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DEVIATION OF TWO CURVATURES*

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In this paper, we determine the deviation of curvatures of two (generalized) connections in terms of the deflection of both connections in question.

Let **FM** denote the category of fibered manifolds and their morphisms. Let $\pi : Y \to X$ be a fibered manifold and let $Y \oplus TX$ denote the fiber product over X. A (generalized) connection Γ on Y means any section $\Gamma : Y \to J^1 Y$ (the first jet prolongation of Y), [3], [6], [8]. Such a connection Γ on Y can be considered as a section $Y \oplus TX \to TY$ linear with respect to TX. In local fiber coordinates x^i, y^α on Y, the equations of Γ are

$$\Gamma \equiv \mathrm{d}y^\alpha = F_i^\alpha(x, y)\,\mathrm{d}x^i,$$

with arbitrary smooth functions F_i^α, [3]. Let VY be the vertical tangent bundle of Y considered as a fibered manifold over X. The vertical prolongation $V\Gamma$ of Γ, [2], [3], [9], is a connection on VY with the following equations

$$V\Gamma \equiv \mathrm{d}y^\alpha = F_i^\alpha(x, y)\,\mathrm{d}x^i, \qquad \mathrm{d}\eta^\alpha = \partial_\beta F_i^\alpha \eta^\beta\,\mathrm{d}x^i,$$

where $\partial_\beta F_i^\alpha = \dfrac{\partial F_i^\alpha}{\partial y^\beta}$ and $\eta^\alpha = \mathrm{d}y^\alpha$ are the induced coordinates on VY, [3].

A 2-fibered manifold is a quintuple $U \xrightarrow{q} Y \xrightarrow{\pi} X$, where $q : U \to Y$ and $\pi : Y \to X$ are fibered manifolds. If $q : U \to Y$ is a vector bundle, $U \xrightarrow{q} Y \xrightarrow{\pi} X$ is called a semi-vector bundle. Let a projectable connection $\bar{\Gamma} : U \to J^1 U$ over $\Gamma : Y \to J^1 Y$ be given. If all maps $\bar{\Gamma}|_{U_y}$ of vector space U_y into $(J^1 U)_{\Gamma(y)}$ are linear for every $y \in Y$, then $\bar{\Gamma}$ is called a semi-linear connection, [3]. Obviously, $V\Gamma$ is a semi-linear connection.

Having another fibered manifold $p : W \to Z$ with a connection $\Delta : W \oplus TZ \to TW$ and an FM-morphism $\varphi : W \to Y$ over $f : Z \to X$ we define the deflection $\mu(\Delta, \Gamma, \varphi) : W \oplus TZ \to VY$ of connections Δ and Γ with respect to φ by

$$\mu(\Delta, \Gamma, \varphi) = T\varphi \circ \Delta - \Gamma \circ (\varphi \oplus Tf).$$

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One sees easily that the vectors $(T\varphi)(\Delta(w, \xi))$ and $\Gamma(\varphi(w), Tf(\xi))$ are over the same vector on the base manifold TX for every $(w, \xi) \in W \oplus TZ$. If the mapping μ vanishes, the connections Δ and Γ are called φ-related. In local fiber coordinates z^p, w^λ on W let the equations of Δ be

$$\Delta \equiv \mathrm{d}w^\lambda = G^\lambda_p(z, w)\, \mathrm{d}z^p$$

and let

$$x^i = f^i(z), \qquad y^\alpha = f^\alpha(z, w),$$

be the equations of φ, then the equations of $\mu(\Delta, \Gamma, \varphi)$ are

$$\eta^\alpha = (\partial_p f^\alpha + \partial_\lambda f^\alpha G^\lambda_p - F^\alpha_i \partial_p f^i)\, \mathrm{d}z^p.$$

We recall, that the curvature of Γ is a section $\Omega_\Gamma : Y \oplus \wedge^2 TX \to VY$, [1], [4]. In local coordinates

$$\Omega_\Gamma \equiv \eta^\alpha = (\partial_j F^\alpha_i + F^\beta_j \partial_\beta F^\alpha_i)\, \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

[6].

For a manifold Z, a vector bundle $p : E \to X$, a linear connection Γ on E and a linear morphism

(1) $$\varphi : \wedge^k TZ \to E \qquad \text{over} \qquad f : Z \to X$$

we define the exterior differential $\mathrm{d}_\Gamma \varphi : \wedge^{k+1} TZ \to E$ by the following way. We construct the induced vector bundle $f^*E \to Z$,

$$f^*E = \{(z, e) \in Z \times E; fz = pe\},$$

and the induced connection $f^*\Gamma$ on f^*E. The connection $f^*\Gamma$ is the unique connection on f^*E for which the deflection $\mu(f^*\Gamma, \Gamma, p_E)$ vanishes, where p_E is the natural projection $f^*E \to E$. The linear morphism φ can be considered as a section $\tilde{\varphi} : Z \to f^*E \otimes \wedge^k T^*Z$ and we take its exterior differential $d_{f^*\Gamma}\tilde{\varphi} : Z \to f^*E \otimes \otimes \wedge^{k+1} T^*Z$, that can be interpreted as a mapping $d_\Gamma \varphi : \wedge^{k+1} TZ \to E$, [7]. In local fiber coordinates x^i, e^α on E and in local coordinates z^p on Z, let

$$x^i = f^i(z), \qquad e^\alpha = \varphi^\alpha_{p_1 \dots p_k}(z)\, \mathrm{d}z^{p_1} \wedge \dots \wedge \mathrm{d}z^{p_k},$$

be the equations of φ and let

$$\mathrm{d}e^\alpha = \Gamma^\alpha_{\beta i}(x)\, e^\beta\, \mathrm{d}x^i,$$

be the equations of Γ. Then

$$\mathrm{d}_\Gamma \varphi \equiv \mathrm{e}^\alpha = (\partial_p \varphi^\alpha_{p_1 \dots p_k} - \Gamma^\alpha_{\beta i} \varphi^\beta_{p_1 \dots p_k} \partial_p f^i)\, \mathrm{d}z^p \wedge \mathrm{d}z^{p_1} \wedge \dots \wedge \mathrm{d}z^{p_k}.$$

Let $W \to Z$ be a fibered manifold with a connection Δ on W. Let $U \to Y \to X$ be a semi-vector bundle with a semi-linear connection $\bar{\Gamma}$ on U over Γ on Y. For any linear morphism $\psi : W \oplus \wedge^k TZ \to U$ over $\varphi : W \to Y$ and over $f : Z \to X$, we define its exterior differential

$$\mathrm{d}_{(\Delta,\bar{\Gamma})}\psi : W \oplus \wedge^{k+1}TZ \to U,$$

with respect to Δ and Γ by the following way, [3].

For every $w \in W$ we take a section ϱ tangent to $\Delta(w)$. Therefore $\psi \circ \varrho$ has the form (1). Now we have the former case, when we consider the induced bundle from the vector bundle U with respect to the mapping $\varphi \circ \varrho$ with the relevant induced connection and we construct

$$\mathrm{d}_{(\varphi\circ\varrho)^*\bar{\Gamma}}(\psi \circ \varrho)\,|_z : \wedge^{k+1}T_zZ \to U_{\varphi(w)}.$$

In local fiber coordinates x^i, y^α, u^ϱ on U and z^p, w^λ on W, let $x^i = f^i(z)$, $y^\alpha = \varphi^\alpha(z, w)$, $u^\varrho = \psi^\varrho_{p_1\dots p_k}(z, w)\,\mathrm{d}z^{p_1} \wedge \dots \wedge \mathrm{d}z^{p_k}$ be the equations of ψ and let

$$\mathrm{d}w^\lambda = G^\lambda_p(z, w)\,\mathrm{d}z^p \qquad \text{or} \qquad \mathrm{d}y^\alpha = F^\alpha_i(x, y)\,\mathrm{d}x^i, \qquad \mathrm{d}u^\varrho = \Gamma^\varrho_{\sigma i}(x, y)\,u^\sigma\,\mathrm{d}x^i$$

be the equations of Δ or Γ, respectively. Then

$$\mathrm{d}_{(\Delta,\bar{\Gamma})}\psi \equiv u^\varrho = (\partial_p\psi^\varrho_{p_1\dots p_k} + G^\lambda_p\,\partial_\lambda\psi^\varrho_{p_1\dots p_k} - \Gamma^\varrho_{\sigma i}\psi^\sigma_{p_1\dots p_k}\,\partial_p f^i)\,\mathrm{d}z^p \wedge \mathrm{d}z^{p_1} \wedge \dots \wedge \mathrm{d}z^{p_k}.$$

By direct calculation, we prove the following assertion.

Theorem. *Let Γ or Δ be a connection on $Y \to X$ or $W \to Z$ with curvature Ω_Γ or Ω_Δ, respectively, and let $\varphi : W \to Y$ be an* **FM**-*morphism over $f : Z \to X$. Then it holds*

$$V\varphi \circ \Omega_\Delta - \Omega_\Gamma \circ (\varphi \oplus \wedge^2 Tf) = -\mathrm{d}_{(\Delta, V\Gamma)}\mu(\Delta, \Gamma, \varphi).$$

One interesting example is the case when Δ and Γ are related connections, then the diagram commutes:

$$\begin{tikzcd} VW \arrow[r, "V\varphi"] & VY \\ W \oplus \wedge^2 TZ \arrow[u, "\Omega_\Delta"] \arrow[r, "\varphi \oplus \wedge^2 Tf"] & Y \oplus \wedge^2 TX, \arrow[u, "\Omega_\Gamma"'] \end{tikzcd}$$

see [5].

REFERENCES

[1] **Dekrét, A.**: *Horizontal structures on fibered manifolds*, Math. Slovaca, **27** (1977), 257—265.

[2] **Dekrét, A.**: *On connections and covariant derivate on fibre manifolds*, Proceedings of the Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, IM PAN, Warszawa 1979, 40—55.

[3] **Kolář, I.**: *Connections in 2-fibered manifolds*, to appear in: Arch. Math., Brno.

[4] **Kolář, I.**: *Higher order torsion of spaces with Cartan connection*, Cahiers Topologie Géom. Différentielle, **12** (1971), 137—146.

[5] Kolář, I.: *Induced connections and connection morphisms*, Proceedings of the Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, IM PAN, Warszawa 1979, 125—134.
[6] Kolář, I.: *On generalized connections*, to appear in Beitr. Algebra Geom.
[7] Koszul, J. L.: *Fibre bundles and differential geometry*, Tata Institute of Fundamental Research, Bombay 1960.
[8] Libermann, P.: *On "Fibre parallelism" and locally reductive spaces*, Global Analysis Appl., Internat. Sem. Course, Trieste 1972, Vol. III., (1974), 13—23.
[9] Vilms, J.: *Curvature of nonlinear connections*, Proc. AMS, **19** (1968), 1125—1129.

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