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# NORMAL SUBGROUPS AS IDEALS

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## 1. INTRODUCTION

We can define *an ideal system on a grupoid* $(S, .)$ as a system $\{A_x : A \subseteq S\}$ fulfilling conditions:

1. $A \subseteq A_x$,
2. $A \subseteq B_x \Leftarrow A_x \subseteq B_x$,
3. $A_x . B \subseteq A_x$,
4. $A_x . B \subseteq (A . B)_x$.

This definition was introduced by K. Aubert [1] in the case where the grupoid operation is both commutative and associative.

We say that $A = A_x$ is *an ideal* and the grupoid operation is called *an ideal operation.*

In [6] F. Voráč studied the structure of a group $G$ with normal subgroups acting as ideals and the commutator operation acting as the ideal operation. In the following we say that such a group $G$ is a *K-group.* Voráč characterized the structure of a $K$-group $G$ with the aid of the centralizers of the elements of factor groups of $G$ (see [6], Theorem 2.7). Later on it was shown in [5] that a $K$-group is necessarily nilpotent. In this paper we show that the class of the nilpotent K-group $G$ is at most three. Furthermore, we show that if the commutator operation is associative in $G$, then $G$ is a K-group. In most cases also the converse result holds.

In this paper $G$ denotes a multiplicative group, $G'$ denotes the commutator subgroup of $G$ and $Z(G)$ the centre of $G$. The centralizer of an element $g$ in $G$ is denoted by $C_G(g)$ and $N(A)$ means the normal subgroup generated by a subset $A$ of $G$. Finally, by $*$ we denote the commutator operation.

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[1]) This paper was written while the latter author was visiting the University of Brno.

## 2. BASIC LEMMAS

Let $G$ be a group. From the properties of normal subgroups and the commutator operation it follows that $G$ is a K-group if and only if $N(A) * B \subseteq \subseteq N(A * B)$ for all $A, B \subseteq G$. We first establish a result of Voráč [6], p. 242.

**Lemma 2.1.** *If $G$ is a K-group, then $C_G(g)$ is a normal subgroup of $G$ for all $g \in G$.*

We also need [6], p. 241.

**Lemma 2.2.** *Let $N$ be a normal subgroup of $G$. If $(a * g) * b \in N$, for all $a \in G$ and for all $g * b \in N$, then $G$ is a K-group.*

**Lemma 2.3.** *Let $G$ be a K-group, $g \in G$ and let $x \in C_A(g)$, then $g * (x * z) = 1$, for all $z \in G$.*

Proof. By lemma 2.1, $z^{-1}xz \in C_G(g)$, so $z^{-1}xzg = gz^{-1}xz$, hence $g^{-1}z^{-1}xzgz^{-1}x^{-1}z = 1$. It follows that $g^{-1}x(x^{-1}z^{-1}xz)\,g(x^{-1}z^{-1}xz)^{-1}x^{-1} = 1$, so $g^{-1}(x * z)\,g(x * z)^{-1} = 1$ and the proof is complete.

As a direct consequence we have

**Lemma 2.4.** *Let $G$ be a K-group, then $(x * y) * y = 1$, for all $x, y \in G$.*

Proof. Now $y \in C_G(y)$, so by lemma 2.3, $y * (y * x) = 1$. Then, clearly, $(x * y) * y = 1$ and the proof is complete.

Finally, we give two results of Levi [4].

**Lemma 2.5.** *The commutator operation is associative in a group $G$ if and only if* $G' \subseteq Z(G)$ (this result can also be found in [2], p. 87).

**Lemma 2..6** *Let $(x * y) * y = 1$ for all $x, y \in G$. Then $G$ is nilpotent of class at most three. Furthermore, if $G$ has no elements of order three, then $G$ is nilpotent of class two.*

A modern treatment of lemma 2.6 is given in [2], p. 288.

## 3. MAIN RESULTS

Now we are able to establish

**Theorem 3.1.** *A group $G$ is a K-group if and only if $(x * y) * y = 1$ for all $x, y \in G$.*

Proof. Let $G$ be a K-group. Now lemma 2.4 applies and an assertion is true. Then suppose that $(x * y) * y = 1$, for all $x, y \in G$. Furthermore, suppose that $N$ is a normal subgroup of $G$ and let $g * b \in N$. Thus $(g * b) * a \in N$, for all $a \in G$. By steps (2) and (3) in the proof of Theorem 6.5 of [2], p. 288–289, we get $)g * b) * a = [(g * a) * b]^{-1} = [(a * g)^{-1} * b]^{-1} = (a * g) * b$, so we can use lemma 2.2. The proof is complete.

Now, by lemma 2.6, a K-group $G$ is nilpotent of class at most three. Lemma 2.5 provides us with

**Theorem 3.2.** *Let G be a group such that G has no elements of order three. Then G is a K-group if and only if the commutator operation is associative.*

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