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METRICS AND TOLERANCES

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§ 1

A reflexive and symmetric binary relation T on a non-empty set A is called a *tolerance relation* (or shortly *tolerance*) on A and the ordered pair (A, T) is called a *tolerance space*. By the symbol I we denote the identity relation on A, i.e. such a relation that xIy if and only if $x = y$ for any x and y from A. Denote $T^0 = I, T^1 = T$, $T^{n+1} = T \,.\, T^n$ for each positive integer n.

Definition 1. Let (A, T) be § 1. a tolerance space. A non-empty subset B of A is called *T-connected in A*, if for any $x \in B$, $y \in B$ there exists a positive integer p such that xT^py. If A is T-connected in (A, T), then (A, T) is called a *connected tolerance space*.

Proposition 1. *Let (A, T) be a tolerance space, let B be a T-connected set in A and let $\delta_T(x, y)$ be an integer-valued function on $B \times B$ given by the rule*

$$\text{(P)} \qquad \begin{aligned} &\delta_T(x, y) = 0 \Leftrightarrow xT^0y, \\ &\delta_T(x, y) = p \Leftrightarrow xT^py \quad \text{and} \quad \neg\, xT^qy \quad \text{for } q < p. \end{aligned}$$

Then $\delta_T(x, y)$ is an integer-valued metric on B.

Proposition 2. *Let (A, μ) be a quasimetric space and ε a positive real number. The relation $T_{\mu(\varepsilon)}$ defined on A by the rule*

$$\text{(Q)} \qquad xT_{\mu(\varepsilon)}y \Leftrightarrow \mu(x, y) \leqq \varepsilon$$

is a tolerance on A and the tolerance space $(A, T_{\mu(\varepsilon)})$ is $T_{\mu(\varepsilon)}$-connected.

Definition 2. Let (A, T) be a tolerance space, let B be a T-connected set in A. The metric δ_T on B is called *induced by the tolerance T*. Let $\varepsilon > 0$ and let (A, μ) be a quasimetric space. Then the tolerance $T_{\mu(\varepsilon)}$ is called *induced by the quasimetric μ with the unit ε*.

Proposition 3. *Let (A, T) be a connected tolerance space, δ_T a metric induced by the tolerance T and $T_{\delta_T(1)}$ the tolerance induced by the metric δ_T with the unit $\varepsilon = 1$. Then $T = T_{\delta_T(1)}$.*

Proposition 4. *Let (A, μ) be a quasimetric space, let $0 < \varepsilon \leqq 1$, let $T_{\mu(\varepsilon)}$ be the tolerance induced by the quasimetric μ with the unit ε and δ_T the metric induced by the tolerance $T_{\mu(\varepsilon)}$. Then $\delta_T(x, y) \geqq \mu(x, y)$ for any $x \in A$, $y \in A$.*

Proposition 5. *Let (A, π) be a metric space with an integer-valued metric π, let $T_{\pi(1)}$ be a tolerance on A induced by the metric π with the unit $\varepsilon = 1$ and δ_T the metric induced by the tolerance $T_{\pi(1)}$. Then $\pi = \delta_T$.*

Proposition 6. *Let (A, μ) be a quasimetric space and $\varepsilon_1, \varepsilon_2$ positive real numbers. If $\varepsilon_1 < \varepsilon_2$, then $T_{\mu(\varepsilon_1)} \subseteq T_{\mu(\varepsilon_2)}$. If $x \in A$, $y \in A$ and $\varepsilon_1 < \mu(x, y) < \varepsilon_2$, then $T_{\mu(\varepsilon_1)} \neq T_{\mu(\varepsilon_2)}$, i.e.*

$$\varepsilon_1 < \varepsilon_2 \Rightarrow T_{\mu(\varepsilon_1)} \subset T_{\mu(\varepsilon_2)}.$$

Remark. Evidently each equivalence on A is a tolerance on A. By Definition 1 it is evident that for an equivalence E on A a set B such that $\emptyset \neq B \subseteq A$ *is E-connected* in A if and only if there exists a partition class $[a] \in A/E$ such that $B \subseteq [a]$. Therefore if x, y, z are elements of $[a]$, then for δ_E the *triangle inequality* holds. Further, if $x = y$, evidently $\delta_E(x, y) = 0$ and for $x \in [a]$, $y \in [a]$, $x \neq y$ we have $\delta_E(x, y) = 1$, because the transitivity of E implies $T_k \subseteq T$ for $k = 0, 1, 2, \ldots$ This implies that if T is an equivalence on A, xTy, yTz, $x \neq y$, $y \neq z$, then the *sharp triangle inequality*

$$\delta_T(x, z) < \delta_T(x, y) + \delta_T(y, z) \tag{T}$$

holds, because $\delta_T(x, z) \leqq 1$ and $\delta_T(x, y) + \delta_T(y, z) = 2$. We shall show that also the converse assertion holds.

Proposition 7. *Let (A, T) be a connected tolerance space with at least three elements and let δ_T be the metric induced by the tolerance T. If for any three elements (pairwise distinct) x, y, z of A the sharp triangle inequality* (T) *holds, then T is an equivalence on A.*

Proof. Let x, y, z be pairwise distinct elements of A and let xTy, yTz. Then by (P) we have $\delta_T(x, y) = 1$, $\delta_T(y, z) = 1$ and (T) implies $\delta_T(x, z) < 2$, i.e. $\delta_T(x, z) \leqq 1$. As $x \neq z$, we have $\delta_T(x, z) \neq 0$, because δ_T is a metric (by Proposition 1), therefore $\delta_T(x, z) = 1$ and (P) implies xTz. As x, y, z were chosen arbitrarily, T is transitive, i.e. it is an equivalence.

§ 2

Definition 3. Let (A, μ) be a quasimetric space and $\{\varepsilon_i\}_{i=1}^{\infty}$ a decreasing sequence of positive real numbers. Denote

$$T_{\lim} = \bigcap_{i=1}^{\infty} T_{\mu(\tau_i)},$$

where $T_{\mu(\varepsilon_i)}$ is a tolerance on A induced by the quasimetric μ with the unit ε_i.

Proposition 8. *Let (A, μ) be a quasimetric space, let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive real numbers and $\varepsilon = \lim_{i\to\infty} \varepsilon_i$. Then $T_{\mu(\varepsilon)} = T_{\lim}$ and $T_{\lim}$ is a tolerance on A.*

Proof. Evidently $T_{\lim}$ is a reflexive and symmetric relation on A, i.e. it is a tolerance. If $xT_{\mu(\varepsilon)}y$, then $\mu(x, y) \leqq \varepsilon$, therefore $\mu(x, y) \leqq \varepsilon_i$ for each i. Hence $xT_{\mu(\varepsilon_i)}y$ for each i and $xT_{\lim}y$. We have proved $T_{\mu(\varepsilon)} \subseteq T_{\lim}$. Conversely, if $xT_{\lim}y$, then $xT_{\mu(\varepsilon_i)}y$ for each i and thus $\mu(x, y) \leqq \varepsilon_i$ and $\mu(x, y) \leqq \varepsilon = \lim_{i\to\infty} \varepsilon_i$, which means $xT_{\mu(\varepsilon)}y$. Hence $T_{\lim} \subseteq T_{\mu(\varepsilon)}$.

Proposition 9. *Let (A, μ) be a quasimetric space and $\{\varepsilon_i\}_{i=1}^{\infty}$ a decreasing sequence of positive real numbers such that $\lim_{i\to\infty} \varepsilon_i = 0$. Then the following two assertiens are equivalent:*

(1) *μ is a metric.*

(2) *$T_{\lim} = I$.*

Proof. Let μ be a metric and $xT_{\lim}y$ for some $x \in A$, $y \in A$. Then $xT_{\mu(\varepsilon_i)}y$ for each ε_i, thus $\mu(x, y) \leqq \lim_{i\to\infty} \varepsilon_i = 0$. As μ is a metric, $\mu(x, y) = 0$ implies $x = y$ and hence $T_{\lim} \subseteq I$. Evidently $I \subseteq T_{\lim}$, therefore $T_{\lim} = I$. Now let $T_{\lim} = I$ and $\mu(x, y) = 0$ for some $x \in A, y \in A$. Then $\mu(x, y) = \lim_{i\to\infty} \varepsilon_i$, i.e. $xT_{\lim}y$, hence by (2) we have $x = y$ and μ is a metric.

Proposition 10. *Let (A, μ) be a quasimetric space and $T_{\mu(\varepsilon)}$ a tolerance induced by the quasimetric μ with the unit ε. Then for $\varepsilon = 0$ the relation $T_{\mu(0)}$ is an equivalence on A and $A/T_{\mu(0)}$ is a metric space.*

Proof. If $xT_{\mu(0)}y$, $yT_{\mu(0)}z$, then $\mu(x, y) = 0$, $\mu(y, z) = 0$ and this implies $0 \leqq \leqq \mu(x, z) \leqq \mu(x, y) + \mu(y, z) = 0$, hence $\mu(x, z) = 0$ and $xT_{\mu(0)}z$. The second assertion is evident.

§ 3

Lemma 1. *Let L be a lattice with the least element 0, let T be a compatible tolerance on L (see for example [3]). If $a \in L$, $b \in L$ and $aT^r\,0$, $bT^s.\,0$ for some non-negative integers r, s, then $(a \vee b)\, T^{\max(r,s)}\, 0$, $(a \wedge b)\, T^{\min(r,s)}\, 0$.*

Proof. If T is a compatible relation on L, then (by Theorem 3 in [1]) T^k is also a compatible relation on L for each non-negative integer k. If $aT^r\,0$, $bT^s\,0$, then by Corollary 5 in [1] we have $aT^q\,0$, $bT^q\,0$ for $q = \max(r, s)$ and the compatibility of T^q implies $(a \vee b)\, T^q\, 0$. Further let $p = \min(r, s)$; without less of generality let $p = r$. Then $aT^r\,0 \Rightarrow (a \wedge b)\, T^r(0 \wedge b) = 0$ and thus $(a \wedge b)\, T^p\, 0$.

Definition 3. Let L be a lattice with the least element 0. A tolerance T on L is called *disjunctive*, if $(a \wedge b)\, T^k\, 0$ implies $aT^k\, 0$ or $bT^k\, 0$.

In [2] the concept of a valuation on a lattice is introduced. A real-valued function v on L is called a *valuation*, if for any two elements a, b of L

$$v(a) + v(b) = v(a \wedge b) + v(a \vee b).$$

A valuation is called *order-preserving*, if $a \leqq b$ implies $v(a) \leqq v(b)$ and *positive*, if $a < b$ implies $v(a) < v(b)$ for any a and b. If there exists an order-preserving (or positive) valuation on L, then L is called a *quasimetric* (or *metric* respectively) *lattice*. (see [2], p. 108).

Theorem 1. *Let L be a lattice with the least element* 0 *and let T be a compatible disjunctive tolerance on L such that (L, T) is a connected tolerance space. Then L is a quasimetric lattice.*

Proof. Let v be an integer-valued function on L defined so that $v(a) = 0$ for each $a \in L$ such that $aT\, 0$ and $v(a) = p$ for each $a \in L$ such that $aT^{p+1}\, 0$ and $\neg aT^q\, 0$ for all $q \leqq p$. As (L, T) is connected, v is defined for all elements of L. If $a \leqq b$, then $a \vee b = b$, $a \wedge b = a$ and thus $v(a \wedge b) + v(a \vee b) = v(a) + v(b)$. Now let a, b be two incomparable elements of L, let $v(a) = p$, $v(b) = q$; without loss of generality let $q \leqq p$. Then $aT^{p+1}\, 0$, $bT^{q+1}\, 0$ and by Lemma 1 we have $a \vee bT^{p+1}\, 0$. Suppose $a \vee bT^p\, 0$. From this and from $aT^p a$ we obtain $a = a \wedge (a \vee b)\, T^p\, 0 \wedge a = 0$ and $v(a) < p$, which is a contradiction. Therefore $v(a \vee b) = p$. Further from Lemma 1 we have $(a \wedge b)\, T^{q+1}\, 0$. Let $j \leqq q$; then $\neg aT^j\, 0$, $\neg bT^j\, 0$ and the disjunctivity of T implies $\neg (a \wedge b)\, T^j\, 0$, therefore $v(a \wedge b) = q$. We have $v(a \wedge b) + v(a \vee b) = p + q = v(a) + v(b)$. We have proved that v is a valuation on L. Now let $x \leqq y$, $v(y) = q$. Then $yT^{q+1}\, 0$ and $\neg yT^r\, 0$ for $r \leqq q$. We have $x \wedge y = x$ and from the compatibility of T^{q+1} we obtain

$$xT^{q+1}x, \qquad yT^{q+1}\, 0 \Rightarrow x = (x \wedge y)\, T^{q+1}\, (x \wedge 0) = 0$$

and thus $v(x) \leqq q = v(y)$ and v is order-preserving. This means that L is quasimetric.

Remark. We shall show that in the case when T is not disjunctive the function v defined in this proof is not a valuation. If T is not disjunctive, then there exist elements a, b of L and a non-negative integer s such that $\neg aT^{s+1}\, 0$, $\neg bT^{s+1}\, 0$, $a \wedge bT^{s+1} 0$. Then $v(a \wedge b) \leqq s$. Let $v(a) = p$, $v(b) = q$; then $p \geqq s + 1$, $q \geqq s + 1$. Without loss of generality let $p \geqq q$. We have $aT^{p+1}\, 0$, $bT^{q+1}\, 0$, thus by Lemma 1 $(a \vee b)\, T^{p+1}\, 0$ and $v(a \vee b) \leqq p$. Then

$$v(a) + v(b) = p + q > p + s + 1,$$
$$v(a \wedge b) + v(a \vee b) \leqq p + s$$

and thus $v(a) + v(b) \neq v(a \wedge b) + v(a \vee b)$.

We have proved that the valuation v defined in the proof of Theorem 1 is order-preserving. The following proposition shows, when it is positive.

Proposition 11. *Let L and T be given as in Theorem 1, let v be the valuation defined in the proof of Theorem 1. The valuation v is positive, only if L is a chain embeddable into the chain of all non-negative integers (naturally ordered).*

Proof. Suppose that L is not a chain. Then there exist two elements x, y of L which are incomparable. Let $v(x) = p$, $v(y) = q$ and without loss of generality $p \geqq q$. In the proof of Theorem 1 it is proved that then $v(x \wedge y) = q$, $v(x \vee y) = p$. But then $x \wedge y < y$, $v(x \wedge y) = v(y)$ and v is not positive. Therefore L is a chain. If v is a positive valuation on a chain, it is evidently an embedding of this chain into the chain of all non-negative integers.

Now it seems to be reasonable to consider the valuation in which $v(a) = 0$ only for $a = 0$.

Theorem 2. *Let L be a lattice with the least element 0 and with the property that $a \wedge b = 0$ in L if and only if $a = 0$ or $b = 0$. Let T be a compatible disjunctive tolerance on L such that (L, T) is a connected tolerance space. Then there exists an order-preserving valuation v on L such that $v(a) = 0$ only for $a = 0$.*

Proof. Let v be the valuation from the proof of Theorem 1. Put $v'(0) = 0$, $v'(a) = v(a) + 1$ for each $a \neq 0$. Let x, y be two elements of L. If $x \neq 0$, $y \neq 0$, then also $x \wedge y \neq 0$, $x \vee y \neq 0$ and we have

$$v'(x \wedge y) + v'(x \vee y) = v(x \wedge y) + v(x \vee y) + 2 = v(x) + v(y) + 2 = \\ = v'(x) + v'(y).$$

If $x = 0$, $y \neq 0$, then $x \wedge y = 0$, $x \vee y \neq 0$ and

$$v'(x \wedge y) + v'(x \vee y) = v(x \wedge y) + v(x \vee y) + 1 = v(x) + v(y) + 1 = \\ = v'(x) + v'(y).$$

Analogously for $x \neq 0$, $y = 0$. For $x = y = 0$ the equality is evident. Therefore v' is the required valuation.

Before proving the last theorem, we shall prove a lemma.

Lemma 2. *Let m_1, m_2, n_1, n_2 be four non-negative integers, let $|m_1 - n_1| \leqq 1$, $|m_2 - n_2| \leqq 1$. Then*

$$|\max(m_1, m_2) - \max(n_1, n_2)| \leqq 1,$$
$$|\min(m_1, m_2) - \min(n_1, n_2)| \leqq 1.$$

Proof. If $m_1 \geqq m_2, n_1 \geqq n_2$, then $|\max(m_1, m_2) - \max(n_1, n_2)| = |m_1 - n_1| \leqq \leqq 1$. If $m_1 \geqq m_2$, $n_1 \leqq n_2$, then $|\max(m_1, m_2) - \max(n_1, n_2)| = |m_1 - n_2|$. If $m_1 \geqq n_2$, then $|m_1 - n_2| = m_1 - n_2 \leqq m_1 - n_1 = |m_1 - n_1| \leqq 1$; if $m_1 \leqq n_2$, then $| m_1 - n_2 | = n_2 - m_1 \leqq n_2 - m_2 = | m_2 - n_2 | \leqq 1$. Analogously we do the proof for $m_1 \leqq m_2, n_1 \geqq n_2$ and $m_1 \leqq m_2, n_1 \leqq n_2$. The proof for the minimum is dual.

Theorem 3. *Let L be a quasimetric lattice with the valuation v satisfying $v(x \vee y) = \max(v(x), v(y))$, $v(x \wedge y) = \min(v(x), v(y))$, for any two elements x, y of L. Let T be the tolerance on L defined so that xTy if and only if $v(x \vee y) - v(x \wedge y) \leqq 1$. Then T is a compatible tolerance on L.*

Proof. Let a, b be two elements of L. Let aTb. This means $v(a \vee b) - v(a \wedge b) \leqq 1$ and according the assumption $\max(v(a), v(b)) - \min(v(a)), v(b)) \leqq 1$. But one of the numbers $v(a)$, $v(b)$ is the maximum and the other is the minimum of these two numbers, therefore $| v(a) - v(b) | \leqq 1$. On the other hand, if $| v(a) - v(b) | \leqq 1$, then $\max(v(a), v(b)) - \min(v(a), v(b)) \leqq 1$ and aTb. We have proved that aTb if and only if $| v(a) - v(b) | \leqq 1$. Now let x_1, x_2, y_1, y_2 be four elements of L such

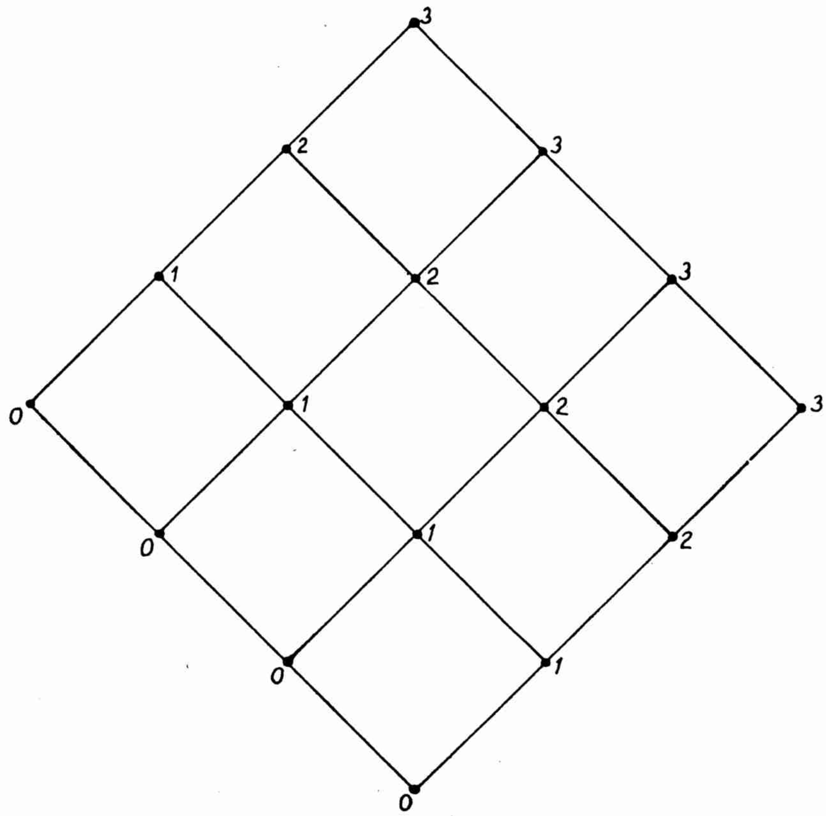

Fig. 1

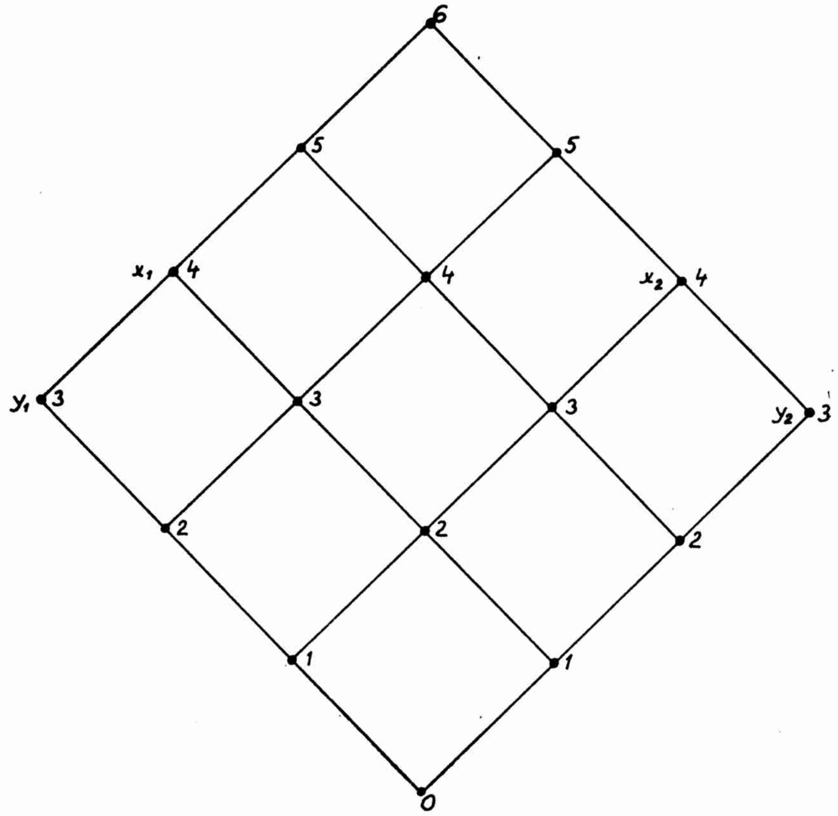

Fig. 2

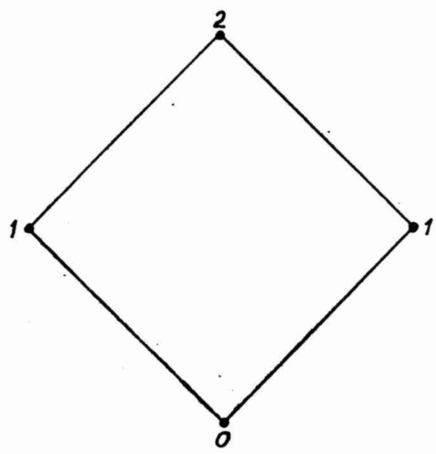

Fig. 3