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TRANSFORMATIONS OF LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The transformation $z(t) = \alpha(t)\, y + \beta(t)\, y'$ of solutions y of a linear second order ordinary differential in general form is considered and a differential equation for $z(t)$ is derived. Previous results concerning such problem are discussed.

Key words. Transformations of ordinary differential equations, Bôcher function, linear combination of solution and its derivative.

Several papers investigating certain special types of transformations between linear differential equations of the second order have appeared recently. There were studied e.g. transformations of the equation

$$y'' + q(t)\, y = 0 \tag{1}$$

onto the equation

$$z'' + Q(t)\, y = 0 \tag{2}$$

by means of the formula $z(t) = \alpha(t)\, y + \beta(t)\, y'$, y being a solution of (1).

In the present paper the transformation of the above mentioned type of the equation

$$y'' + a(t)\, y' + b(t)\, y = 0 \tag{3}$$

onto

$$z'' + A(t)\, z' + B(t)\, z = 0 \tag{4}$$

is considered. Note that the independent variable remains the same, but the derivative y' is included in the transformation. Throughout the paper j denotes an interval (c, d), where $-\infty \leqq c < d \leqq \infty$.

Definition 1. Let $A, B \in C_0(j)$, $a, b \in C^1(j)$, $\alpha, \beta \in C^2(j)$. We say that the formula

$$z(t) = \alpha(t)\, y + \beta(t)\, y' \tag{5}$$

maps the set of solutions of the equation (3) onto the set of solutions of the equation (4) (or shortly transforms (3) onto (4)) if for any solution $y = y(t)$ of (3) there is a unique solution $z = z(t)$ of (4) such that

$$t(t) = \alpha(t)\, y(t) + \beta(t)\, y'(t) \qquad \text{for } t \in j \tag{6}$$

and reversely for any solution $z = z(t)$ of (4) there is a unique solution $y = y(t)$ of (3) complying with (6).

Remark 1. The transformation (5) between (1) and (2) from a little different points of view is investigated and its various priperties are given in [5], [6] and [9]. However, the functions $\alpha(t)$, $\beta(t)$ cannot be chosen arbitrarily in such a case because of vanishing of the term involving the first derivative in (2).

Some applications to the Bessel equation are given in [3], [4], [8], particulary the function $\mu J_\nu(t) + tJ_\nu'(t)$ is investigated in [8]. The case $\alpha = 0$ is used in order to investigate the distribution of zeros of derivatives of solutions of (1) and (3) e.g. in [10], [11]. The case $\alpha(t) = k\beta(t)$ is used in [7] to solve certain boundary value problem.

Lemma 1. *If $a, b \in C^1(j)$, $\alpha, \beta \in C^2(j)$ and the formula $z(t) = \alpha(t)\, y + \beta(t)\, y'$ transforms* (3) *onto* (4), *then*

$$\alpha^2 + \alpha\beta' - \alpha\beta a - \alpha'\beta + \beta^2 b \neq 0 \qquad on \quad j. \tag{7}$$

Proof. Let y_1, y_2 denote a pair of linearly independent solutions of (3). Let z_1, z_2 be defined for $t \in j$ by

$$z_i(t) = \alpha y_i + \beta y_i', \qquad i = 1, 2.$$

The function (5) transforms (3) onto (4) iff z_1, z_2 are linearly independent solutions of (4). Let $w(f, g) = f'g - fg'$ denote the wronskian of the couple f, g. Direct calculation shows that

$$w(z_1, z_2) = (\alpha^2 + \alpha\beta' - \alpha\beta a - \alpha'\beta + \beta^2 b)\, w(y_1, y_2)$$

and (7) follows now immediately.

Theorem 1. *Let $a, b \in C^1(j)$, $\alpha, \beta \in C^2(j)$ and $(D(t) =)\ \alpha^2 + \alpha\beta' - \alpha\beta a - \alpha'\beta + \beta^2 b \neq 0$ on j. The formula $z(t) = \alpha(t)\, y + \beta(t)\, y'$ transforms the equation*

$$y'' + a(t)\, y' + b(t)\, y = 0 \tag{3}$$

onto

$$z'' + A(t)\, z' + B(t)\, z = 0 \tag{4}$$

just if

$$A = a - D'/D, \qquad B = b - (D_1 U_2 - U_1 D_2)/D,$$

where

$$\begin{aligned} U_1(t) &= \alpha + \beta' - \beta a, \\ U_2(t) &= \beta b - \alpha', \end{aligned} \tag{8}$$

$$D_1(t) = 2\alpha' + \beta'' - \beta' a - \beta a',$$
$$D_2(t) = 2\beta' b + \beta b' - a\alpha' - \alpha''.$$

Proof. Let y be a solution of (3) and set $z(t) = \alpha(t)\, y + \beta(t)\, y'$. Then

$$z' = (\alpha' - \beta b)\, y + (\alpha + \beta' - \beta a)\, y',$$
$$z'' = (\alpha'' - \alpha b - 2\beta' b - \beta b' + \beta ab)\, y + $$
$$+ (2\alpha' - \alpha a + \beta'' - 2\beta' a + \beta a^2 - \beta a' - \beta b)\, y'$$

which implies that z is a solution of (4) just if the coefficients A, B comply with

(9) $$\alpha'' - \alpha b - 2\beta' b - \beta b' + \beta ab + A(\alpha' - \beta b) + B\alpha = 0,$$

(10) $$2\alpha' - \alpha a + \beta'' - 2\beta' a + \beta a^2 - \beta a' - \beta b + A(\alpha + \beta_t - \beta a) + B\beta = 0.$$

The linear combination (9) β — (10) α can be expressed in the form

$$-D' + Da - DA = 0,$$

so that

$$A = a - D'/D.$$

Suppose B in the form $B = b + X/D$, X being a sought expression. Substituting the above mentioned form of A and B into (9) and (10) we receive

$$-DD_2 + D'U_2 + \alpha X = 0, \qquad DD_1 - D'U_1 + \beta X = 0$$

and

(11) $$D(-\alpha D_2 + \beta D_1) + D'(\alpha U_2 - \beta U_1) + (\alpha^2 + \beta^2)\, X = 0.$$

Direct calculation shows that

(12) $$D = \alpha U_1 + \beta U_2, \qquad D' = \alpha D_1 + \beta D_2,$$

with respect to (8) and (12) it follows from (11) that

$$(\alpha U_1 + \beta U_2)(-\alpha D_2 + \beta D_1) + (\alpha D_1 + \beta D_2)(\alpha U_2 - \beta U_1) +$$
$$+ (\alpha^2 + \beta^2)\, X = 0,$$

so that

$$(\alpha^2 + \beta^2)(D_1 U_2 - U_1 D_2 + X) = 0$$

and $X = U_1 D_2 - D_1 U_2$.

Remark 2. Only small changes in the proof of Theorem 1 are necessary to receive the following a little different formulation of Theorem 1.

Theorem 1'. *Let* $a, b \in C^1(j)$, $\alpha, \beta \in C^2(j)$. *If the formula* (5) *transforms* (3) *onto* (4) *then* $D(t) \neq 0$ *on* j *and* (8) *holds. Conversely, if* $D(t) \neq 0$ *on* j *and* (8), *hold, then the formula* (5) *transforms* (3) *onto* (4).

Remark 3. Shortly, we can say that the equation

$$z'' + (a - D'/D)\, z' + (b - (D_1 U_2 - U_1 D_2)/D)\, z = 0, \tag{13}$$

where D, D_1, D_2, U_1, U_2 are defined by (8), is a differential equation for the function $z(t) = \alpha(t)\, y + \beta(t)\, y'$. The special case of (13) was derived also by I. Bihary in [1] and presented at the "Colloquium on the Qualitative Theory on Differential Equations" at Szeged in August 1984. The equation of the form (1) is considered and the equation for so-called "Bôcher-function" $\Phi(t) = \varphi_1(t)\, y - \varphi_2(t)\, y'$ is derived there.

However, the coefficient $A(t)$ is presented erroneously. This fact is without influence on the main line of the above mentioned paper but several formulae should be changed slightly.

Remark 4. The derivatives of the function $z(t) = \alpha y + \beta y'$ can be expressed in the form

$$z'(t) = \alpha_1 y + \beta_1 y',$$

where

$$\alpha_1(t) = \alpha' - \beta b, \qquad \beta_1(t) = \alpha + \beta' - a\beta,$$

$$z''(t) = \alpha_2 y + \beta_2 y',$$

where

$$\alpha_2(t) = \alpha_1' - \beta_1 b, \qquad \beta_2(t) = \alpha_1 + \beta_1' - a\beta_1 .$$

By means of this notation it can be easily shown that

$$D = \begin{vmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{vmatrix}, \qquad A = \frac{1}{D} \begin{vmatrix} \alpha & \beta \\ \alpha_2 & \beta_2 \end{vmatrix}, \qquad B = \frac{1}{D} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

Remark 5. In case α, β are constants, the equation (13) has the following explicite form

$$z'' + \left(a - \beta \frac{\beta b' - \alpha a'}{\alpha^2 + \beta^2 b - \alpha\beta a}\right) z' +$$

$$+ \left(b - \beta \frac{\beta(ab' - a'b) - \alpha b'}{\alpha^2 + \beta^2 b - \alpha\beta a}\right) z = 0.$$

This equation is used in [3] to investigate certain higher monotonicity properties of Airy and Bessel functions.

Remark 6. Due to the fact that the general form of linear differential equation of the second order is considered a differential equations for the functions $z_i(t) = \alpha(t)\, y^{(i)}(t) + \beta(t)\, y^{(i+1)}(t)$ or for the i-th derivative of $z(t)$ $(i = 1, 2, \dots)$ can be easily found by means of the method introduced in [11].

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