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A UNIFORM STRUCTURE FOR TOPOLOGICAL SPACES

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We introduce below a uniform structure for topological spaces in order to make it possible to state and prove the classical theorem concerning the fact that a continuous function on a compact space is uniformly continuous.

It is believed that the present notion of a uniform structure is less restrictive than the one introduced in [1] and which was further studied in [2], [3], [4].

Let X be a topological space and I an index set. Let $(\mathscr{A}_i)_{i\in I}$ be a family of open covers $\mathscr{A}_i$ of X. We call an element of $\mathscr{A}_i$ an i-*neighborhood*. Let S be a subset of X. We call $E_i(S)$ the i-*extended neighborhood* of S if and only if:

$$E_i(S) = U\{A \mid A \in \mathscr{A}_i \textit{ and } A \cap S \neq \emptyset\}$$

i.e., $E_i(S)$ is the union of all the i-neighborhoods of every element of S.

Based on the above notions, we introduce:

Definition. *A family $(\mathscr{A}_i)_{i\in I}$ of open covers $\mathscr{A}_i$ of a topological space X is called a uniform structure for X if and only if:*

(1) *For every open set V of X and every $x \in V$ there exists an i-neighborhood A_i such that $x \in A_i$ and $E_i(A_i) \subseteq V$ and*

(2) *For every $i \in I$ and $h \in I$ there exists a $k \in I$ such that $\mathscr{A}_k$ refines both $\mathscr{A}_i$ and $\mathscr{A}_h$.*

Next, we prove:

Theorem. *Let f be a continuous mapping from a compact topological space X with a uniform structure $(\mathscr{A}_i)_{i\in I}$ into a topological space Y with a uniform structure $(\mathscr{B}_j)_{j\in J}$. Then for every $j \in J$ there exists a $k \in I$ such that every* k-*neighborhood is mapped by f into some* j-*neighborhood (i.e., f is uniformly continuous).*

Proof. Let $j \in J$ be given. Since f is continuous, clearly, for every $x \in X$ there exists an open set V of X such that $x \in V$ and V is mapped by f into a j-neighborhood. But then, from (1) it follows that there exists an $i(x)$-neighborhood $A_{i(x)}$ such that

(3) *$i(x) \in I$ and $x \in A_{i(x)}$ and $E_{i(x)}(A_{i(x)}) \subseteq V$.*

Clearly, $(A_{i(x)})_{x\in X}$ is an open cover of X and since X is compact, X is covered by finetely many members, say, $A_{i(x_1)}, A_{i(x_2)}, \ldots, A_{i(x_n)}$ of the cover. But then by (2) there exists a $k \in I$ such that $\mathscr{A}_k$ refines $\mathscr{A}_{i(x_1)}, \mathscr{A}_{i(x_2)}, \ldots, \mathscr{A}_{i(x_n)}$. To complete the proof, we show that every k-neighborhood is mapped by f into some j-neighborhood. Indeed, let A_k be a k-neighborhood and $x \in A_k$. Clearly, x is covered by one of the abovementioned finitely many open sets, say, $A_{i(x_m)}$. Thus, $x \in A_{i(x_m)}$ and in view of the refinement mentioned above $A_k \subseteq E_{i(x_m)}(A_{i(x_m)})$. But then from (3) it follows that $E_{i(x_m)}(A_{i(x_m)}) \subseteq V$. Consequently, $E_{i(x_m)}(A_{i(x_m)})$ as well as A_k is mapped by f into a j-neighborhood, as desired.

Remark. We observe that condition (II) of [1] could be replaced by (2) above.

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