, \\ (t - x_0) u^{-1}(t) u(x), & t \leq x \leq b, \end{cases}$$

belongs to the Sobolev space $\dot{W}_2^1(a, b)^1$ which is identical with the domain of the closure of the form of equation (2). By using this function v the estimate (12) gets the form

$$(20) \quad \int_a^b [p(v')^2 + q(x)v^2] dx \leq \int_{x_0}^t (q - Q)(x - x_0)^2 dx + \int_{x_0}^t Q(x - x_0)^2 dx, \\ a < x_0 < t \leq \tau \leq b.$$

Of course, an analogous estimate holds when the point t is situated to the left of x_0 . Finally, the point x_0 can be identical with one of the endpoints of the interval (a, b) . The following corollary corresponds to the case $a < x_0 < t < b$.

Corollary 4: *Let $P \equiv p \equiv 1$ and assume that u is a nontrivial solution of equation (1) with fixed sign on (a, b) and $u(a) = 0 = u(b)$. Let further $x_0, a < x_0 < b$, and $t, x_0 < t < b$, be points with the properties*

$$\frac{1}{t - x_0} = \frac{u'(t)}{u(t)} \quad \text{and} \quad \frac{1}{x - x_0} \geq \frac{u'(x)}{u(x)}, \quad t \leq x < b.$$

If the inequality

$$(21) \quad \int_{x_0}^{\xi} q(x - x_0)^2 dx \leq \int_t^{\xi} Q(x - x_0)^2 dx$$

holds for all points ξ with $t \leq \xi \leq b$, then every solution v of equation (2) has a zero on $[x_0, b)$.

Proof: It follows from (20) and (21) that

$$\int_{x_0}^b [p(v')^2 + qv^2] dx \leq 0,$$

where v is defined by (19). In the case

¹⁾ $\dot{W}_2^1(a, b)$ is the completion of $C_0^\infty(a, b)$ by using the norm

$$\|\varphi\|_1 = \left(\int_a^b (|\varphi'|^2 + |\varphi|^2) dx \right)^{1/2}.$$

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$$\inf_{\varphi \in C_0^\infty(x_0, b), \|\varphi\|=1} \int_{x_0}^b (p|\varphi'|^2 + q|\varphi|^2) dx < 0,$$

there exists a nontrivial solution of (2) on $[x_0, b]$ with at least two zeros in (x_0, b) and, consequently, every solution of (2) has a zero in (x_0, b) (compare the proof of Theorem 2). Assuming the case

$$\inf_{\varphi \in C_0^\infty(x_0, b), \|\varphi\|=1} \int_{x_0}^b (p|\varphi'|^2 + q|\varphi|^2) dx = 0$$

the (normalized) function $v(x)$, $x_0 \leq x \leq b$, of (19) realizes the infimum. Hence v is a nontrivial solution of (2) on $[x_0, b]$ which has the zero x_0 . This proves Corollary 4.

In the case $x_0 = a$ we obtain the following result.

Corollary 5: *Let the suppositions of Corollary 4 be fulfilled for $x_0 = a$. Then every solution v of equation (2) has a zero in (a, b) , or v has the following properties:*

- i) v is a constant multiple of $x - a$ on $[a, t]$,
- ii) v is a constant multiple of u on $[t, b]$.

$t = a$: If

$$(22) \quad \frac{1}{x-a} \geq \frac{u'(x)}{u(x)}, \quad a < x < b,$$

and the inequality

$$(23) \quad \int_a^\xi q(x-a)^2 dx \leq \int_a^\xi Q(x-a)^2 dx$$

holds for all ξ , $a < \xi < b$, then every solution v of (2) has a zero in (a, b) or v is a constant multiple of u .

The proof of Corollary 5 is analogous to the proof of Corollary 4.

Example: Every solution of the equation

$$(24) \quad -u'' + q(x)u = 0, \quad q \neq -1, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2},$$

has a zero in $(-\frac{\pi}{2}, \frac{\pi}{2})$ if there exists a point c , $-\frac{\pi}{2} \leq c \leq \frac{\pi}{2}$, $c \neq 0$, such that

$$(25) \quad \max_{-\frac{\pi}{2} \leq x_1 \leq c \leq x_2 \leq \frac{\pi}{2}} \int_{x_1}^{x_2} (q+1)(x-c-\cot c)^2 dx \leq 0$$

or if

$$(26) \quad \sup_{-\frac{\pi}{2} \leq x_1 < 0 < x_2 \leq \frac{\pi}{2}} \left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} q dx \right) \leq -1.$$

Proof: Compare equation (24) with the equation

$$-u'' - u = 0, \quad u\left(-\frac{\pi}{2}\right) = 0 = u\left(\frac{\pi}{2}\right),$$

and take $u = \cos x$. In the case where $|c| < \frac{\pi}{2}$, $c \neq 0$, apply Corollary 2. Condition (17) is fulfilled for $t_1 = t_2 = c$ and $x_0 = c + \cot c$. Then (25) corresponds to (16) with $Q \equiv -1$. In the case where $c = -\frac{\pi}{2}$ apply Corollary 5 under the supposition $t = a$. In this case the condition (25) has the form

$$\max_{-\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\xi} (q+1) \left(x + \frac{\pi}{2}\right)^2 dx \leq 0.$$

An analogous condition is valid in the case $c = \frac{\pi}{2}$. Inequality (26) corresponds to (18) of Corollary 3.

Corollary 6: Let $P \equiv p \equiv 1$ and consider the solution u of equation (1) determined by the initial values $u(c) = \alpha > 0$, $u'(c) = \beta > 0$, $a < c < b$. If the inequalities

$$(27) \quad -\frac{\pi^2}{(b-a)^2} \leq \frac{\int_{x_1}^{x_2} Q(x-c+\alpha\beta^{-1})^2 dx}{\int_{x_1}^{x_2} (x-c+\alpha\beta^{-1})^2 dx} \leq 0$$

hold for all numbers x_1, x_2 with

$$\max(a, c - \alpha\beta^{-1}) \leq x_1 < c < x_2 \leq b,$$

then the solution u does not vanish in at least one of the intervals (a, c) or (c, b) . In the case where $u(c) = \alpha > 0$, $u'(c) = 0$, the same conclusion is true when

$$(28) \quad -\frac{\pi^2}{(b-a)^2} \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} Q dx \leq 0$$

for all x_1, x_2 with $a \leq x_1 < c < x_2 \leq b$.

Proof: Assume that u has a zero a' in (a, c) and a zero b' in (c, b) . We may assume that u is positive on (a', b') . Now apply the Corollaries 2–5. First let $u'(c) > 0$. It follows from

$$\frac{1}{c - x_0} = \frac{u'(c)}{u(c)} = \frac{\beta}{\alpha}$$

that $x_0 = c - \alpha\beta^{-1}$. Thus, replacing a by a' and b by b' , Corollary 2 can be applied when $c - \alpha\beta^{-1} < a'$. The points t_1 and t_2 can be determined such that (17) is fulfilled with $a = a'$ and $b = b'$. Now it follows from (27) that (16) is fulfilled by setting

$$q(x) = -\frac{\pi^2}{(b-a)^2}.$$

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The solution $v = \sin\left(\pi \frac{x-a}{b-a}\right)$ of equation (2), however, does not vanish on $[a', b']$ contradictory to the conclusion of Corollary 2. Assume now $a' < c - \alpha\beta^{-1}$ and apply Corollary 4 with $a = a'$ and $b = b'$. The point t , $c \leq t < b'$, can be determined and (27) implies (21) with

$$q(x) = -\frac{\pi^2}{(b-a)^2}.$$

Thus, considering the solution $v = \sin\left(\pi \frac{x-a}{b-a}\right)$ of equation (2) we again obtain a contradiction. Finally, in the case $a' = c - \alpha\beta^{-1}$ apply Corollary 5 with $a = a'$ and $b = b'$. Analogously, the assertion of Corollary 6 under the supposition $\beta = 0$ follows from Corollary 3. This completes the proof of Corollary 6.

The case $u(c) = \alpha > 0$, $u'(c) = \beta < 0$ can be handled analogously.

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