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DAS ITERATIONSVERFAHREN FÜR EINE PARTIELLE DIFFERENTIALGLEICHUNG VIERTER ORDNUNG

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Das Ableiten der partiellen Differentialgleichung der Längs- oder Torsionsschwingungen von Stäben führt bei der Benutzung der energetischen Methode und wenn wir dabei die Deformation des Querschnittes in seiner Ebene in Erwägung ziehen, zu einer partiellen Differentiagleichung vierter Ordnung [1], die man in der Form

$$u_{xxtt} = A(t, x)\, u_{xx} + B(t, x)\, u_{tt} \tag{1}$$

schreiben kann.

Diese Differentialgleichung wurde für gewisse Nebendingungen gelöst [3], [4], wobei das Iterationsverfahren benützt wurde. In den folgenden Absätzen ist für eine allgemeinere Form der partiellen Differentialgleichung vierter Ordnung unter gewissen Nebenbedingungen ein Existenzsatz mittles des Iterationsverfahrens bewiesen und die Eindeutigkeit der Lösung bestimmt, wobei die Differentialgleichung (1) ein Sonderfall dieser Differentialgleichung ist.

Satz 1. Der Existenzsatz.

Es sei die Differentialgleichung

$$u_{xxtt} = f(t, x, u, u_{xx}, u_{tt}) \tag{2}$$

gegeben. Die Funktion $f(t, x, u, v, w)$ sei in dem Gebiete $D = \{\alpha < x < \beta; \gamma < t < \delta\}$ und für beliebige u, v, w stetig und erfülle in jedem kompakten Teilgebiete $D_0 \subset D$ in Bezug auf u, v, w die Lipschitz-Bedingung

$$\begin{aligned} &| f(t, x, u_2, v_2, w_2) - f(t, x, u_1, v_1, w_1) | \leqq \\ &\leqq M(| u_2 - u_1 | + | v_2 - v_1 | + | w_2 - w_1 |). \end{aligned} \tag{3}$$

Fener sei $\alpha < \xi_0 \leqq \xi_1 < \beta$, $\gamma < \tau < \delta$, und die Nebenbedingungen seien längs der Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$, so vorgeschrieben, dass die partielle Differentialgleichung

$$u_{xxtt} = 0 \tag{4}$$

genau eine Lösung in dem Gebiete D hat, die diese Nebenbedingungen erfüllt, die der Funktionenklasse $C^2(D)$ gehört und die wir $u_0(t, x)$ bezeichnen.

Dann existiert mindestens eine Lösung der partiellen Differentialgleichung (2), *die den vorgeschriebenen Nebenbedingungen genügt und man erhält diese Lösung mittels des Iterationsverfahrens von Picard als den Grenzwert der Folge*

$$u_n(t, x) = u_0(t, x) + F_{n-1}(t, x), \tag{5}$$

wo die Bezeichnung

$$F_k(t, x) = \int_{\tau}^{t}\int_{\tau}^{t_1}\int_{\xi_0}^{x}\int_{\xi_1}^{x_1} f\left(t_2, x_2, u_k, \frac{\partial^2 u_k}{\partial x_2^2}, \frac{\partial^2 u_k}{\partial t_2^2}\right) dx_2\, dx_1\, dt_2\, dt_1 \tag{6}$$

benützt wurde.

Beweis. Als erste Annäherung wählen wir die Funktion $u_0(t, x)$, die den Voraussetzungen nach gegebene Nebenbedingungen erfüllt und der Funtionenklasse $C^2(D)$ gehört.

Sei $u_{n-1}(t, x)$ die $(n-1)$-te Näherungsfunktion der gesuchten Lösung, die der Klasse $C^2(D)$ gehört. Dann gehört wegen der Stetigkeit des Integranden auch die n-te Näherungsfunktion

$$u_n(t, x) = u_0(t, x) + F_{n-1}(t, x)$$

der Klase $C^2(D)$.

Beweisen wir, dass die Folge von Näherungsfunktionen $u_n(t, x)$ in jedem kompakten Teilgebiete $D_0 \subset D$, wo $D_0 = \{\alpha_0 \leqq x \leqq \beta_0;\ \gamma_0 \leqq t \leqq \delta_0\}$ ist und $\alpha_0 \leqq \leqq \xi_0 \leqq \xi_1 \leqq \beta_0$, $\gamma_0 \leqq \tau \leqq \delta_0$ gilt, gleichmässig konvergiert.

Die Funktion $f\left(t, x, u_0, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial t^2}\right)$ ist in D_0 stetig und damit auch beschränkt. Es sei $|f| \leqq A$; dann ist für $n = 1$:

$$|u_1(t, x) - u_0(t, x)| = |F_0(t, x)| \leqq \frac{1}{4} A\, |t - \tau|^2\, |x - \xi_0|\, |x - \xi_1|,$$

$$\left|\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2}\right| = |F_{0xx}(t, x)| \leqq \frac{1}{2} A\, |t - \tau|^2,$$

$$\left|\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right| = |F_{0tt}(t, x)| \leqq \frac{1}{2} A\, |x - \xi_0|\, |x - \xi_1|,$$

wo die Bezeichnung

$$\begin{aligned} F_{kxx}(t, x) &= \int_{\tau}^{t}\int_{\tau}^{t_1} f\left(t_2, x, u_k, \frac{\partial^2 u_k}{\partial x^2}, \frac{\partial^2 u_k}{\partial t_2^2}\right) dt_2\, dt_1 \\ F_{ktt}(t, x) &= \int_{\xi_0}^{x}\int_{\xi_1}^{x} f\left(t, x_2, u_k, \frac{\partial^2 u_k}{\partial x_2^2}, \frac{\partial^2 u_k}{\partial t^2}\right) dx_2\, dx_1 \end{aligned} \tag{7}$$

benützt wurde.

Es sei entweder $\xi = \xi_0$ wenn $|x - \xi_0| \geqq |x - \xi_1|$,
oder $\xi = \xi_1$ wenn $|x - \xi_0| \leqq |x - \xi_1|$

für $\alpha_0 \leqq x \leqq \beta_0$.

Dann ist $|x - \xi_1|\, |x - \xi_0| \leqq |x - \xi|^2$.

Es sei ferner $2K = \max\{2, [(\beta_0 - \alpha_0) + (\delta_0 - \gamma_0)]^2\}$. Dann ist

$$(|x - \xi| + |t - \tau|)^2 \leqq 2K \tag{8}$$

und gelten die Beziehungen

$$
(9)\qquad
\begin{aligned}
&|\, u_1(t, x) - u_0(t, x) \,| \leqq \frac{1}{2} AK(|\, t - \tau \,| + |\, x - \xi \,|)^2 \\
&\left| \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} \right| \leqq \frac{1}{2} AK(|\, t - \tau \,| + |\, x - \xi \,|)^2 \\
&\left| \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} \right| \leqq \frac{1}{2} AK(|\, t - \tau \,| + |\, x - \xi \,|)^2
\end{aligned}
$$

Wir beweisen nun, dass für beliebige natürliche Zahl n folgende Beziehungen

$$
(10)\qquad
\begin{aligned}
&|\, u_n(t, x) - u_{n-1}(t, x) \,| \leqq \frac{A}{3M} \frac{(3MK)^n}{(2n)!} (|\, t - \tau) + |\, x - \xi \,|)^{2n} \\
&\left| \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_{n-1}}{\partial x^2} \right| \leqq \frac{A}{3M} \frac{(3MK)^n}{(2n)!} (|\, t - \tau \,| + |\, x - \xi \,|)^{2n} \\
&\left| \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_{n-1}}{\partial t^2} \right| \leqq \frac{A}{3M} \frac{(3MK)^n}{(2n)!} (|\, t - \tau \,| + |\, x - \xi \,|)^{2n}
\end{aligned}
$$

gelten, wobei M die Lipschitz-Konstante ist.
Für $n = 1$ ist die Gültigkeit der Beziehungen (10) wahr, da sie mit den Beziehungen (9) übereinstimmen.

Nehmen wir an, dass (10) für $n = m$ gilt; dann ist für $n = m + 1$:

$$
\begin{aligned}
|\, u_{m+1} - u_m \,| &= |\, F_m(t, x) - F_{m-1}(t, x) \,| \\
&\leqq M \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} \left\{ |\, u_m - u_{m-1} \,| + \left| \frac{\partial^2 u_m}{\partial x_2^2} - \frac{\partial^2 u_{m-1}}{\partial x_2^2} \right| + \right. \\
&\qquad \left. + \left| \frac{\partial^2 u_m}{\partial t_2^2} - \frac{\partial^2 u_{m-1}}{\partial t_2^2} \right| \right\} \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\leqq \frac{A}{3M} \frac{(3MK)^m}{(2m)!} \cdot 3M \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} (|\, t_2 - \tau \,| + |\, x_2 - \xi \,|)^{2m} \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\leqq \frac{A}{3M} \frac{(3MK)^{m+1}}{(2m+2)!} (|\, t - \tau \,| + |\, x - \xi \,|)^{2m+2}
\end{aligned}
$$

Ferner ist für die zweite partielle Ableitung nach x:

$$
\begin{aligned}
\left| \frac{\partial^2 u_{m+1}}{\partial x^2} - \frac{\partial^2 u_m}{\partial x^2} \right| &= |\, F_{mxx}(t, x) - F_{m-1,\, xx}(t, x) \,| \\
&\leqq M \int_\tau^t \int_\tau^{t_1} \left\{ |\, u_m - u_{m-1} \,| + \left| \frac{\partial^2 u_m}{\partial x^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} \right| + \left| \frac{\partial^2 u_m}{\partial t_2^2} - \frac{\partial^2 u_{m-1}}{\partial t_2^2} \right| \right\} \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\leqq \frac{A}{3M} \frac{(3MK)^{m+1}}{(2m+2)!} (|\, t - \tau \,| + |\, x - \xi \,|)^{2m+2}
\end{aligned}
$$

Ähnlich erhalten wir für die zweite partielle Ableitung nach t:

$$\left| \frac{\partial^2 u_{m+1}}{\partial t^2} - \frac{\partial^2 u_m}{\partial t^2} \right| = | F_{mtt}(t, x) - F_{m-1, tt}(t, x) |$$

$$\leqq \frac{A}{3M} \frac{(3MK)^m}{(2m)!} . 3M \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} (| t - \tau | + | x_2 - \xi |)^{2m} \, dx_2 \, dx_1$$

$$\leqq \frac{A}{3M} \frac{(3MK)^{m+1}}{(2m+2)!} (| t - \tau | + | x - \xi |)^{2m+1}$$

Die Folge der Funktionen

$$u_n(t, x) = u_0(t, x) + [u_1(t, x) - u_0(t, x)] + [u_2(t, x) - u_1(t, x)] + \ldots + [u_n(t, x) - \\ - u_{n-1}(t, x)]$$

und die Folgen ihrer zweiten partiellen Ableitungen nach x und t konvergieren also für $n \to \infty$ gleichmässig in jedem kompakten Teilgebiete $D_0 \subset D$ gegen eine Funktion $u(t, x)$ und gegen ihre partielle Ableitungen $u_{xx}(t, x)$, $u_{tt}(t, x)$. Da D_0 ein beliebiges Teilrechteck ist, ist die Funktion $u(t, x)$ in dem ganzen Gebiete D vorhanden und einschliesslich ihrer zweiten partiellen Ableitungen nach x und t stetig.

Wenn wir in (5) zum Grenzwert für $n \to \infty$ übergehen, erhalten wir

$$\text{(11)} \qquad u(t, x) = u_0(t, x) + \int_{\tau}^{t} \int_{\tau}^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f(t_2, x_2, u, u_{xx}, u_{tt}) \, dx_2 \, dx_1 \, dt_2 \, dt_1.$$

Wegen der Voraussetzungen über die Funktion $f(t, x, u, v, w)$ existiert auch die vierte partielle Ableitung u_{xxtt}, ist stetig und genügt der Differentialgleichung (2). Da die Funktion $u_0(t, x)$ die vorgeschriebenen Nebenbedingungen längs der Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$ erfüllt, erfüllt sie auch die Funktion $u(t, x)$, wie aus der Beziehung (11) klar ist.

Satz 2. *Unter den Voraussetzungen des Satzes 1. gibt es in dem Gebiete D genau eine Lösung der partiellen Differentialgleichung (2), die den vorgeschriebenen Nebenbedingungen längs der Charakteristiken $x = \xi_0$, $x = \xi_1$, $t = \tau$ genügt.*

Beweis. Wir nehmen an, dass die Differentialgleichung (2) zwei Lösungen $u(t, x)$ und $\bar{u}(t, x)$ hat, die als Grenzwert der Folge der Näherungsfunktionen von Picard (5) mit derselben Anfangsfunktion $u_0(t, x)$ geschaffen sind.

Es ist also

$$\text{(12)} \qquad \begin{aligned} u(t, x) &= u_0(t, x) + \int_{\tau}^{t} \int_{\tau}^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f(t_2, x_2, u, u_{xx}, u_{tt}) \, dx_2 \, dx_1 \, dt_2 \, dt_1, \\ \bar{u}(t, x) &= u_0(t, x) + \int_{\tau}^{t} \int_{\tau}^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f(t_2, x_2, \bar{u}, \bar{u}_{xx}, \bar{u}_{tt}) \, dx_2 \, dx_1 \, dt_2 \, dt_1. \end{aligned}$$

Wir werden beweisen, dass diese Lösungen in jedem kompakten Teilrechteck $D_0 \subset D$, in dem die Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$ liegen, übereinstimmen.

Wegen der Lipschitz-Bedingung ist

$$(13)\qquad |f(t, x, u, u_{xx}, u_{tt}) - f(t, x, \bar{u}, \bar{u}_{xx}, \bar{u}_{tt})| \leqq M(u - \bar{u}| + |u_{xx} - \bar{u}_{xx}| + + |u_{tt} - \bar{u}_{tt}|).$$

Wir führen die Funktion

$$(14)\qquad \omega(t, x) = (|u - \bar{u}| + |u_{xx} - \bar{u}_{xx}| + |u_{tt} - \bar{u}_{tt}|)\, e^{-E}$$

ein, wobei $E = K(|x - \xi| + |t - \tau|)$ ist, ξ dieselbe Bedeutung wie in dem Beweise des Satzes 1. hat, und $K = \left(\frac{\sqrt{2M}}{\sqrt{1 + 2M} - \sqrt{2M}}\right)^{1/2}$ ist.

Wenn wir dieselbe Bezeichnung wie in dem Satze 1. benützen, wobei der Querstrich den Wert mit der Funktion $\bar{u}(t, x)$ bedeutet, gelten wegen (12) und (13) die Beziehungen

$$\begin{aligned}\omega(t, x) &= e^{-E}\{|F(t, x) - \bar{F}(t, x)| + |F_{xx}(t, x) + \bar{F}_{xx}(t, x)| + \\ &\quad + |F_{tt}(t, x) - \bar{F}_{tt}(t, x)|\} \\ &\leqq Me^{-E}\{\int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^x \int_{\xi_1}^{x_1} e^E\, \omega(t_2, x_2)\, dx_2\, dx_1\, dt_2\, dt_1 + \\ &\quad + \int_\tau^t \int_\tau^{t_1} e^E \omega(t_2, x)\, dt_2\, dt_1 + \int_{\xi_0}^x \int_{\xi_1}^{x_1} e^E \omega(t, x_2)\, dx_2\, dx_1\}\end{aligned}$$

Wenn wir $\mu = \max\limits_{(t,x) \in D_0} \omega(t, x)$ bezeichnen, ist

$$\begin{aligned}\omega(t, x) &\leqq M\, e^{-E} \mu \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^x \int_{\xi_1}^{x_1} e^E\, dx_2\, dx_1\, dt_2\, dt_1 + \int_\tau^t \int_\tau^{t_1} e^E\, dt_2\, dt_1 + \int_{\xi_0}^x \int_{\xi_1}^{x_1} e^E\, dx_2\, dx_1\} \\ &\leqq M\, e^{-E}\mu \,.\left(\frac{2\, e^E}{K^2} + \frac{e^E}{K^4}\right) = \frac{1}{2}\, \mu.\end{aligned}$$

Weil $\omega(t, x) \leqq \frac{1}{2} \mu$ in ganzem Gebiete D_0 gilt, ist auch $\max\limits_{(t,x) \in D_0} \omega(t, x) = \mu \leqq \frac{1}{2} \mu$. Daraus folgt, dass $\mu = 0$, damit auch $\omega(t,x) = 0$ und in beliebigem kompaktem Teilrechteck D_0 auch $u(t, x) = \bar{u}(t, x)$ gilt.

Nehmen wir nun an, dass die Funktion f auf der rechten Seite der Differentialgleichung (2) nicht von u abhängt. Dann gelten folgende zwei Sätze, wobei wir dieselbe Bezeichnungen wie in den vorangehenden Absätzen benützen.

Satz 3. *Es sei die Differentialgleichung*

$$(15)\qquad u_{xxtt} = f(t, x, u_{xx}, u_{tt})$$

gegeben. Die Funktion $f(t, x, v, w)$ sei in dem Gebiete D und für beliebige v, w stetig und erfülle in jedem kompakten Teilgebiete $D_0 \subset D$ in Bezug auf v, w die Lipschitz-Bedingung

$$(16)\qquad |f(t, x, v_2, w_2) - f(t, x, v_1, w_1)| \leqq M(|v_2 - v_1| + |w_2 - w_1|).$$

Ferner seien die Voraussetzungen über die Lösung $u_0(t, x)$ der Differentialgleichung (4) des Satzes 1. erfüllt.

Dann existiert mindestens eine Lösung der partiellen Differentialgleichung (15), die den vorgeschriebenen Nebenbedingungen längs der Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$ genügt und die der Grenzwert der Folge von Näherungsfunktionen

$$(17)\qquad u_n(t, x) = u_0(t, x) + \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f\left(t_2, x_2, \frac{\partial^2 u_{n-1}}{\partial x_2^2}, \frac{\partial^2 u_{n-1}}{\partial t_2^2}\right) \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1$$

ist.

Der Beweis dieses Satzes ist durchaus derselbe, wie des Satzes 1., nur anstatt der Beziehungen (10) beweisen wir, dass für die n-te Näherungsfunktion die Beziehungen

$$(18)\qquad \begin{aligned} | u_n(t, x) - u_{n-1}(t, x) | &\leqq \frac{A}{2M} \frac{(2MK)^n}{(2n)!} (| x - \xi | + | t - \tau |)^{2n} \\ \left| \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_{n-1}}{\partial x^2} \right| &\leqq \frac{A}{2M} \frac{(2MK)^n}{(2n)!} (| x - \xi | + | t - \tau |)^{2n} \\ \left| \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_{n-1}}{\partial t^2} \right| &\leqq \frac{A}{2M} \frac{(2MK)^n}{(2n)!} (| x - \xi | + | t - \tau |)^{2n} \end{aligned}$$

gelten.

Weil wir für die Funktion f die Lipschitz-Bedingung nur in Bezug auf v und w voraussetzen, gilt für die Eindeutigkeit der Lösung folgender Satz:

Satz 4. *Mittels der Anfangs- und Randbedingungen längs der Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$ seien die Werte $u(\tau, x)$, $u_t(\tau, x)$, $u(t, \xi_0)$, $u_x(t, \xi_0)$ in den Intervallen $I_1 = \{\alpha < x < \beta\}$, $I_2 = \{\gamma < t < \delta\}$ eindeutig gegeben und so, dass diese Funktionen der Klasse $C^2(I_1)$ und $C^2(I_2)$ gehören. Dann erfüllt die Funktion $u_0(t, x)$, die das Integral der Differentialgleichung (4) ist, das diesen Anfangs- und Randbedingungen genügt, die Voraussetzungen des Satzes 1.*

Es seien ferner die Voraussetzungen des Satzes 3. über die Funktion $f(t, x, v, w)$ erfüllt. Dann hat die Differentialgleichung (15) genau eine Lösung, die als Grenzwert der Funktionen $u_n(t, x)$ in (17) für $n \to \infty$ gegeben ist und die den vorgeschriebenen Anfangs- und Randbedingungen genügt.

Beweis. Wie in dem Satze 2. nehmen wir an, dass die Differentialgleichung (15) zwei Lösungen

$$(19)\qquad \begin{aligned} u(t, x) &= u_0(t, x) + \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f(t_2, x_2, u_{xx}, u_{tt}) \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\ \bar{u}(t, x) &= u_0(t, x) + \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^{x} \int_{\xi_1}^{x_1} f(t_2, x_2, \bar{u}_{xx}, \bar{u}_{tt}) \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \end{aligned}$$

hat, für welche wegen der Voraussetzungen über die Anfangs- und Randbedingungen in dem Gebiete D gilt:

$$(20)\qquad \begin{aligned} u(\tau, x) &= \bar{u}(\tau, x); \qquad & u_t(\tau, x) &= \bar{u}_t(\tau, x) \\ u(t, \xi_0) &= \bar{u}(t, \xi_0); \qquad & u_x(t, \xi_1) &= \bar{u}_x(t, \xi_1) \end{aligned}$$

Wir führen die Funktion

$$(21)\qquad \omega(t, x) = (| u - \bar{u} | + | u_{xx} - \bar{u}_{xx} | + | u_{tt} - \bar{u}_{tt} |) \, \mathrm{e}^{-E}$$

ein, wobei $E = K(|x - \xi| + |t - \tau|)$, ξ dieselbe Bedeutung wie in dem Satze 1. hat und $K = 2\sqrt{M}$ ist.

Aus den Beziehungen (19), (21), und aus der Lipschitz-Bedingung (16) erhalten wir für die Funktion $\omega(t, x)$ die Beziehung

$$\omega(t, x) \leqq M\, e^{-E} \{ \int_\tau^t \int_\tau^{t_1} \int_{\xi_0}^x \int_{\xi_1}^{\xi_1} (|u_{xx} - \bar{u}_{xx}| + |u_{tt} - \bar{u}_{tt}|)\, dx_2\, dx_1\, dt_2\, dt_1 +$$

$$+ \int_\tau^t \int_\tau^{t_1} (|u_{xx} - \bar{u}_{xx}| + |u_{tt} - \bar{u}_{tt}|)\, dt_2\, dt_1 +$$

$$+ \int_{\xi_0}^x \int_{\xi_1}^{x_1} (|u_{xx} - \bar{u}_{xx}| + |u_{tt} - \bar{u}_{tt}|)\, dx_2\, dx_1$$

Mit Rücksicht auf (20) ist

$$\omega(t, x) \leqq M\, e^{-E} \{ \int_\tau^t \int_\tau^{t_1} \omega(t_2, x)\, e^E\, dt_2\, dt_1 + \int_{\xi_0}^x \int_{\xi_1}^{x_1} \omega(t, x_2)\, e^E\, dx_2\, dx_1 \}$$

Wenn wir $\mu = \max\limits_{(t,x) \in D_0} \omega(t, x)$ bezeichnen, erhalten wir, dass in dem ganzen kompakten Teilrechteck D_0 die Beziehung

$$\omega(t, x) \leqq M\mu \frac{2}{K^2} = \frac{1}{2}\mu$$

gilt. Wie in dem Satze 2. folgt daraus, dass $\omega(t, x) = 0$ in beliebigem kompaktem Gebiete D_0 ist und damit auch $u(t, x) = \bar{u}(t, x)$ in dem ganzen Gebiete D gilt.

Satz 5. *Es seien die Funktionen $\varphi_0(x)$, $\varphi_1(x)$ und $\psi_0(t)$, die der Klasse $C^2(I_1)$ und $C^2(I_2)$ gehören, und in dem Intervalle I_2 stetige Funktion $\psi_1(t)$ gegeben. Ferner sei $\alpha < \xi_0 < \xi_1 < \beta$, $\gamma < \tau < \delta$ und so, dass*

$$\varphi_0(\xi_0) = \psi_0(\tau), \qquad \varphi_1(\xi_0) = \psi_0'(\tau)$$

gilt, und die Funktion $A(t, x)$ für $x = \xi_1$ in dem Intervalle I_2 stetig ist.
Dann gibt es in dem Gebiete D genau eine Lösung $u_0(t, x)$ der Differentialgleichung (4), die längs der Charakteristiken $t = \tau$, $x = \xi_0$, $x = \xi_1$ die Anfangsbedingungen

$$u(\tau, x) = \varphi_0(x), \qquad u_t(\tau, x) = \varphi_1(x) \tag{22}$$

und die Randbedingungen

$$u(t, \xi_0) = \psi_0(t) \tag{23}$$

$$u_{xtt}(t, \xi_1) - A(t, \xi_1)\, u_x(t, \xi_1) = \psi_1(t) \tag{24}$$

erfüllt. Diese Lösung erfüllt die Voraussetzungen des Satzes 1.

Ferner seien die übrigen Voraussetzungen des Satzes 3. über die Funktion

$$f(t, x, u_{xx}, u_{tt}) = A(t, x)\, u_{xx} + B(t, x)\, u_{tt}$$

erfüllt. Dann gibt es in dem Gebiete D genau eine Lösung der Differentialgleichung (1), die die Anfangsbedingungen (22) und die Randbedingungen (23), (24) erfüllt. Diese

Lösung erhalten wir als Grenzwert der Folge der Näherungsfunktionen (17), wo die Funktion $u_0(t, x)$ in der Form

$$\begin{aligned} u_0(t, x) = \varphi_0(x) + \psi_0(t) - \psi_0(\tau) + (t - \tau)\,[\varphi_1(x) - \varphi_1(\xi_0)] + \\ + (x - \xi_0)\,[g(t) - \varphi_0'(\xi_1)] - (x - \xi_0)\,(t - \tau)\,\varphi_1'(\xi_1) \end{aligned} \tag{25}$$

gegeben ist, wobei die Funktion $g(t)$ eine Lösung der Differentialgleichung

$$g''(t) - A(t, \xi_1)\,g(t) = \psi_1(t) \tag{26}$$

ist, welche den Anfangsbedingungen

$$g(\tau) = \varphi_0'(\xi_1), \qquad g'(\tau) = \varphi_1'(\xi_1) \tag{27}$$

genügt.

Beweis. Durch direkte Integration der Differentialgleichung (4) erhalten wir, dass die Funktion $u_0(t, x)$, die in der Form (25) gegeben ist, die einzige Lösung dieser Gleichung ist, die die Nebenbedingungen (22), (23), (24) erfüllt. Wegen der Voraussetzungen über die Funktionen $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(t)$, $\psi_1(t)$, $A(t, \xi_1)$ ist diese Lösung der Funktionenklasse $C^2(D)$.

Da auch die übrigen Voraussetzungen der Sätze 3. und 4. erfüllt sind, gibt es genau eine Lösung der Differentialgleichung (1), die den Nebenbedingungen (22), (23), (24) genügt und die wir als Grenzwert der Folge der Näherungsfunktionen (17) erhalten, wobei für $u_0(t, x)$ die Funktion (25) gesetzt wird.

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SUR UNE CLASSE DE VARIÉTÉS PSEUDO-ISOTROPES

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INTRODUCTION

Une C^∞-variété immergée dans une variété lorentzienne V_L a été définie dans [9], [10], [11] comme étant *pseudo-isotrope* (notée V_{ps}) si la représentation sphérique par rapport à un vecteur normal spatial est *amétrique*. Dans le travail actuel on étudie des pareilles variétés de dimension ou codimension 2, immergées dans un espace de Minkowski $\mathcal{M}^{2+n}$. Dans le cas de l'immersion $x : V^2_{ps} \to \mathcal{M}^{2+n}$ on suppose en plus que toutes les formes de connexion normales spatiales associées à x sont identiquement nulles (une pareille connexion est dénommée une *connexion normale spatiale triviale*). L'immersion $x : V^2_{ps} \to \mathcal{M}^{2+n}$ met alors en évidence une 1-forme ω et un champ isotrope I dénommés respectivement *forme caractéristique* et *champ caractéristique* associé à x. Deux cas importants se présentent suivant que le $\mathcal{M}$-index V^2_{ps} est 1 ou 0. Dans le second cas ($\mathcal{M}$-index $V^2_{ps} = 0$) la variété V^2_{ps} est *cylindrique* et présente certaines analogies avec un type de variétés cylindriques étudié dans l'espace E^{2+n} par K. Shiohama [13].

Dans le cas de l'immersion $x : V^2_{ps} \to M^{2+n}$ on définit sur V^2_{ps} une certaine structure *presque cosymplectique* $\mathbf{C}_{pc}(\Omega, \alpha)$, *cosymplectique* $\mathbf{C}(\Omega, \alpha)$ si $n = 2m + 1$ ou *symplectique* $S(\Omega)$ si $n = 2m$. Différentes propriétés afférentes à ces structures qui font intervenir le champ I et la forme ω sont mises en évidence.

1. V^2_s étant une C^∞-variété spatiale 2-dimensionnelle (la métrique de V^2 est par définition définie négative), soit $x : V^2_s \to \mathcal{M}^{2+n}$ une immersion de V^2_s dans un espace de Minkowski $\mathcal{M}^{2+n}$ (de signature hyperbolique). Si $F(V^2_s)$ et $F(\mathcal{M}^{2+n})$ sont respectivement les faisceaux des repères orthonormés de V^2_s et $\mathcal{M}^{2+n}$, soit $B \subset V^2_s \times F(V^2_s)$ le *fibré principal* des repères adaptés tels que les vecteurs e_i $(i, j, k = 1, 2)$ soient tangents à V^2_s et les vecteurs e_{i^*} $(i^*, j^*, k^* = 3, 4, \ldots, n + 2)$ soient normaux en $x(p)$ $(p \in V^2_s)$. Notons par e_{r^*} $(r^*, s^*, t^* = 1, 2, \ldots, n + 1)$ les vecteurs *spatiaux* d'un repère quelconque $b \in B$ et par $T_p(V^2_s)$ le plan tangent en p à V^2_s. Si ω^i sont les formes duales de e_i, induites par x, on peut écrire

$$\mathrm{d}p = -\omega^i \otimes e_i \tag{1}$$

et V^2_s est alors structurée par la connexion

$$\begin{aligned} \nabla e_{r^*} &= \omega^A_{r^*} \otimes e_A, \\ \nabla e_{n+2} &= -\omega^{r^*}_{n+2} \otimes e_{r^*}, \end{aligned} \tag{2}$$

où $\omega_A^B = \gamma_{Ai}^B \omega^i (A, B, C = 1, 2, \ldots, n + 2)$ sont les formes de connexion induites par x. Eu égard à (1) et (2) le premier et le second groupe des équations de structure qui résultent de l'immersion x s'écrivent respectivement

$$(3) \qquad d \wedge \omega^i = \omega^j \wedge \omega_j^i,$$

$$(4) \qquad \begin{aligned} d \wedge \omega_{s^*}^{r^*} &= \varepsilon_A \omega_{s^*}^A \wedge \omega_A^{r^*}, \qquad (\varepsilon_{r^*} = 1, \varepsilon_{n+2} = -1) \\ d \wedge \omega_{s^*}^{n+2} &= \omega_{s^*}^{r^*} \wedge \omega_{r^*}^{n+2}. \end{aligned}$$

2. Une variété 2-dimensionnelle telle que la représentation sphérique par rapport à un vecteur normal genre espace soit *amétrique* (isotrope) a été définie dans [9] comme étant une *variété pseudo-isotrope*. Il est manifeste que l'on peut considérer différents types de pareilles variétés suivant que l'on impose à un ou plusieurs vecteurs normaux genre espace d'admettre un représentation sphérique amétrique. Dans ce travail nous étudierons le cas où pour *tous* les vecteursnormaux genre espace e_r $(r, s, t = 3, 4, \ldots, n + 1)$ on a

$$(5) \qquad \langle \nabla e_r, \nabla e_r \rangle = 0.$$

Nous supposons de plus que les formes de connexion normales spatiales ω_r^s associées à l'immersion x sont identiquement nulles. Nous dirons qu'une pareille variété est alors à *connexion normale spatiale triviale*. Une variété douée d'une pareille connexion et qui satisfait à (5) sera notée par V_{ps}^2.

Eu égard à la connexion (2) la relation de définition (5) permet de poser

$$(6) \qquad \begin{gathered} \omega_r^{n+2} = \omega_r, \\ \omega_1^r = \omega_r \cos \varphi_r, \qquad \omega_2^r = \omega_r \sin \varphi_r, \qquad \varphi_r \in \mathscr{D}(V_{ps}^2). \end{gathered}$$

La différentiation extérieure des équations

$$(7) \qquad \omega^r = 0,$$

$$(8) \qquad \omega_s^r = 0$$

donne

$$(9) \qquad \varphi_r = \varphi, \qquad \omega_r = t_r \omega, \qquad t_r \neq 0 \in \mathscr{D}(V_{ps}^2)$$

où l'on a posé

$$(10) \qquad \omega = \omega^1 \cos \varphi + \omega^2 \sin \varphi.$$

Les relations (2), (6), (8), (9) et (10) permettent d'écrire

$$(11) \qquad \nabla e_r = -t_r \omega \otimes I$$

où I est un *vecteur isotrope* (réel) défini par

$$(12) \qquad I = \cos \varphi e_1 + \sin \varphi e_2 - e_{n+2}.$$

L'égalité (12) montre que I est la différence du champ tangentiel $t = \cos \varphi e_1 + \sin \varphi e_2 \in T_p(V_{ps}^2)$ et du vecteur normal temporel. La forme ω et le vecteur I seront dénommés respectivement la *forme caractéristique* et le *vecteur caractéristique* associés à l'immersion $x : V_{ps}^2 \to \mathscr{M}^{2+n}$.

3. Comme on peut toujours diagonaliser l'une des secondes formes fondamentales associées à l'immersion d'une variété dans un espace quelconque, nous supposons que cette opération est effectuée pour la seconde forme fondamentale correspondant au vecteur temporel. Dans ce cas on peut écrire

$$\omega_1^{n+2} = \lambda\omega^1, \qquad \omega_2^{n+2} = \mu\omega^2, \qquad \lambda, \mu \in \mathscr{D}(V_{ps}^2) \tag{13}$$

et eu égard à (13) la différentiation extérieure de (6) donne:

$$\begin{aligned}(\omega_1^2 + \mathrm{d}\varphi) \wedge \omega + (\lambda\omega^1 \sin\varphi - \mu\omega^2 \cos\varphi) \wedge \omega. \\ (\mathrm{d} \ln t_r - \lambda\omega^1 \cos\varphi - \mu\omega^2 \sin\varphi) \wedge \omega + d \wedge \omega = 0.\end{aligned} \tag{14}$$

Il est facile de voir qu'en général on a

$$\begin{aligned} d \wedge \omega &= (\omega_1^2 + \mathrm{d}\varphi) \wedge {}_*\omega, \\ d \wedge {}_*\omega &= (\omega_1^2 + \mathrm{d}\varphi) \wedge \omega, \end{aligned} \tag{15}$$

où

$${}_*\omega = -\omega^1 \sin\varphi + \omega^2 \cos\varphi$$

est la *forme adjointe* de ω par rapport à la métrique de V_s^2. Ces équations montrent aussitôt:

$$\omega_1^2 + \mathrm{d}\varphi = 0 \Leftrightarrow \Delta\omega = 0 \tag{16}$$

et à l'aide de (14) il résulte

$$\lambda + \mu = 0. \tag{17}$$

Cette relation exprime que si (16) à lieu, alors la variété V_{ps}^2 est *minimale* dans la direction du vecteur normal temporel, d'où la

Proposition. *Si la forme caractéristique d'une variété V_{ps}^2 est harmonique, alors la variété est plate et minimale dans la direction du vecteur normal temporel.*

Remarques. 1. Considérons la *structure symplectique* $S(V_{ps}^2, \Omega)$ définie sur V_{ps}^2 par la 2-forme $\Omega = \omega^1 \wedge \omega^2$ (élément d'aire) et les isomorphismes $X \in T(V_{ps}^2) \to X \lrcorner \Omega \in \wedge^1 (V_{ps}^2)$ définis par $\omega = h \lrcorner \Omega$ et ${}_*\omega = {}_*h \lrcorner \Omega$. Le *crochet de Poisson* relativement à $S(V_{ps}^2, \Omega)$, étant comme on sait (le crochet de Poisson s'obtient en transportant le crochet de Lie de $T(V_{ps}^2)$ dans $\wedge^1(V_{ps}^2)$) défini par [3]

$$(\omega, {}_*\omega) = [h, {}_*h] \lrcorner \Omega \tag{18}$$

on trouve après calculs

$$(\omega, {}_*\omega) = -{}_*(\omega_1^2 + \mathrm{d}\varphi). \tag{19}$$

De la il résulte

$$(\omega, {}_*\omega) = 0 \Leftrightarrow \omega_1^2 + \mathrm{d}\varphi = 0 \tag{20}$$

et par conséquent le crochet de Poisson de la forme caractéristique et son adjointe est nulle si et seulement si cette forme est *harmonique*.

2. Une forme α sur une variété différentiable compacte définit une *transformation infinitésimale conforme* si le tenseur $t(\alpha) = 0$ (voir Lichnerowicz [4]). Puisque l'on a (sous forme invariante)

$$t(\alpha)_{ij} = D_j\alpha_i + D_i\alpha_j + \frac{2}{n} g_{ij}\delta\alpha, \tag{21}$$

$$D_i\alpha_j = \frac{\partial\alpha_j}{\partial\omega^i} - \gamma_{ji}^k\alpha_k, \tag{22}$$

en effectuant le calcul pour une V_s^2 compacte et pour la forme ω on trouve

$$t(\omega)_{ij} = 0 \Leftrightarrow \omega_1^2 + d\varphi = 0. \tag{23}$$

On peut donc dire que ω définit une transformation infinitésimale conforme si et seulement si ω est harmonique.

3. Si $\omega_1^2 + d\varphi = 0$ et $\lambda = \mu = 0$, alors on trouve après un calcul élémentaire que I est un *champ parallèle* [15].

4. Etudions maintenant certaines propriétés de rigidité d'une variété V_{ps}^2. Eu égard aux relations (1) et (2) les secondes formes fondamentales sont exprimées par

$$\varphi_r = -\langle dp, \nabla e_r\rangle = t_r\omega \otimes \omega, \tag{24}$$

$$\varphi_{n+2} = -\langle dp, \nabla e_{n+2}\rangle = -(\lambda\omega^1 \otimes \omega^1 + \mu\omega^2 \otimes \omega^2). \tag{25}$$

Ces expressions montrent que *l'invariant arithmétique* de Chern [2] est égal à deux ($\omega \neq 0$) et que les secondes formes fondamentales qui correspondent aux vecteurs spatiaux e_r sont *développables*. L'espace $\mathscr{M}^{2+n}$ étant plat et les courbures de Lipschitz—Killing correspondantes aux vecteurs e_r étant nulles, on trouve que la courbure intrinsèque K_{in} de la variété en p a pour expression

$$K_{in} = \lambda\mu. \tag{26}$$

Cette expression de K_{in} montre que si la variété V_{ps}^2 est minimale dans la direction du vecteur normal temporel e_{n+2} alors elle est *anti-convexe*. Si l'on suppose que la seconde forme fondamentale afférente à e_{n+2} est *conforme* à la métrique de V_{ps}^2, on trouve à l'aide de (13)

$$\lambda = \mu \to \lambda = \text{const.}$$

Dans ce cas la variété V_{ps}^2 est *convexe*.

Considérons maintenant le champ normal

$$X = \xi^r e_r + \xi^{n+2} e_{n+2} \in T_p^{\perp}(V_{ps}^2). \tag{27}$$

En nous rapportant à la définition donnée dans [6] du $\mathscr{M}$*-index* d'une variété immergée, on trouve dans le cas qui nous occupe

$$\mathscr{M}\text{-}index\ V_{ps}^2 = \dim \{\xi^r(\gamma_{ij}^r) + \xi^{n+2}(\gamma_{ij}^{n+2})\} \tag{28}$$

où

$$\sum_r \xi^r\gamma^r + \xi^{n+2}\gamma^{n+2} = 0, \tag{29}$$

γ^r, γ^{n+2} traces des matrices (γ^r_{ij}), (γ^{n+2}_{ij}). On obtient après calculs

$$\xi^r t_r + \xi^{n+2}(\lambda + \mu) = 0 \tag{30}$$

ce qui donne les deux cas:

1) $\lambda + \mu \neq 0$:

$$\mathscr{M}\text{-}index\ V^2_{ps} = \dim \xi^r t_r \left\{ \begin{pmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{pmatrix} - \frac{1}{\lambda + \mu} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right\}; \tag{31}$$

2) $\lambda + \mu = 0$:

$$\mathscr{M}\text{-}index\ V^2_{ps} = \dim \xi^{n+2} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}. \tag{32}$$

Il résulte donc:

1) $\mathscr{M}$-*index* $V^2_{ps} = 1$ si λ ou $\mu \neq 0$;
2) $\mathscr{M}$-*index* $V^2_{ps} = 0 \Leftrightarrow \lambda = \mu = 0$.

Nous énoncons la

Proposition. *Soit V^2_{ps} une variété spatiale, pseudo-isotrope par rapport à tous les vecteurs normaux spatiaux et à connexion normale spatiale triviale. Alors:*

(i) *l'invariant arithmétique de Chern associé a l'immersion est en général égal à deux;*
(ii) *les secondes formes fondamentales afférentes aux vecteurs normaux spatiaux sont développables et conformes entre elles;*
(iii) *si la seconde forme fondamentale afférente au vecteur normal temporel est conforme á la métrique de la variété, alors elle est nécessairement homothétique;*
(iv) *le $\mathscr{M}$-index $V^2_{ps} = 1$ pour λ* ou *$\mu \neq 0$ et le $\mathscr{M}$-index $V^2_{ps} = 0$ pour $\lambda = \mu = 0$.*

5. Soit $t = \cos \varphi e_1 + \sin \varphi e_2$ la composante tangentielle du vecteur caractéristique I (voir n° 2). Si nous considérons la structure symplectique $S(\Omega, V^2_{ps})$ on trouve que la *parenthése de Poisson pfaffienne* de ω (l'amplication p: $\wedge^1(V^2_{ps}) \to T(V^2_{ps})$, $\omega \to p(\omega)$ définie par $p(\omega) \lrcorner\ \Omega = -\omega$ [8]) est le champ

$$\bar{t} = -\sin \varphi e_1 + \cos \varphi e_2$$

orthogonal à t. Le *shape operator* $S_{\bar{t}}(t)$ des vecteurs t et $\bar{t}$ $(S_{\bar{t}}(t))$ opérateur bilinéaire auto-adjoint de $T_p(V^2_{ps})$ dans $T^{\perp}_p(V^2_{ps})$) étant défini par [5]

$$S_{\bar{t}}(t) = \gamma^{i^*}_{ij} \bar{t}^i t^j e_{i^*} \tag{33}$$

on trouve à l'aide de (6) et (13):

$$S_{\bar{t}}(t) = (\mu - \lambda)\, e_{n+2}.$$

Ainsi pour une V^2_{ps}, *le shape operator $S_{\bar{t}}(t)$ de la composante tangentielle du vecteur caractéristique I et de son orthogonal est colinéaire au vecteur normal temporel. En particulier, si la variété est sphérique on a $S_{\bar{t}}(t) = 0$.*

6. Soit l'application $\mathscr{A}$: $p \to \bar{p} = p + f e_{n+2}$ $(f \in \mathscr{D}(V^2_{ps}))$. A l'aide de (1) et (2) on trouve

$$d\bar{s} \otimes d\bar{s} = -(f\lambda - 1)^2 \omega^1 \otimes \omega^1 - (f\mu - 1)^2 \omega^2 \otimes \omega^2 + df \otimes df - -f^2 \Sigma\, t^2_r \omega \otimes \omega. \tag{34}$$

En posant $\Sigma\, t_r^2 = 1$ et en supposant que la seconde forme fondamentale afférente au vecteur normal temporel est homothétique à la métrique de la variété, on trouve à l'aide de (14) que ω est un *cobord*. Dans ce cas on peut toujours choisir la fonction f de manière à avoir

$$\mathrm{d}f \otimes \mathrm{d}f = f^2\omega \otimes \omega. \tag{35}$$

Dans ce cas on déduit de ce qui précède que l'application est conforme.

7. Dans le cas d'une variété V_{ps}^2 avec $\mathscr{M}$-*index* $V_{ps}^2 = 0$ la relation (26) montre alors que V_{ps}^2 est *plate*. Les courbes $\omega = 0$ sont des *asymptotiques doubles* et il est facile de voir à l'aide de (1) et (2) que ce sont des *droites*. On conclut de tout ce qui vient d'être dit que dans le cas considéré et si en outre V_{ps}^2 est *complète* alors elle peut être considérée comme une surface *cylindrique*.

En nous rapportant à l'application $\mathscr{A}$ du numéro précédent on voit que dans ce cas $\mathscr{A}$ est une *isométrie*, résultat qui pouvait être prévisible géométriquement. On a donc la

Proposition. *Si la variété est complète et de* $\mathscr{M}$-*index* $V_{ps}^2 = 0$, *alors c'est une variété cylindrique.*

8. Soit $X = c(\cos\varphi e_1 + \sin\varphi e_2) \in T_p(V_{ps}^2)$ un champ tangentiel homothétique au champ t. Conformément à la définition donnée dans [1] [12] [14] ce champ définit une *autotransformation infinitésimal équivalente* si

$$\Delta(p, X) = 0, \tag{36}$$

Δ étant le paramètre mixte de Beltrami relatif aux formes invariantes ω^1, ω^2. A l'aide de la connexion (1), (2) et de (15) on trouve que la relation (36) devient

$$(\omega_1^2 + \mathrm{d}\varphi) \wedge \omega = 0 \Leftrightarrow \delta\omega = 0 \tag{37}$$

d'où la

Proposition. *La condition nécessaire et suffisante pour que tout champ homothétique à la composante tangentielle du vecteur caractéristique* I *définisse une autotransformation infinitésimale équivalente de* V_{ps}^2 *est que la forme caractéristique soit cofermée.*

9. Envisageons maintenant le cas d'une variété pseudo-isotrope de *codimension 2*. Conformément aux notations du n° 1 les indices parcourent ici les valeurs suivantes:

$$\begin{gathered} 1 \leqq i, j, k \leqq n, \qquad n+1 \leqq i^*, j^*, k^* \leqq n+2, \\ 1 \leqq r^*, s^*, t^* \leqq n+1. \end{gathered} \tag{38}$$

La variété V_{ps}^n étant par hypothèse pseudo-isotrope on devra écrire

$$(\nabla e_{n+1})^2 = 0. \tag{39}$$

Eu égard à la connexion (2) cette relation permet de poser

$$\omega_i^{n+1} = \lambda_i\omega, \qquad \Sigma\,\lambda_i^2 = 1, \qquad \lambda_i \in \mathscr{D}(V_{ps}^n) \tag{40}$$

où $\omega = \omega_{n+1}^{n+2}$ est l'unique forme de connexion normale (ou forme de torsion) associée à l'immersion x. Mais V_{ps}^n étant une variété intégrale du système linéaire

$$\omega^{n+1} = \omega^{n+2} = 0, \tag{41}$$

on déduit par différentiation extérieure de (41)

(42) $$\omega = \gamma \lambda_i \omega^i$$

où $\gamma \in \mathscr{D}(V^n_{ps})$ est un facteur de proportionalité. Comme dans le cas d'une variété bidimensionnelle nous supposerons (ce qui est toujours possible) que la seconde forme fondamentale afférente au vecteur normal temporel est diagonale. On peut donc écrire

(43) $$\omega_i^{n+2} = \mu_i \omega^i, \qquad \mu_i \in \mathscr{D}(V^n_{ps}) \qquad \text{(ne pas sommer!)}.$$

D'autre part, à l'aide des équations de structure (3) et (4), on déduit de (40) par différentiation extérieure:

(44) $$\lambda_i d \wedge \omega = (\lambda_j \omega_i^j + \mu_i \omega^i - \mathrm{d}\lambda_i) \wedge \omega.$$

Compte tenu de (40) on déduit de (44) par un calcul élémentaire

(45) $$d \wedge \omega = \sum_i \lambda_i \mu_i \omega^i \wedge \omega.$$

En convenant d'appeler, comme au n° 2, ω la *forme caractéristique* de V^n_{ps}, la relation (45) montre bien que cette forme est *complètement intégrable*. Il est bon de signaler qu'un résultat analogue a été obtenu dans l'étude des hypersurfaces pseudo-isotropes immergées dans $\mathscr{M}^{n+2}$ [9].

10. Eu égard aux hypothèses faites au numéro précédent les deux secondes formes fondamentales associées à l'immersion x sont

(46) $$\begin{aligned} \varphi_{n+1} &= -\langle \mathrm{d}p, \mathrm{d}e_{n+1}\rangle = \frac{1}{\gamma}(\omega)^2, \\ \varphi_{n+2} &= -\langle \mathrm{d}p, \mathrm{d}e_{n+2}\rangle = -\Sigma\, \mu_i(\omega^i)^2. \end{aligned}$$

Comme dans le cas d'une surface V^2_{ps} on constate que la seconde forme fondamentale afférente au vecteur normal spatial est *développable*. De plus, il est facilede voir que $\frac{1}{n}\gamma$ est la *courbure moyenne* en p qui correspond à ce vecteur. Le vecteur isotrope caractéristique I s'écrit maintenant

(47) $$I = \Sigma\, \lambda_i e_i - e_{n+2}$$

et nous noterons la composante tangentielle de I aussi par t.

La différentiation extérieure de (43) démontre que si la seconde forme fondamentale afférente au vecteur normal temporel est conforme à la métrique de la variété V^n_{ps}, alors cette forme est nécessairement homothétique et un calcul élémentaire montre que dans ce cas la variété a une courbure sectionnelle constante. Il résulteaussi de (42) et (45) que ω est fermée si et seulement si V^n_{ps} est ombilicale dans la direction de e_{n+2}.

Dans le cas où la variété V^n_{ps} est *compacte*, la courbure moyenne (ou le facteur de proportionalité) est encore susceptible de l'interprétation de la formule intégrale suivante. Notons par η l'élément de volume de la variété. Le *carré scalaire global* de la forme caractéristique ω est alors défini par

(48) $$\langle \omega, \omega \rangle = \int_{V^n_{ps}} (\omega, \omega)\, \eta$$

où

$$\omega \wedge {}_*\omega = (\omega, \omega)\, \eta. \tag{49}$$

On a toujours $(\omega, \omega) \geqq 0$. Mais puisque

$${}_*\omega = \Sigma\, (-1)^{i+1} \gamma \lambda_i \omega^1 \wedge \omega^2 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n,$$

il vient

$$\langle \omega, \omega \rangle = \int_{V^n_{ps}} \gamma^2 \eta. \tag{50}$$

11. Considérons le champ normal $N \in T^{\perp}_p(V^n_{ps})$ défini par

$$N = f(Sh\varphi e_{n+1} + Ch\varphi e_{n+2}), \qquad f, \varphi \in \mathscr{D}(V^n_{ps}). \tag{51}$$

L'immersion x est comme on le sait *substantielle* [15] s'il n'existe pas de vecteur normal N tel que $\nabla N = 0$. A l'aide de la connexion (2) on trouve pour tout N que $\nabla N \neq 0$ si $\omega \neq 0$. D'autre part, on a

$$\nabla N = (\nabla N)_t + (\nabla N)_n$$

où $(\nabla N)_t \in T_p(V^n_{ps})$ et $(\nabla N)_n \in T^{\perp}_p(V^n_{ps})$ et la condition nécessaire et suffisante pour que N soit *parallèle dans le faisceau normal* est $(\nabla N)_n = 0$ [16]. A l'aide de la connexion (2) on trouve que

$$(\nabla N)_n = 0 \Leftrightarrow \mathrm{d}f = 0 \qquad \text{et} \qquad \omega + \mathrm{d}\varphi = 0. \tag{52}$$

On a donc la

Proposition. *Etant donnée une variété V^n_{ps} pseudo-isotrope de codimension 2, l'immersion pseudo-isotrope $x : V^n_{ps} \to \mathscr{M}^{2+n}$ est toujours substantielle. En outre, la condition nécessaire et suffisante pour qu'il existe un champ normal parallèle dans le faisceau normal est que le module de ce champ soit constant et que la forme caractéristique de la variété soit un cobord.*

12. Dans l'hypothèse où $n = 2m + 1$ définissons sur V^n_{ps} une *structure presque cosymplectique* $C_{pc}(\Omega, \alpha)$ telle que

$$\begin{aligned} \Omega &= \omega^1 \wedge \omega^2 + \ldots + \omega^{n-2} \wedge \omega^{n-1}, \\ \alpha &= \omega^n. \end{aligned} \tag{53}$$

En vertu du lemme de Reeb il résulte immédiatement de (53) que le *champ canonique* $E(E \lrcorner\, \Omega = 0, E \lrcorner\, \alpha = 1)$ associé à la structure $C_{pc}(\Omega, \alpha)$ est e_n.

Rappelons que l'application

$$l : \wedge^1 (V^n_{ps}) \to T(V^n_{ps}), \qquad \varphi \to l(\varphi)$$

où

$$\begin{aligned} l(\varphi) \lrcorner\, \alpha &= 0, \\ l(\varphi) \lrcorner\, \Omega &= (E \lrcorner\, \varphi)\, \alpha - \varphi \end{aligned} \tag{54}$$

définit *la parenthèse de Lagrange pfaffienne* relativement à la structure $C_{pc}(\Omega, \alpha)$. Dans le cas de la structure $C_{pc}(\Omega, \alpha)$ défini par (53) et pour $\varphi = \omega$, on trouve

$$l(\omega) = \gamma(-\lambda_2 e_1 + \lambda_1 e_2 + \ldots - \lambda_{n-1} e_{n-2} + \lambda_{n-2} e_{n-1}). \tag{55}$$

Cherchons maintenant s'il existe un vecteur $W \in T_p^{\perp}(V_{ps}^n)$ qui soit *principal* pour $l(\omega)$ dans le sens d'Ōtsuki. Conformément à la définition donnée dans [7] on doit écrire

$$\forall Z \in T_p(V_{sp}^n) : S_{l(\omega)}Z = \langle l(\omega), Z \rangle\, W. \tag{56}$$

En faisant usage de (40), (43) et (55) on trouve que le champ W existe si et rien que si $\mu_i = \mu_j = \mu$. Dans ces conditions on obtient

$$W = \mu e_{n+2}.$$

Par ailleurs $(HX)_t$ étant un *champ horizontal* quelconque par rapport à la structure $C_{pc}(\Omega, \omega^n)$ et en plus orthogonal au champ t, on trouve à l'aide des équations (40), (43) que le shape operator $S_{(HX)_t}(t)$ est *colinéaire* au vecteur normal temporel. On peut donc énoncer la

Proposition. *Soit $x : V_{ps}^n \to \mathcal{M}^{2+n}$ une immersion pseudo-isotrope et soit t la composante tangentielle du vecteur caractéristique (isotrope) associé à l'immersion x. On peut définir sur V_{ps}^n une structure presque cosymplectique $\mathbf{C}_{pc}(\Omega, \alpha)$ de manière que le shape operator de tout champ orthogonal à t et horizontal relativement à $\mathbf{C}_{pc}(\Omega, \alpha)$ soit colinéaire au vecteur normal temporel. En particulier, si la seconde forme fondamentale afférente au vecteur temporel est conforme à la métrique de V_{ps}^n, alors la parenthèse de Lagrange pfaffienne de la forme caractéristique associée a x est un champ principal pour un vecteur bien déterminé, colinéaire au vecteur normal temporel.*

13. En supposant maintenant que la structure $\mathbf{C}(\Omega, \omega^n)$ est *cosymplectique*, exprimons que le champ $l(\omega)$ définit un *automorphisme infinitésimal* de cette structure. On doit écrire

$$\mathcal{L}_{l(\omega)}\Omega = 0, \qquad \mathcal{L}_{l(\omega)}\omega^n = 0. \tag{57}$$

En faisant usage de (55) on trouve que la condition nécessaire et suffisante pour que (57) soit satisfaite est que la forme *semi-basique* Bω définie par

$$B\omega = \omega - \gamma\lambda_n\omega^n \tag{58}$$

soit fermée. Nous énoncons la

Proposition. *Dans le cas où la structure $\mathbf{C}(\Omega, \omega^n)$ du numéro 12 est cosymplectique, la condition nécessaire et suffisante pour que la parenthèse de Lagrange pfaffienne de la forme caractéristique définisse un automorphisme de la structure $\mathbf{C}(\Omega, \omega^n)$ est que la forme semi-basique $B\omega = \omega - \gamma\lambda_n\omega^n$ soit un cocycle.*

14. Dans le cas où $n = 2m$ supposons que la forme

$$\Omega = \omega^1 \wedge \omega^2 + \ldots + \omega^{n-1} \wedge \omega^n \tag{59}$$

définit une *structure symplectique* $S(\Omega, V_{ps}^{2m})$ sur V_{ps}^{2m}. Cherchons dans quel cas la parenthèse de Poisson pfaffienne $p(\omega)$ de la forme caractéristique définit un automorphisme infinitésimal de la structure $S(\Omega, V_{ps}^{2m})$. On doit écrire

$$\mathcal{L}_{p(\omega)}\Omega = 0, \tag{60}$$

et à l'aide de (42) on trouve

$$\mathcal{L}_{p(\omega)}\Omega = -d \wedge \omega, \tag{61}$$

d'où la

Proposition. *Si la variété pseudo-isotrope est de dimension paire et symplectique, alors la condition nécessaire et suffisante pour que la parenthèse de Poisson pfaffienne de la forme caractéristique définisse un automorphisme infinitésimal de la structure est que la forme caractéristique soit un cocycle (ou que la variété soit ombilicale dans la direction du vecteur normal temporel).*

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GRAMMATICAL LEVELS AND SUBGRAMMARS OF CONTEXT-FREE GRAMMARS

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1. In this paper the concept of a grammatical level is used to define subgrammars of context-free grammars (CFG's) and this approach is compared to that of Małuszyński. Moreover, some results concerning the number of subgrammars are derived.

2. As far as CFG's are concerned, we shall use throughout this paper Ginsburg's [1] notation and terminology except that we allow for a CFG to have a finite set of initial symbols.

By a context-free rule p we mean an ordered pair $p = (A, \alpha)$ where A is a symbol and α is a word over a finite alphabet. Instead of $p = (A, \alpha)$ we shall usually write p in the form $A \to \alpha$ and we shall denote $\mathscr{L}(p) = A, \mathscr{R}(p) = \alpha$.

If P is a finite set of context-free rules, then we define

$\mathscr{V}(P)$ to be the smallest alphabet such that $P \subset \mathscr{V}(P) \times \mathscr{V}(P)^*$

$$\mathscr{L}(P) = \{\mathscr{L}(p);\ p \in P\},$$
$$\mathscr{R}(P) = \{\mathscr{R}(p);\ p \in P\},$$
$$\Sigma(P) = \mathscr{V}(P) - \mathscr{L}(P).$$

On a set P of context-free rules the relations $\leqq$, $*\leqq$ and $\equiv$ are defined as follows:

$p_1 \leqq p_2$ if either $\mathscr{L}(p_1) = \mathscr{L}(p_2)$ or the symbol $\mathscr{L}(p_2)$ occurs in the word $\mathscr{R}(p_1)$,
$*\leqq$ is the reflexive and transitive closure of the relation $\leq$,
$\equiv\ = *\leqq \cap *\leqq^{-1}$ is an equivalence relation on P.

Let $\bar{P} = P/\equiv$ be a quotient set of P relative to the relation $\equiv$. For $P_1, P_2 \in \bar{P}$ let $P_1 *\leqq P_2$ if there are $p_1 \leqq P_1$ and $p_2 \in P_2$ such that $p_1 \leqq^* p_2$. $*\leqq$ is a quasiordering on $\bar{P}$.

3. If $P' \subset P$ for a CFG $G = < V, \Sigma, P, S >$, then we define

$$\mathscr{S}_G(P') = \mathscr{L}(P') \cap \mathscr{R}(P - P') \cup S \cap \mathscr{V}(P').$$

Definition 1. Let $G = < V, \Sigma, P, S >$ be a CFG with S being a set of initial symbols. For each $P' \subseteqq P$ let $\mathscr{G}(P') = \langle \mathscr{V}(P'), \Sigma(P'), P', \mathscr{S}_G(P')\rangle$. If $P' \in \bar{P}$, then $\mathscr{G}(P')$ is said to be a grammatical level of G. If $\bar{M} \subseteq \bar{P}$, then $\mathscr{H}(M) = \mathscr{G}(\bigcup_{\bar{m} \in \bar{M}} \bar{m})$ is said to be a quasisubgrammar of G. Let $\Gamma(G)$ and $Q(G)$ be the sets of all grammatical levels and quasisubgrammars of G.

If $G_1 = \langle V_1, \Sigma_1, P_1, S_1\rangle \in \Gamma(G)$ and $G_2 = \langle V_2, \Sigma_2, P_2, S_2\rangle \in \Gamma(G)$ for a CFG G, then we write $G_1 *\leqq G_2$ if $P_1 *\leqq P_2$.

In the class $Q(G)$ the operations $\vee$ and $\wedge$ are defined as follows. If G_1 and G_2 are CFG's defined as above, then

$$G_1 \vee G_2 = \langle \mathscr{V}(P_1 \cup P_2),\ \Sigma\,(P_1 \cup P_2),\ P_1 \cup P_2,\ S_G(P_1 \cup P_2)\rangle$$
$$G_1 \wedge G_2 = \langle \mathscr{V}(P_1 \cap P_2),\ \Sigma\,(P_1 \cap P_2),\ P_1 \cap P_2,\ S_G(P_1 \cap P_2)\rangle$$

Hence $G_0 \in Q(G)$ if, and only if $G_0 = \bigvee\limits_{i=1}^{k} G_i$ for some $G_i \in \Gamma(G)$.

Proposition 1. The algebra $\langle Q(G), \vee, \wedge \rangle$ *is a lattice.*

Proof. The mapping $\mathscr{H}$ is a bijection of the set $2^{\bar{P}}$ onto $Q(G)$ which maps set union $\cup$ in $\vee$ and set intersection $\cap$ in $\wedge$. Thus $\mathscr{H}$ is the isomorphism of $2^{\bar{P}}$ onto $Q(G)$ and therefore $\langle Q(G), \vee, \wedge\rangle$ is the lattice as well as the algebra $\langle 2^{\bar{P}}, \cup, \cap\rangle$.

4. A subset A of a partially ordered set N is said to be a final segment (antichain) of N if $x \in A$, $y \in N$, $x \leqq y$ implies $y \in A$ (if $x \in A$, $y \in A$, $x \leqq y$ implies $x = y$).

If $G = \langle V, \Sigma, P, S\rangle$ is a CFG nad $\bar{M}$ is a final segment of $\bar{P}$, then $\mathscr{H}(\bar{M})$ is said to be a subgrammar of G, Denote by $\Theta(G)$ the set of all subgrammars of G.

Since intersection and union of two final segments of $\bar{P}$ is again a final segment of $\bar{P}$, we have immediately

Proposition 2. *The algebra* $\langle \Theta(G), \vee, \wedge\rangle$ *is the sublattice of the lattice* $\langle Q(G), \vee, \wedge\rangle$.

Theorem 3. $G_0 = \langle V_0, \Sigma_0, P_0, S_0\rangle$ *is a subgrammar of a CFG* $G = \langle V, \Sigma, P, S\rangle$ *if, and only if* $G_0 = \bigvee\{G';\, G' \in \psi\}$ *where* ψ *is a final segment of* $\Gamma(G)$.

Proof. $\bar{M} \subseteq \bar{P}$ is a final segment of $\bar{P}$ if, and only if $\{\mathscr{G}(\bar{m});\, \bar{m} \in \bar{M}\}$ is a final segment of $\Gamma(G)$. Hence, G_0 is a subgrammar of G if, and only if there is a final segment $\bar{M}$ of $\bar{P}$ such that $G_0 = \mathscr{H}(\bar{M}) = \mathscr{G}(\bigcup\limits_{\bar{m}\in\bar{M}} \bar{m}) = \bigvee\{\mathscr{G}(\bar{m});\, \bar{m} \in \bar{M}\}$ and $\{\mathscr{G}(\bar{m});\, \bar{m} \in \bar{M}\}$ is a final segment of G. Hence the theorem.

Theorem 4. *The number of subgrammars of a CFG G is equal to the number of antichains of the set* $\Gamma(G)$.

Proof. Let $G = \langle V, \Sigma, P, S\rangle$. If $\bar{E} \subseteq \bar{P}$ ($\bar{A} \subseteq \bar{P}$) is a final segment of $\bar{P}$ (an antichain of $\bar{P}$), then let $\nu(\bar{E})$ be the set of all minimal elements of $\bar{E}$ (let $\mu(\bar{A})$ be the smallest final segment of $\bar{P}$ containing $\bar{A}$). Clearly $\mu\nu$ and $\nu\mu$ are identity mappings on the sets of all final segments and antichains of $\bar{P}$. Hence ν is a bijection of the set of all final segments of $\bar{P}$ onto the set of all antichains of $\bar{P}$. Now the theorem follows from the fact that there exists a bijection of the set of final segments of $\bar{P}$ onto the set $\Theta(G)$ and there is a bijection of the set of antichains of $\bar{P}$ onto the set of antichains of $\Gamma(G)$.

It is evident that the number of subgrammars does not depend on the set of initial symbols.

5. It was shown in Gruska [2] that for any n there is a context-free language $L_n \subset \{a, b\}^*$ such that any CFG generating L_n has at least n grammatical levels. From that and from Theorem 4 it folows immediately.

Theorem 5. *For any integer n there is a context-free language $L_n \subset \{a, b\}^*$ such that any CFG generating L_n has at least n subgrammars.*

One can even prove a little more, namely, that for any n the language $L_n' = \{a^{k_1}ba^{k_2}b \ldots a^{k_n}bbaab^{k_n}ab^{k_{n-1}}a \ldots b^{k_1}a;\ 0 \leqq k_i < \infty,\ i = 1, 2, \ldots, n\}$ can be generated by a CFG having n subgrammars but not by a CFG with less than n subgrammars.

It was shown in Gruska [3] that it is undecidable for an arbitrary CFG G whether or not the language $L(G)$ can be generated by a CFG having exactly one grammatical level. From that it follows.

Theorem 6. *It is undecidable for an arbitrary CFG G and an arbitrary integer n whether or not there is a CFG generating the language $L(G)$ and having not more than n subgrammars.*

6. In a slightly different way the concept of a grammatical level was defined by Małuszyński [4]. He defines that a quadruple $G' = \langle V', \Sigma', P', S' \rangle$ is a subgrammar of a CFG $G = \langle V, \Sigma, P, S \rangle$ if the following conditions are satisfied.

(1) $V' \subset V$ and if $A \in V' \cap (V - \Sigma)$ and $A \to xby \in P$, $b \in V$, then $b \in V'$ (i. e. V' is a "final segment" of symbols)

(2) $$\Sigma' = \Sigma \cap V'$$

(3) $$P' = \{A \to x;\ A \in V', A \to x \in P\}$$

(4) $$S' = S \cap V' \cup \{a;\ a \in V' \text{ and there exists a rule } A \to xay \in P - P'\}$$

According to this definition it may happen that a subgrammar of a CFG is not a context-free grammar since the set S' may contain terminal symbols. If, however, the condition (4) is modified to have the form

(4a) $$S' = S \cap V' \cup \{B;\ B \in V' - \Sigma' \text{ and there exists a rule } A \to xBy \text{ in } P - P'\}$$

then the two definitions are equivalent. (To be precise we should add the condition $V' = \mathscr{V}(P')$.) To show it we proceed as follows. If the conditions (1) to (3) and (4a) are satisfied with $V' = \mathscr{V}(P')$, then from (1) it follows that P' is a final segment of $\bar{P}$ and (4a) implies $S' = \mathscr{S}_G(P')$ and therefore G' is a subgrammar of G. If, on the other hand, G' is a subgrammar, then P' is a final segment of $\bar{P}$ and therefore the conditions (1) to (3) hold and (4a) follows directly from the definition of $\mathscr{S}_G(P')$.

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EXTENSIONS OF MAPPINGS OF FINITE BOOLEAN ALGEBRAS TO HOMOMORPHISMS

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This paper is a continuation of the paper [1], in which a matrix representation of homomorphic mappings in Boolean algebra has been given. It is obvious that investigations of homomorphisms are essentially simplified by this representation. As in paper [1] also here, I do not deal with an abstract Boolean algebra, but with its isomorphic representation — the B-modul (see [2]).

Let $\mathfrak{M}_m$ be the m-dimensional B-modul, i. e. Boolean algebra with 2^m elements and $\mathfrak{M}_n$ the n-dimensional B-modul, i. e. Boolean algebra with 2^n elements. Let us denote by φ_i the B-vectors from $\mathfrak{M}_m$, by ψ_i the B-vectors from $\mathfrak{M}_n$. We denote o the zero vector in both B-moduls, since there is no danger of misunderstanding.

By the matrix representaion it is possible to solve the following problem:

Let there be given mapping of a subset of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ by relations:

$$\text{(S)} \qquad \begin{array}{cccc} \varphi_i \in \mathfrak{M}_m, & \psi_i \in \mathfrak{M}_n, & \varphi_i \to \psi_i & i = 1, 2, \ldots, k < 2^m \\ & o \in \mathfrak{M}_m & o \to o \in \mathfrak{M}_n & \end{array}$$

We have to determine the homomorphic mappings α of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ which fulfil the relations (S).

This problem is the so called problem of extension of mappings to homomorphisms. We shall show bellow that this problem can have no, one or more solutions. Let the relations (S) be prescribed; for the sake of brevity let us introduce the following notation:

$$\varphi_i = (f_1^{(i)}, f_2^{(i)}, \ldots, f_m^{(i)}), \qquad \psi_i = (g_1^{(i)}, g_2^{(i)}, \ldots, g_n^{(i)}), \qquad i = 1, 2, \ldots, k.$$

Now there holds the following theorem:

Theorem 1. Let A be a B-matrix of the type m/n, representing a homomorphic mapping α of $\mathfrak{M}_m$ into $\mathfrak{M}_n$, this mapping fulfilling conditions (S).

$$\text{If } (f_p^{(1)}, f_p^{(2)}, \ldots, f_p^{(k)}) \neq (g_q^{(1)}, g_q^{(2)} \ldots, g_q^{(k)}),$$

then $a_{pq} = 0$ (a_{pq} is an element of the matrix A in therow and q-th column).

Proof: Let the assumption of the theorem hold and $a_{pq} = 1$. Then the p-th row of B-matrix A is a B-vector

$$a_p = (a_{p1}, \ldots, a_{p,\,q-1}, 1, a_{p,q+1}, \ldots, a_{pn});$$

for the p-th vector of the base of modul $\mathfrak{M}_m$, i.e. for $e^{(p)}$ holding $\alpha(e^{(p)}) = a_p$. Of course, by the definition 5 (see [1]), there holds $\psi_1 = f_1^{(1)}a_1 + \ldots + f_{p-1}^{(1)}a_{p-1} + f_p^{(1)}a_p + \ldots + f_m^{(1)}a_m$.

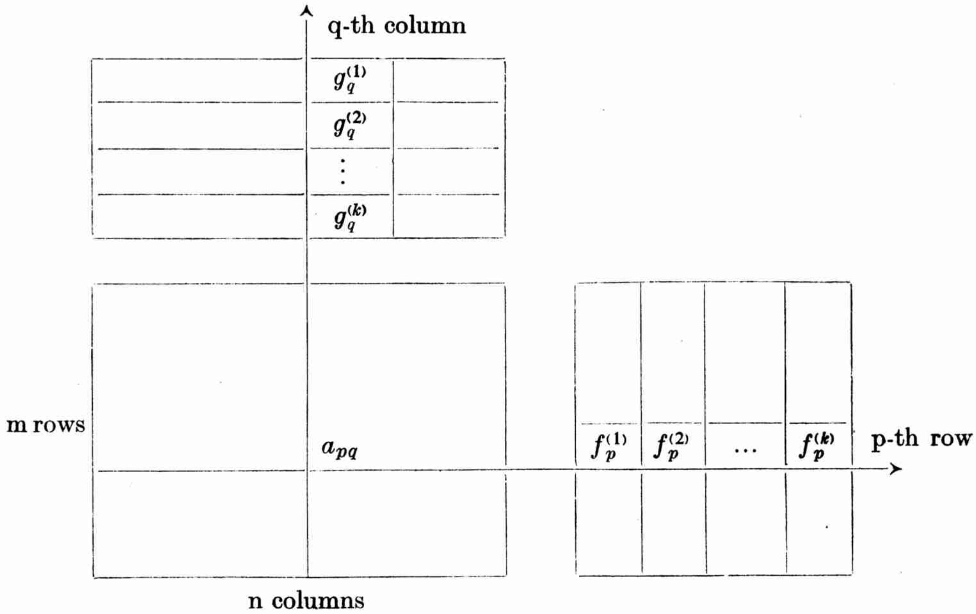
By the assumption, the q-th coordinate of the B-vector a_p is equal to *1*. But there can be at most one unity in each column of the matrix A (see theorem 1 in [1]). Thus the vector a_p has a unity only in the q-th column. Then the q-th coordinate of the vector ψ_1 is equal to $f_p^{(1)} . 1 = f_p^{(1)}$ and from this $g_q^{(1)} = f_p^{(1)}$. Analogously for all i, thus

$$(f_p^{(1)}, \ldots, f_p^{(k)}) = (g_q^{(1)}, \ldots, g_q^{(k)})$$

which is a contradiction.

This theorem implies a simple algorithm:

We shall construct a matrix with m rows and n columns where all $m . n$ places are still not filled in. Behind this matrix, we shall write vertically vectors $\varphi_1, \varphi_2, \ldots, \varphi_k$, over this matrix, we shall write horizontally $\psi_1, \psi_2, \ldots, \psi_k$, see diagram:



Let us write the prescribed diagram and fill the coordinates of B-vectors φ_i, ψ_i. Now we compare the sequences of 0 and 1 in the vectors (in the first diagram in frames). If these sequences are different, we fill 0 as an element in the crossing of this row and column. In the opposite case we can fill 1 of course, in each column at most one. By this way we receive all desired matrices. It is possible to fill 1 in all places a_{pq}, where

$$(f_p^{(1)}, \ldots, f_p^{(k)}) = (g_q^{(1)}, \ldots, g_q^{(h)}).$$

The matix C derived in this way will be called *the decomposition matrix*. The set of all matrices representing all desired homomorphic mappings form the decomposition if this matrix C (see definition 4 in [1]). Thus number of all possible homomorphisms is the number of matrices in the decomposition of the matrix C. It is the product of the sums of unities in the non zero columns of the matrix C.

It can happen that the images of vestors φ_i derived from the matrix C (constructed in usual way) are not ψ_i as given by the prescription (S). In this case the homomorphic mapping with the desired property does not exist (as it is easy to show).

This algorithm solves the problem of existence, number and construction of homomorphic extension of the given mapping of a finite Boolean algebra into another Boolean algebra.

Example 1. Find the homomorphic mappings of $\mathfrak{M}_5$ into $\mathfrak{M}_4$ fulfilling the relations:

$$\begin{aligned} \varphi_1 &= (0\ 1\ 1\ 0\ 1) \rightarrow \psi_1 = (1\ 0\ 1\ 1) \\ \varphi_2 &= (0\ 1\ 0\ 1\ 0) \rightarrow \psi_2 = (0\ 0\ 1\ 0) \\ o &= (0\ 0\ 0\ 0\ 0) \rightarrow \ o = (0\ 0\ 0\ 0) \end{aligned}$$

Then the prescribed diagram is:

$$\begin{array}{r|c|c|c|} \psi_1 = & (1\ 0\ 1\ 1) & & \\ \psi_2 = & (0\ 0\ 1\ 0) & \varphi_1 & \varphi_2 \\ \hline & & 0 & 0 \\ & & 1 & 1 \\ & & 1 & 0 \\ & & 0 & 1 \\ & & 1 & 0 \\ \hline \end{array}$$

Now we fill it by the given algorithm:

$$\begin{array}{rc|c|c|} & 1\,0\,1\,1 & & \\ & 0\,0\,1\,0 & & \\ \hline C = & \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 1\,0\,0\,1 \\ 0\,0\,0\,0 \\ 1\,0\,0\,1 \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix} \\ \hline \end{array}$$

$$s = 2\,.\,1\,.\,1\,.\,2 = 4$$

Thus there are just 4 matrices representing 4 possible homomorphic mappings. These matrices form the decomposition of the matrix C:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The image of each B-vector from $\mathfrak{M}_5$ will be found by means of matrices A_i.

The mappings constructed in example 1 are evidently mappings which map 1 onto 1, which is easy to prove. Now in the following example we shall show the form of a matrix representing mapping without this property.

Example 2. Find homomorphisms of $\mathfrak{M}_6$ into $\mathfrak{M}_4$ fulfilling relations:

$$(\mathrm{S}_1) \qquad \begin{aligned} \varphi_1 &= (1\ 1\ 0\ 0\ 0\ 0) \rightarrow (0\ 1\ 1\ 0) = \psi_1 \\ \varphi_2 &= (0\ 0\ 1\ 0\ 0\ 1) \rightarrow (0\ 0\ 1\ 0) = \psi_2 \\ o &= (0\ 0\ 0\ 0\ 0\ 0) \rightarrow (0\ 0\ 0\ 0) = o \end{aligned}$$

Write directly the diagram:

$$\begin{array}{r|c|c}
\begin{matrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\end{matrix} & & \\
\hline
C = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} & \begin{matrix}1\\ 1\\ 0\\ 0\\ 0\\ 0\end{matrix} & \begin{matrix}0\\ 0\\ 1\\ 0\\ 0\\ 1\end{matrix} \\
\hline
\end{array}$$

From the decomposition of the matrix C we get 8 different matrices representing the homomorphic mappings. The third column of each of these matrices is a zero column, hence the mappings are necessarily of the type "into", and the desired homomorphic mapping fulfilling (S_1) does not exist because vectors ψ_1, ψ_2 have not preimages in each possible homomorphisms.

It is easy to prove the theorem:

Theorem 2. The matrix A of type m/n which has in h columns only 0 ($1 \leqq h \leqq n$) and in all remaining $n - h$ columns just one unity and k_j unities in the j-th row represents the homomorphic mapping of $\mathfrak{M}_m$ into $\mathfrak{M}_n$. Let $r_j = \max(1, k_j) - 1$, $r = \sum_{j=1}^{n} r_j$. Then there exist just $2^n - 2^{n-h-r}$ vectors in the $\mathfrak{M}_n$, which have not a preimage in $\mathfrak{M}_m$.

It is evident that by the matrix representation it is easy to determine whether the given homomorphism is of the type "onto" or "into". We can prove the theorem:

Theorem 3. Given the relations (S). If the decomposition matrix C has at most one unity in each row, then only one of the following three alternatives is holding:

1. There does not exist a homomorphic mapping of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ fulfilling (S).
2. There exists a homomorphic mapping of $\mathfrak{M}_m$ *into* $\mathfrak{M}_n$ fulfilling (S), but there does not exist a homomorphic mapping of the type "onto" fulfilling (S).
3. There exists a homomorphic mapping $\mathfrak{M}_m$ *onto* $\mathfrak{M}_n$ fulfilling (S), but there does not exist a homomorphic mapping of the type "into" fullfilling (S) which is not of the type "onto".

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VERALLGEMEINERUNG DER GONIOMETRISCHEN FUNKTIONEN

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In dieser Behandlung betrachten wir die goniometrischen Funktionen in engerem Sinne des Wortes; d. h. wir behandeln nur die Funktionen $\cos t$, $\sin t$, weil die übrigen als die von jenen zusammegesetzten betrachtet werden können.

Dieses Studium bietet unmittelbare Analogien der goniometrischen Funktionen und zwar sowohl aus den analytischen als auch den metrischen Gründen.

Die ganze Behandlung besteht aus fünf thematischen Absätzen, A bis E, deren Inhalt immer im voraus angedeutet wird.

A. Analytische Analogien und deren Prinzipien.

1] Definition. $f(t)$ sei eine Funktion im Argument t, mit unbeschränkter Anzahl der Ableitungen im gewissen Intervall; n sei eine natürl. Zahl.

Die Funktion mit der Ableitungsperiode n ist eine Funktion $f(t)$, für die gilt:

$$D_n f(t) = f(t), \qquad D_k f(t) \neq f(t), \qquad k = 1, 2, \ldots, n-1 \tag{1}$$

2] Vereinbarung. Die charakteristische Gleichung, die der dif. Gleichung (1) entspricht, bezeichnen wir $\lambda^n = 1$. Die Wurzeln dieser Gleichung bezeichnen wir $\varepsilon_{n,k} = \varepsilon_k$, $k = 0, 1, \ldots, n-1$, wo (wie wohl bekannt ist) gilt:

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \, . \sin \frac{2k\pi}{n}, \qquad k = 0, 1, \ldots, n-1 \tag{2}$$

3] Definition. Eine Spezialbasis der partikulären Lösungen der Gleichung (1), die wir bezeichnen:

$$f_{n,i}(t), \qquad i = 0, 1, \ldots, n-1 \tag{3}$$

besteht aus den Funktionen, die folgende Bedingungen erfüllen:

$$\begin{cases} f_{n,0}(0) = 1; \quad f_{n,j}(0) = 0, \quad j = 1, 2, \ldots, n-1 \\ Df_{n,j}(t) = f_{n,j+1}(t), \quad j = 0, 1, \ldots, n-1. \end{cases} \tag{4}$$

4] Vereinbarung. $s_{n,k}$ sei die Bezeichnung für die Summe $\sum_{j=0}^{n-1} (\varepsilon_{n,j})^k$, $k = 0, 1, 2, \ldots$ also für die sogenannte k-te symmetrische Funktion, die den Wurzeln der Gleichung $\lambda^n = 1$ entspricht.

5] Lemma. Es gilt:

$$s_{n,k} = 0 \quad \text{für} \quad k \neq C \,.\, n \tag{5}$$
$$\quad = n \quad \text{für} \quad k = C \,.\, n$$

wo C eine ganze Zahl ist.

Beweis. Betrachten wir ein willkürliches Polynom n-ten Grades: $P_n(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$. $S_{n,k}$ sei die Bezeichnung für die k-te symmetrische Funktion der Wurzeln des Polynoms $P_n(x)$. Dann gilt folgende Rekurenz: $S_{n,k} + a_1 \,.\, S_{n,k-1} + a_2 \,.\, S_{n,k-2} + \ldots + a_{k-1} \,.\, S_{n,1} + k \,.\, a_k = 0$, siehe z. B. [1], Seite 39. Daraus sieht man sofort, dass für das Polynom $\lambda^n - 1 = 0$ gilt: 1) $s_{n,k} = 0$ für $k = 1, 2, \ldots, n - 1$. 2) $s_{n,k} = n$ für $k = 0$. Es gelte weiter $k = n + h$, wo $h = 0, 1, 2, \ldots$. Dann gilt: $s_{n,n+h} = \sum_{i=0}^{n-1} (\varepsilon_i)^{n+h} = \sum_{i=0}^{n-1} (\varepsilon_i)^n \,.\, (\varepsilon_i)^h = \sum_{i=0}^{n-1} (\varepsilon_i)^h = s_{n,h}$. W.z.b.w.

6] Vereinbarung. Mit dem Symbol V_n bezeichnen wir jene Vandermonde's Determinante n-ten Grades, in deren j-ter Spalte einzelne Potenzen der Wurzeln $\varepsilon_{n,j} = \varepsilon_j$, $j = 0, 1, \ldots, n - 1$, der Gleichung $\lambda^n = 1$ sind, siehe (6). Wegen der Übereinstimmung mit den Wurzeln $\varepsilon_{n,j}$ wollen wir die Zeilen- und Spaltenindexen bei V_n von der Null anfangend bezeichnen.

$$V_n = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \varepsilon_0 & \varepsilon_1 & \ldots & \varepsilon_{n-1} \\ \varepsilon_0^2 & \varepsilon_1^2 & \ldots & \varepsilon_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ \varepsilon_0^{n-1} & \varepsilon_1^{n-1} & \ldots & \varepsilon_{n-1}^{n-1} \end{vmatrix} \tag{6}$$

Ersetzen wir nun in der Determinante V_n ihre k-te Spalte durch die erste Spalte der Einsermatrix $\mathbf{E}_n$: die heimit enstandene Determinante bezeichnen wir $V_{n,k}$; $k = 0, 1, \ldots, n - 1$.

7] Lemma. Es sei gegeben das System von n linearen, nicht homogenen Gleichunggen mit n Unbekannten, in folgender Matrixform:

$$\mathbf{V}_n \,.\, \mathbf{z} = \mathbf{j}; \tag{7}$$

$\mathbf{V}_n$ ist die der Determinante V_n zugehörige Matrix; $\mathbf{z}$ ist der Spaltenvektor von den Unbekannten: $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1})$, und $\mathbf{j}$ ist der Vektor aus der ersten Spalte der Matrix $\mathbf{E}_n$; $\mathbf{j} = (1, 0, 0, \ldots, 0, 0)$.

Das eben angeführte System hat folgende Lösung:

$$z_i = \frac{1}{n}; \qquad i = 0, 1, \ldots, n - 1 \tag{8}$$

Beweis. Beachten wir zuerst, dass das System (7) gerade eine Lösung hat: V_n ist nämlich eine Vandermond'sche Determinante und keiner von ihren Faktoren, d. h. $(\varepsilon_j - \varepsilon_k)$, $j \neq k$, wird gleich der Null, da einzelne Wurzeln der Gleichung $\lambda^n = 1$ voneinander verschieden sind. Der Wert der Determinante verändert sich nicht, wenn wir zu den Elementen der k-ten Spalte die entsprechenden Elemente von allen übrigen Spalten addieren. Dann bekommen wir:

$$\begin{array}{c} \quad \quad k\text{-te Spalte} \\ V_n = \begin{vmatrix} 1 & \dots & n & \dots & 1 \\ \varepsilon_0^1 & \dots & s_{n,1} & \dots & \varepsilon_{n-1}^1 \\ \varepsilon_0^2 & \dots & s_{n,2} & \dots & \varepsilon_{n-1}^2 \\ \vdots & & \vdots & & \vdots \\ \varepsilon_0^{n-1} & \dots & s_{n,n-1} & \dots & \varepsilon_{n-1}^{n-1} \end{vmatrix} \end{array}$$

Aber nach dem Lemma 5. gilt: $s_{n,1} = s_{n,2} = \dots = s_{n,n-1} = 0$. Wenn wir also aus den Elementen der k-ten Spalte die Nummer n ausklammern, bekommen wir: $V_n = n \,.\, V_{n,k}$; $k = 0, 1, \dots, n-1$. Wenn wir nun weiter, zwecks der Lösung des Systems (7), die Cramer'sche Regel anwenden, dann gilt:

$$z_k = \frac{V_{n,k}}{V_n} = \frac{\frac{1}{n} \,.\, V_n}{V_n} = \frac{1}{n}, \qquad k = 0, 1, \dots, n-1$$

8] **Satz.** Die Funktion $f_{n,i}(t)$; $i = 0, 1, \dots, n-1$; kann man durch folgende Formel ausdrücken:

$$f_{n,i}(t) = \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_{n,j}^i \,.\, e^{\varepsilon_{n,j} \cdot t}; \qquad i = 0, 1, \dots, n-1 \tag{9}$$

B e w e i s. Die allgemeine Lösung der Gl. (1), wenn wir ε_i anstatt $\varepsilon_{n,i}$ schreiben, ist:

$$z_0 \,.\, e^{\varepsilon_0 \cdot t} + z_1 \,.\, e^{\varepsilon_1 \cdot t} + \dots + z_{n-1} \,.\, e^{\varepsilon_{n-1} \cdot t}$$

wo z_i die unbestimmten Koeffizitenten sind. Wenn wir nun von der allgemeinen zur partikulären Gleichung übergehen, wobei die Bedingungen (4) zu erfüllen sind, gelangen wir offenbar zu dem System (7), so dass, mit Hinsicht auf das Lemma 7., die Glütigkeit unseres Satzes ersichtlich wird.

9] **Bemerkung.** Die Formeln (9) für $n = 2$ sind die wohl bekannten Formeln für den hyperbolischen Sinus und Cosinus.

10] **Satz.** Die Funktionen $f_{n,i}(t)$; $i = 0, 1, \dots, n-1$; kann man als folgende Potenzreihen ausdrücken; $t \in (-\infty, \infty)$:

$$\begin{aligned} f_{n,0}(t) &= \sum_{k=0}^{\infty} \frac{t^{nk}}{(nk)!} \\ f_{n,j}(t) &= \sum_{k=1}^{\infty} \frac{t^{nk-j}}{(nk-j)!}, \qquad j = 1, 2, \dots, n-1 \end{aligned} \tag{10}$$

B e w e i s. Nach dem Satz 8. gilt:

$f_{n,j}(t) = \frac{1}{n} \sum_{r=0}^{n-1} \varepsilon_r^j \, e^{\varepsilon_r \cdot t} = \frac{1}{n} \sum_{r=0}^{n-1} \sum_{i=0}^{\infty} \frac{\varepsilon_r^j (\varepsilon_r \,.\, t)^i}{i!} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{r=0}^{n-1} \frac{\varepsilon_r^{j+i}}{n} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \frac{s_{n,i+j}}{n}$, wo nach 7. gilt: 1) $s_{n,i+j} = 0$ für $i + j \neq n \,.\, k$, 2) $s_{n,i+j} = n$ für $i + j = n \,.\, k$, wo

$k = 0, 1, 2, \ldots$, so dass gilt: $f_{n,j}(t) = \sum_{k=1}^{\infty} \frac{t^{nk-j}}{(nk-j)!}$; dabei für $j = 0$ gilt: $f_{n,0}(0) =$ $= 1$, siehe (4). W.z.b.w.

11] **Satz.** Es gilt folgende Formel für $t \in (-\infty, \infty)$:

$$\sum_{i=0}^{n-1} f_{n,i}(t) = e^t \tag{11}$$

Beweis. Nach 7. und 8. gilt: $\sum_{i=0}^{n-1} f_{n,i}(t) = \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} \frac{1}{n} \varepsilon_k^i \, e^{\varepsilon_k \cdot t} = \frac{1}{n} \sum_{k=0}^{n-1} e^{\varepsilon_k \cdot t} \, . \sum_{i=0}^{n-1} \varepsilon_k^i = \frac{1}{n} \sum_{k=0}^{n-1} e^{\varepsilon_k \cdot t} \, . \sum_{i=0}^{n-1} \varepsilon_i^k = \frac{1}{n} \sum_{k=0}^{n-1} e^{\varepsilon_k \cdot t} \, . \, s_{n,k} = \frac{1}{n} e^{\varepsilon_0 \cdot t} \, . \, s_{n,0} = e^t.$

12] **Satz.** *Es gilt folgende Analogie des Moivre'schen Satzes:*

$$\sum_{i=0}^{n-1} f_{n,i}(rt) = \Big[\sum_{i=0}^{n-1} f_{n,i}(t)\Big]^r \tag{12}$$

wo r eine beliebige reale Zahl ist; $n \geqq 1$; $t \in (-\infty, \infty)$.

Beweis. Nach 11. gilt für ein beliebiges reelle Argument z: $\sum_{i=0}^{n-1} f_{n,i}(z) = e^z$. Setzen wir hier $z = r \, . \, t$ ein, so bekommen wir: $\sum_{i=0}^{n-1} f_{n,i}(rt) = e^{rt}$.

Nun potenzieren wir die Gl. (11) zum r-ten Grad; dann geht, durch den Vergleich mit der letzten Gleichung, unser Satz sofort hervor.

13] **Satz.** *Zwischen den Funktionen $f_{n,i}(t)$; $i = 0, 1, \ldots, n-1$, gelten folgende Relationen, die mit den additiven Relationen der goniometrischen Funktionen analogisch sind:*

$$f_{n,k}(x + y) = \sum_{i=0}^{n-1} f_{n,i}(x) \cdot f_{n,k-i}(y), \qquad k \in \{0, 1, \ldots, n-1\} \tag{13}$$

wo man im Falle $k - i < 0$ den Index $n + k - i$ anstatt $k - i$ benützt.

Beweis. Unterscheiden wir hier zwei Arten der Faktoren:

1) Wenn $p = q$ ist, dann gilt $\sum_{i=0}^{n-1}\sum_{p=0}^{n-1} \frac{1}{n^2} \cdot \varepsilon_p^i \, . \, \varepsilon_p^{k-i} \, . \, e^{\varepsilon_p \cdot x} \, . \, e^{\varepsilon_p \cdot y} = \sum_{p=0}^{n-1} \frac{1}{n^2} \cdot \big(\sum_{i=1}^{n-1} \varepsilon_p^{i+k-i} \, . \, e^{\varepsilon_p(x+y)}\big) = \sum_{p=0}^{n-1} \frac{1}{n^2} \cdot n \, . \, \varepsilon_p^k \, . \, e^{\varepsilon_p(x+y)} = f_{n,k}(x + y)$

2) Wenn $p \neq q$ ist, dann gilt: $\sum_{i=0}^{n-1}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1} \frac{1}{n^2} \cdot \varepsilon_p^i \, . \, \varepsilon_q^{k-i} \, . \, e^{\varepsilon_p \cdot x + \varepsilon_q \cdot y} = \sum_{p=0}^{n-1}\sum_{q=0}^{n-1} \frac{1}{n^2} \cdot e^{\varepsilon_p \cdot x + \varepsilon_q \cdot y} \, . \, \varepsilon_{kq} \cdot \sum_{i=0}^{n-1} \varepsilon_i^{p-q} = 0$, weil $\varepsilon_p^i \, . \, \varepsilon_q^{k-i} = \varepsilon_q^k \, . \, \varepsilon_{i(p-q)} = \varepsilon_{kq} \cdot \varepsilon_i^{p-q}$; dabei $p - q \neq$ $\neq 0$. W.z.b.w.

14] **Bemerkung.** Aus den angeführten Formeln (13) entstehen auch weitere Identitäten, z. B. wenn wir $y = -x$ setzen, mit Hinsicht auf (4).

15] Lemma. Es sei $\varepsilon_j = \varepsilon_{n,j}$ eine beliebige Wurzel der Gl. $\lambda^n = 1$ und i sei ein beliebiges Element der Menge $\{0, 1, \ldots, n - 1\}$. Dann gilt:

$$f_{n,i}(\varepsilon_j \,.\, z) = \varepsilon_{n-ij} \cdot f_{n,i}(z) \tag{14}$$

Beweis. Durch den Einsatz $t = \varepsilon_j \,.\, z$ bekommen wir nach (9): $f_{n,i}(\varepsilon_j z) = = \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon_k^i \,.\, e^{\varepsilon_k \varepsilon_j z} = \varepsilon_j^{-i} \,.\, \varepsilon_j^i \,.\, \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon_k^i \, e^{\varepsilon_k \varepsilon_j . z}$. Da es gilt: $\varepsilon_j^i \,.\, \varepsilon_k^i = \varepsilon_i^j \,.\, \varepsilon_i^k = \varepsilon_i^{j+k} = = \varepsilon_{j+k}^i$, und weiter: $\varepsilon_j^{-i} = \varepsilon_{-ij} = \varepsilon_{n-ij}$, kann man den obigen Ausdruck folgenderweise umformen: $f_{n,i}(\varepsilon_j \,.\, z) = \varepsilon_{n-ij} \cdot \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon_{k+j}^i \,.\, e^{\varepsilon_{k+j} . z}$. Aber die Folge $\{\varepsilon_j^i, \varepsilon_{1+j}^i, \ldots, \varepsilon_{n-1+j}^i\}$ bildet, mit Hinsicht auf die Identität $\varepsilon_{n+k}^i = \varepsilon_k^i$, eine zyklische Permuation der Grundfolge $\{\varepsilon_0^i, \varepsilon_1^i, \ldots, \varepsilon_{n-1}^i\}$. Demnach gilt: $\sum_{k=0}^{n-1} \varepsilon_{k+j}^i \,.\, e^{\varepsilon_{k+j} . z} = = \sum_{k=0}^{n-1} \varepsilon_k^i \,.\, e^{\varepsilon_k . z}$, so dass die zu beweisende Relation gilt.

16] Satz. (Der Satz über die Zerlegung der Formel (12) nach den Wurzeln der Gleichung $\lambda^n = 1$).

Wenn r in der Gleichung (12) eine ganze Zahl bedeutet, dann zerfällt diese Gleichung — nach der Substitution $t = \varepsilon_1 \,.\, z$ und nach der folgenden Durchführung des angezeigten Potenzierens auf der rechten Seite — in n Teilgleichungen; dabei wird die $(n - k)$-te Teilgleichung durch gerade alle jenen Glieder zusammengesetzt, die den Faktor ε_k enthalten. $(k = 0, 1, \ldots, n - 1)$. Die $(n—k)$-te Teilgleichung drückt hiemit die Funktion des r-fachen Argumentes, d. i. $f_{n,k}(rz)$, durch die zugehörigen Funktionen des einfachen Argumentes aus.

Beweis. In der Gleichung (13) setzen wir $y = (r - 1) \,.\, x$ ein; dann nach der Substitution $t = \varepsilon_1 \,.\, z$, siehe (14), bekommen wir: $\varepsilon_{n-k} \cdot f_{n,k}(rt) = \sum_{i=0}^{n-1} \varepsilon_{n-i} \cdot f_{n,i}(t) \,.\, \cdot\, \varepsilon_{n-k+i} \cdot f_{n,k-i}((r - 1)\, t)$, wo gilt: $\varepsilon_{n-i} \,.\, \varepsilon_{n-k+i} = \varepsilon_n \,.\, \varepsilon_{n-i-k+i} = \varepsilon_{n-k}$, so dass gilt:

$$\varepsilon_{n-k} \cdot f_{n,k}(rt) = \varepsilon_{n-k} \cdot \sum_{i=0}^{n-1} f_{n,i}(t) \cdot f_{n,k-i}((r - 1)\, t) \tag{A}$$

Wenn wir dies schrittweise für $k = 0, 1, \ldots, n - 1$, durchführen, und alle diesen Gleichungen summieren, so bekommen wir:

$$\sum_{k=0}^{n-1} \varepsilon_{n-k} \cdot f_{n,k}(rt) = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \varepsilon_{n-k} \cdot f_{n,i}(t) \cdot f_{n,k-i}((r - 1)\, t) \tag{B}$$

Anderseits, wenn wir die Gl. (12) für das Argument $\varepsilon_1 t$ statt t schreiben, bekommen wir nach (14):

$$\sum_{k=0}^{n-1} \varepsilon_{n-k} \cdot f_{n,k}(rt) = \Big[\sum_{k=0}^{n-1} \varepsilon_{n-k} \cdot f(t)\Big]^r \tag{C}$$

Nun stimmen die linken Seiten der Gleichungen (B), (C) überein und ausserdem kann die rechte Seite von (B) (schrittweise) ausschliesslich durch die Funktionen des einfachen Argumentes ausgedrückt werden. Daraus ist aber ersichtlich, dass die rechten Seiten von (B), (C), nur zwei verschiedene Arten von demselben Ausdruck sind. Die Gl. (B) kann man aber in n Teilgleichungen der Art (A) zerlegen, d. h. alle Glieder von einer und derselben Wurzel ε_k bilden immer eine Teilgleichung. W.z.b.w.

17] **Beispiel.** *Applikation des Satzes* 16. *für* $n = 3$.

$\sum_{j=0}^{2} f_{3,j}(2x) = \left[\sum_{j=0}^{2} f_{3,j}(x)\right]^2$; nach der Subst. $x = \varepsilon_1 t$: $f_{3,0}(2t) + \varepsilon_2 . f_{3,1}(2t) + \varepsilon_1 . f_{3,2}(2t) =$ $= [f_{3,0}(t) + \varepsilon_2 . f_{3,1}(t) + \varepsilon_1 . f_{3,2}(t)]^2 = f_{3,0}^2(t) + \varepsilon_1 . f_{3,1}^2(t) + \varepsilon_2 . f_{3,2}^2(t) + \varepsilon_2 . 2 .$ $. f_{3,0}(t) . f_{3,1}(t) + \varepsilon_1 . 2 . f_{3,0}(t) . f_{3,2}(t) + 2 . f_{3,1}(t) . f_{3,2}(t)$. Daraus, nach dem Vergleich aller Glieder von derselben Wurzel ε_j:

a) $$f_{3,0}(2t) = f_{3,0}^2(t) + 2 . f_{3,1}(t) . f_{3,2}(t)$$

b) $$f_{3,1}(2t) = f_{3,2}^2(t) + 2 . f_{3,0}(t) . f_{3,1}(t)$$

c) $$f_{3,2}(2t) = f_{3,1}^2(t) + 2 . f_{3,0}(t) . f_{3,2}(t)$$

Dies aber entspricht dem Satz 13. für $n = 3$, wenn wir $x = y = t$ setzen.

B. Die geometrischen Analogien in einer Ebene mit dem System von n Halbachsen. Prinzipielle Verhältnisse.

18] **Definition.** Eine Ebene mit dem Koordinatensystem von n Halbachsen benennen wir kurz „*die Ebene* α“. Die Ebene der komplexen Zahlen (die *Gauss'sche* Ebene) benennen wir kurz „*die Ebene* γ“.

Die Deckung zweier Koordinatenebenen bedeutet, dass diese beiden Ebenen dicht aufeinanderliegen, wie zwei Nachbarseiten in einem Buch. Jede zwei Punkte, die bei der Deckung im Kontakt sind, werden immer mit demselben Buchstaben bezeichnet. Die Anfangspunkte der beiden Koordinatenebenen müssen wechselseitig im Kontakt sein.

Bringen wir die Deckung der Ebene α mit γ zustande: In der Ebene γ seien einzelne Wurzeln ε_j der Gl. $\lambda^n = 1$ jenach der Vereinbarung 2. bezeichnet. Ausserdem seien hier auch die zugehörigen Radiusvektoren $\bar{\varepsilon}_j = \overrightarrow{0\varepsilon_j}$ bezeichnet; $j = 0, 1, \ldots$ $\ldots, n - 1$.

Das Koordinatsystem von n Halbachsen in der Ebene α ist das System von n orientierten Halbgeraden $X_0, X_1, \ldots, X_{n-1}$, die aus dem gemeinsamen Anfangspunkt 0 in solcher Richtung führen, dass die Halbachse X_j mit dem Radiusvektor $\bar{\varepsilon}_j \in \gamma$ zusammenfällt; $j = 0, 1, \ldots, n - 1$. Dabei haben X_j, $\bar{\varepsilon}_j$, auch dieselbe Orientierung.

Ein beliebiger Punkt $P \in \alpha$ wird angegeben durch einen geordneten Satz von n reellen, nicht negativen Zahlen $(a_0, a_1, \ldots, a_{n-1})$, so dass der entsprechende Punkt $P \in \gamma$ als der Endpunkt des orientierten Vektorpolygons $a_0\bar{\varepsilon}_0 \wedge a_1\bar{\varepsilon}_1 \wedge \ldots \wedge a_{n-1} .$ $. \bar{\varepsilon}_{n-1}$ fungiert, das in dem Anfangspunkt 0 beginnt. $(\bar{\varepsilon}_j \parallel a_j\bar{\varepsilon}_j)$.

Mit dem Symbol r bezeichnen wir den Radius OP (oder den Absolutwert des Radiusvektors $\overrightarrow{OP}$).

19] **Bemerkung.** Einem und demselben Punkt $P \in \alpha$ kann man ersichtlich eine unendliche Menge der n-gliedrigen Sätze von Koordinaten zuteilen, d. h. jeder von diesen Sätzen genügt nach der Def. 18.. In dieser Menge existiert aber gerade ein

Koordinatensatz, der einem willkürlich gewählten Parameterswert t entspricht. Darüber im Absatz C. Einstweilen setzen wir einen willkürlichen Koordinatensatz des Punktes P voraus.

20] **Definition.** Es sei ein willkurlicher Punkt $P \in \alpha$, der durch einen beliebigen Koordinatensatz $(a_0, a_1, \ldots, a_{n-1})$ angegeben ist; $n \geqq 3$. Die zusammenhängende gebrochene Linie $\in \alpha$, die aus den obigen Koordinaten so gebildet wird, dass sie sich bei der Deckung α mit γ überall mit dem orientierten Vektorpolygon $a_0\bar{\varepsilon}_0 \wedge \wedge a_1\bar{\varepsilon}_1 \wedge \ldots \wedge a_{n-1}\bar{\varepsilon}_{n-1}$ deckt, bildet, zusammen mit dem Radius OP, ein $(n + 1)$-eck, welches wir als „*das Koordinaten-*$(n + 1)$*-eck des Punktes* $P \in \alpha$" bezeichnen. Dabei benennen wir die Seite OP „*die Hypothenuse*" und die Seite a_j „*die* j*-te Kathete*", $j = 0, 1, \ldots, n - 1$.

21] **Satz.** Es sei ein Punkt $P \in \alpha$ durch seinen beliebigen Koordinatensatz $(a_0, a_1, \ldots, a_{n-1})$ gegeben; $n \geqq 3$. Dann gilt

a) Für $n = 2k + 1$, $k = 1, 2, 3, \ldots$

$$r^2 = \sum_{i=0}^{2k} \Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big) a_i \,.\, a_{i+j} \,.\, \cos\frac{2\pi j}{2k+1}, \tag{15}$$

b) Für $n = 2k$, $k = 2, 3, 4, \ldots$

$$r^2 = \sum_{i=0}^{2k-1} \Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big) a_i \,.\, a_{i+j} \,.\, \cos\frac{\pi j}{k}, \tag{16}$$

wobei in den beiden Fällen gilt $a_{cn+j} = a_j$, wo c eine ganze Zahl ist.

Beweis. Bilden wir das Skalarprodukt $\overrightarrow{OP} \,.\, \overrightarrow{OP}$; wir bekommen:

$$r^2 = \Big(\sum_{i=0}^{n-1} a_i\bar{\varepsilon}_i\Big) \,.\, \Big(\sum_{j=0}^{n-1} a_j\bar{\varepsilon}_j\Big)$$

Erwägen wir, wieviele verschiedene Paare entstehen auf der rechten Seite dieser Gleichung. Es gilt offenbar: 1) Zwei Faktoren mit wechselseitig gleichen, bzw. ungleichen Indexen, treffen immer gerade einmal, bzw. zweimal zusammen. 2) Es gilt: $(a_i\bar{\varepsilon}_i) \,.\, (a_{i+j}\bar{\varepsilon}_{i+j}) = a_i \,.\, a_{i+j} \,.\, \cos\frac{2\pi \,.\, j}{n}$.

Überlegen wir nun wieviele Paare $a_i \,.\, a_{i+j} \,.\, \cos\frac{2\pi j}{n}$ existieren für einzelne Indexe j: ordnen wir dem Indexensatz $(0, 1, 2, \ldots, n - 1)$ seine j-te zyklische Permutation zu; hiemit bekommen wir:

$$\begin{aligned}\{(0, j), (1, 1 + j), \ldots, (n - 1 - j, n - 1), (n - j, 0),\\ (n - j + 1, 1), \ldots, (n - 1, j - 1)\}.\end{aligned} \tag{17}$$

Betrachten wir nun zwei Fälle: a) $n = 2k + 1$, $k = 1, 2, 3, \ldots$: Dann geht die Folge (17), schrittweise für $j = 1, 2, \ldots, k$, in die zugehörige Folge über, die in jedem Fall aus $(2k + 1)$ verschiedenen Paaren besteht. b) $n = 2k$, $k = 2, 3, 4, \ldots$: Dann geht (17) in die zugehörige Folge über, die bei den Fällen $j \neq k$ immer aus $2k$

verschiedenen Paaren besteht. In dem Fall $j = k$ erscheint jedoch jedes Paar gerade zweimal, so dass summarisch gilt:

ad a) $$r^2 = \sum_{i=0}^{2k} a_i^2 + 2 \,.\, \sum_{i=0}^{2k} \sum_{j=1}^{k} a_i \,.\, a_{i+j} \,.\, \cos \frac{2\pi j}{2k+1}$$

ad b) $$r^2 = \sum_{i=0}^{2k-1} a_i^2 + 2 \,.\, \sum_{i=0}^{2k-1} \sum_{j=1}^{k-1} a_i \,.\, a_{i+j} \,.\, \cos \frac{\pi j}{k} - \sum_{i=0}^{2k-1} a_i \,.\, a_{i+k}$$

Daraus bekommen wir, nach kurzer Umformung, die obigen Formeln.

Die Beziehung $a_{cn+j} = a_j$, wo c eine ganze Zahl ist, geht daraus hervor, dass immer a_i als Koeffizient beim Vektor $\bar{\varepsilon}_i$ fungiert.

22] **Definition.** In einem Koordinaten-$(n + 1)$-eck, der einem beliebigen Punkt $P \in \boldsymbol{\alpha}$, gehört, fungieren folgende metrischen Funktionen:

$$\varphi_{n,j} = \frac{a_j}{r}; \qquad j = 0, 1, \ldots, n-1 \tag{18}$$

Mit Worten: $\varphi_{n,j}$ ist das Verhältnis der j-ten Kathete $(j = 0, 1, \ldots, n-1)$ zur Hypothenuse, in einem Koordinaten-$(n + 1)$-eck eines beliebigen Punktes $P \in \boldsymbol{\alpha}$, $P \neq 0$.

23] **Bemerkung.** Eine so definierte Funktion $\varphi_{n,,j}$ ändert sich im allgemein mit der Änderung der Lage der zuständigen Gipfelpunkte des Koordinaten-$(n + 1)$-eckes; also nicht nur mit der Änderung der Lage des Punktes P.

24] **Satz.** Zwischen den metrischen Funktionen $\varphi_{n,i}$, $i = 0, 1, \ldots, n-1$, die einem Koordinaten-$(n + 1)$-eck gehören, gelten folgende Relationen:

a) Für $n = 2k + 1$, $k = 1, 2, 3, \ldots$

$$1 = \sum_{i=0}^{2k} \left(\sum_{j=0}^{k} + \sum_{j=1}^{k}\right) \varphi_{n,i} \cdot \varphi_{n,i+j} \cdot \cos \frac{2\pi j}{2k+1} \tag{19}$$

b) Für $n = 2k$, $k = 2, 3, 4, \ldots$

$$1 = \sum_{i=0}^{2k-1} \left(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\right) \varphi_{n,i} \cdot \varphi_{n,i+j} \cdot \cos \frac{\pi j}{k} \tag{20}$$

Beweis. Wenn wir die Gleichung (15), bzw. (16), durch die Grösse r^2 dividieren, dann bekommen wir nach kurzer Umformung und nach dem Einsatz von (18) sofort die zu beweisenden Relationen.

C. Die Verbindung der gefundenen Relationen, d. h. der analytischen mit den metrischen.

25] **Satz.**

a) Für $n = 2k + 1$, $k = 1, 2, 3, \ldots$, und für $t \in (-\infty, \infty)$, gilt folgende Identität.

$$e^{2\left(\cos \frac{2\pi}{2k+1}\right) . t} = \sum_{i=0}^{2k} \left(\sum_{j=0}^{k} + \sum_{j=1}^{k}\right) f_{n,i}(t) \cdot f_{n,i+j}(t) \,.\, \cos \frac{2\pi j}{2k+1} \tag{21}$$

b) Für $n = 2k$, $k = 2, 3, 4, \ldots$, und für $t \in (-\infty, \infty)$, gilt folgende Identität:

$$e^{2\left(\cos\frac{\pi}{k}\right).t} = \sum_{i=0}^{2k-1}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big) f_{n,i}(t) \cdot f_{n,i+j}(t) \cdot \cos\frac{\pi j}{k} \tag{22}$$

Beweis ad a) ε_i bedeutet hier die i-te Wurzel der Gleichung $\lambda^{2k+1} = 1$. Weiter bezeichnen wir $\varepsilon_{2k+1-j} = \varepsilon_{-j}$. Dann gilt, nach (9) und nach (2):

$$\sum_{i=0}^{2k}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big) f_{2k+1,i}(t) \cdot f_{2k+1,i+j}(t) \cdot \cos\frac{2\pi j}{2k+1} = \sum_{i=0}^{2k}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)\left(\frac{1}{2k+1}\sum_{r=0}^{2k}\varepsilon_r^i e^{\varepsilon_r t}\right) \cdot$$
$$\cdot\left(\frac{1}{2k+1}\sum_{s=0}^{2k}\varepsilon_s^{i+j} e^{\varepsilon_s t}\right)\cdot\frac{1}{2}(\varepsilon_j + \varepsilon_{-j}) = \frac{1}{2}\cdot\frac{1}{(2k+1)^2}\sum_{r=0}^{2k}\sum_{s=0}^{2k} e^{(\varepsilon_r+\varepsilon_s)t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)\varepsilon_{js}\,\cdot$$
$$\cdot(\varepsilon_j + \varepsilon_{-j})\sum_{i=0}^{2k}\varepsilon_i^{r+s} = |\text{ mit Hinsicht auf (5) }| = \frac{1}{2}\cdot\frac{1}{2k+1}\sum_{s=0}^{2k} e^{(\varepsilon_s+\varepsilon_{-s})t}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)\cdot$$
$$\cdot\,\varepsilon_{js}(\varepsilon_j + \varepsilon_{-j}) = |\text{ zerlgen wir }| = \left\{\frac{1}{2}\cdot\frac{1}{2k+1}e^{(\varepsilon_0+\varepsilon_0)t}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)\varepsilon_0(\varepsilon_j - \varepsilon_{-j})\right\} +$$
$$+\left\{\frac{1}{2}\cdot\frac{1}{2k+1}\sum_{p=1}^{k} e^{(\varepsilon_p+\varepsilon_{-p})t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)(\varepsilon_{jp} + \varepsilon_{-jp})(\varepsilon_j + \varepsilon_{-j})\right\}.$$

Die Ausdrücke in den geschweiften Klammern werden wir nun nacheinander umformen:

1) $$\frac{1}{2}\cdot\frac{1}{2k+1}e^{2t}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)(\varepsilon_j + \varepsilon_{-j}) = \frac{1}{2}\cdot\frac{1}{2k+1}e^{2t}\cdot[2\varepsilon_0 + 2(\varepsilon_1 + \varepsilon_{-1}) +$$
$$+ 2(\varepsilon_2 + \varepsilon_{-2}) + \ldots + 2(\varepsilon_k + \varepsilon_{-k})] = \frac{1}{2k+1}e^{2t}\sum_{s=0}^{2k}\varepsilon_s = 0,\text{ siehe (5).}$$

2) $$\frac{1}{2}\cdot\frac{1}{2k+1}\sum_{p=1}^{k} e^{(\varepsilon_p+\varepsilon_{-p})t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k}\Big)[(\varepsilon_j^{p+1} + \varepsilon_{-j}^{p+1}) + (\varepsilon_j^{p-1} + \varepsilon_{-j}^{p-1})] = \frac{1}{2}\cdot$$
$$\cdot\frac{1}{2k+1}\sum_{p=1}^{k} e^{(\varepsilon_p+\varepsilon_{-p})t}\cdot\Big[2\sum_{u=0}^{2k}\varepsilon_u^{p+1} + 2\sum_{v=0}^{2k}\varepsilon_v^{p-1}\Big] = |\text{ mit Hinsicht auf (5) }| = \frac{1}{2}\cdot$$
$$\cdot\frac{1}{2k+1}e^{(\varepsilon_1+\varepsilon_{-1})t}\cdot 2(2k+1) = e^{2\left(\cos\frac{2\pi}{2k+1}\right)t}.$$

Beweis ad b) ε_j bedeutet hier die j-te Wurzel der Gleichung $\lambda^{2k} = 1$. Weiter bezeichnen wir $\varepsilon_{2k-j} = \varepsilon_{-j}$. Dann gilt, nach (9) und nach (2):

$$\sum_{i=0}^{2k-1}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big) f_{2k,i}(t) \cdot f_{2k,i+j}(t) \cdot \cos\frac{\pi j}{k} = \sum_{i=0}^{2k-1}\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big)\left(\frac{1}{2k}\cdot\sum_{r=0}^{2k-1}\varepsilon_r^i e^{\varepsilon_r t}\right)\left(\frac{1}{2k}\cdot\right.$$
$$\left.\cdot\sum_{s=0}^{2k-1}\varepsilon_s^{i+j} e^{\varepsilon_s t}\right)\cdot\frac{1}{2}(\varepsilon_j + \varepsilon_{-j}) = \frac{1}{2}\cdot\frac{1}{(2k)^2}\cdot\sum_{r=0}^{2k-1}\sum_{s=0}^{2k-1} e^{(\varepsilon_r+\varepsilon_s)t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big)\varepsilon_{js}(\varepsilon_j + \varepsilon_{-j}).$$
$$\cdot\sum_{i=0}^{2k-1}\varepsilon_i^{r+s} = |\text{ mit Hinsicht auf (5) }| = \frac{1}{2}\cdot\frac{1}{2k}\cdot\sum_{s=0}^{2k-1} e^{(\varepsilon_s+\varepsilon_{-s})t}\cdot\Big(\sum_{j=0}^{k}\sum_{j=1}^{k-1}\Big)\varepsilon_{js}(\varepsilon_j + \varepsilon_{-j}) =$$
$$= |\text{ zerlegen wir }| = \left\{\frac{1}{2}\cdot\frac{1}{2k}\cdot e^{(\varepsilon_0+\varepsilon_0)t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big)\varepsilon_0\cdot(\varepsilon_j + \varepsilon_{-j})\right\} + \left\{\frac{1}{2}\cdot\frac{1}{2k}\cdot\right.$$
$$\left.\cdot\, e^{(\varepsilon_k+\varepsilon_k)t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big)\cdot\varepsilon_{jk}\cdot(\varepsilon_j + \varepsilon_{-j})\right\} + \left\{\frac{1}{2}\cdot\frac{1}{2k}\cdot\sum_{p=1}^{k-1} e^{(\varepsilon_p+\varepsilon_{-p})t}\cdot\Big(\sum_{j=0}^{k} + \sum_{j=1}^{k-1}\Big)(\varepsilon_{jt} +\right.$$
$$\left.+\,\varepsilon_{-jt})(\varepsilon_j + \varepsilon_{-j})\right\}.$$

Die Ausdrücke in den geschweiften Klammern werden wir nun nacheinander umformen:

1) $\frac{1}{2}\cdot\frac{1}{2k}\cdot e^{2t}\{2\varepsilon_0 + 2(\varepsilon_1+\varepsilon_{-1}) + 2(\varepsilon_2+\varepsilon_{-2}) + \ldots + 2(\varepsilon_{k-1}+\varepsilon_{-k+1}) + 2\varepsilon_k\} = \frac{1}{2k}\cdot$
$\cdot\, e^{2t}\,.\sum\limits_{s=0}^{2k-1}\varepsilon_s = 0$, siehe (5).

2) $\frac{1}{2}\cdot\frac{1}{2k}\cdot e^{-2t}\,.\,\{2\varepsilon_0 - 2(\varepsilon_1+\varepsilon_{-1}) + 2(\varepsilon_2+\varepsilon_{-2}) \pm \ldots + (-1)^{k-1}\,.\,2(\varepsilon_{k-1} +$
$+ \varepsilon_{-k+1}) + (-1)^k\,.\,2\varepsilon_k\} = \frac{1}{2}\cdot\frac{1}{2k}\cdot e^{-2t}\,.\,2\,.\sum\limits_{s=0}^{2k-1}(-1)^s\,.\,\varepsilon_s = \frac{1}{2}\cdot e^{-2t}\,.\,\{(-1)\,.$
$\cdot\sum\limits_{s=0}^{2k-1}\varepsilon_s + 2\,.\sum\limits_{s=0}^{k-1}\varepsilon_{2s}\} = 0$ nach (5), denn $\sum\limits_{s=0}^{k-1}\varepsilon_{2s}$ die Summe aller Wurzeln der Gleichung $\lambda^k = 1$ ist.

3) $\frac{1}{2}\cdot\frac{1}{2k}\sum\limits_{p=1}^{k-1}e^{(\varepsilon_p+\varepsilon_{-p})t}\,.\,(\sum\limits_{j=0}^{k}+\sum\limits_{j=1}^{k-1})\,[(\varepsilon_j^{p+1}+\varepsilon_{-j}^{p+1}) + (\varepsilon_j^{p-1}+\varepsilon_{-j}^{p-1})] = \frac{1}{2}\cdot$
$\cdot\frac{1}{2k}\sum\limits_{p=1}^{k-1}e^{(\varepsilon_p+\varepsilon_{-p})t}\,.\,2[(\varepsilon_0^{p+1}+\varepsilon_1^{p+1}+\varepsilon_{-1}^{p+1}+\varepsilon_2^{p+1}+\varepsilon_{-2}^{p+1}+\ldots+\varepsilon_{k-1}^{p+1}+\varepsilon_{k+1}^{p+1}+$
$+\varepsilon_k^{p+1}) + (\varepsilon_0^{p-1}+\varepsilon_1^{p-1}+\varepsilon_{-1}^{p-1}+\ldots+\varepsilon_{k-1}^{p-1}+\varepsilon_{k+1}^{p-1}+\varepsilon_k^{p-1})] = \frac{1}{2}\cdot\frac{1}{2k}\cdot e^{(\varepsilon_1+\varepsilon_{-1})t}\,.$
$.\,2\,.\,2k = e^{2\left(\cos\frac{\pi}{k}\right)t}$.

26] **Definition.** Mit dem Symbol $g_{n,j}(t)$; $j = 0, 1, \ldots, n-1$; $n \geqq 3$, bezeichnen wir folgende Funktionen:

$$g_{n,j}(t) = f_{n,j}(t)\,.\,e^{-\left(\cos\frac{2\pi}{n}\right).t}, \tag{23}$$

wo $f_{n,j}(t)$ nach (9) bestimmt werden.

27] **Satz.** Zwischen den Funktionen $g_{n,j}(t)$; $j = 0, 1, \ldots, n-1$; $n \geqq 3$, existieren folgende Relationen:

a) Für $n = 2k+1$, $k = 1, 2, 3, \ldots$ und für alle reellen Zahlen t, gilt:

$$1 = \sum_{i=0}^{2k}\left(\sum_{j=0}^{k}+\sum_{j=1}^{k}\right)g_{n,i}(t)\,.\,g_{n,i+j}(t)\,.\,\cos\frac{2\pi j}{n} \tag{24}$$

b) Für $n = 2k$, $k = 2, 3, 4 \ldots$ und für alle reellen Zahlen t, gilt:

$$1 = \sum_{i=0}^{2k-1}\left(\sum_{j=0}^{k}+\sum_{j=1}^{k-1}\right)g_{n,i}(t)\,.\,g_{n,i+j}(t)\,.\,\cos\frac{2\pi j}{n} \tag{25}$$

Beweis. Multiplizieren wir die Gleichung (21), bzw. (22), mit dem Faktor $\exp\left\{-2\left(\cos\frac{2\pi}{n}\right).\,t\right\}$. Dann bekommen wir, nach kurzer Umformung mit Hinsicht auf (23), sofort die Relation (24), bzw. (25).

Beachten wir nun die Übereinstimmung der Form der Gleichungen (19), (24), bzw. (20), (25). Auf Grund dieser Übereinstimmung finden wir einen reellen Zusammenhang zwischen den Werten der analytischen Funktionen $g_{n,i}(t)$ und den Werten der entsprechenden metrischen Funktionen $\varphi_{n,i}$, $i = 0, 1, \ldots, n-1$. Dies kann man folgenderweise ausdrücken:

28] **Korolar.** *Es sei $n \geqq 3$. Einem realen Radius $r > 0$ und einem realen Argument t kann man gerade einen Punkt $P \in \alpha$, $OP = r$, zuordnen, u.z. mit jenem Koordinatensatz dieses Punktes, für den es gilt:*

$$a_i = r \,.\, \varphi_{n,i} = r \,.\, g_{n,i}(t); \qquad i = 0, 1, \ldots, n-1 \tag{26}$$

29] **Vereinbarung.** Für die Grössen aus dem Korolar ↑ führen wir die Bezeichnung $P(a_0, a_1, \ldots, a_{n-1})$, bzw. $P(r, t)$ ein, und benenen wir sie „*der metrische-*", bzw. „*der polare Koordinatensatz des Punktes $P \in \alpha$*".

30] **Korolar.** *Es sei Index $n \geqq 3$. Jedem metrischen Koordinatensatz $P(a_0, a_1, \ldots$ $\ldots, a_{n-1})$, der einem bestimmten Punkt $P \in \alpha$ gehört, können wir gerade einen polaren Koordinatensatz $P(r, t)$ zuordnen, und umgekehrt.*

Beweis. Den Radius r ordnen wir nach (15) oder (16) zu. Dann gilt es nach (26), (23) und (11):

$$\sum_{i=0}^{n-1} a_i = r \,.\, \sum_{i=0}^{n-1} g_{n,i}(t) = r \,.\, e^{-\left(\cos\frac{2\pi}{n}\right).t} \,.\, \sum_{i=0}^{n-1} f_{n,i}(t) = r \,.\, e^{\left(1-\cos\frac{2\pi}{n}\right).t}$$

Daraus, für jedes n, bekommen wir einen reellen Wert t. Das rückwertige Verfahren führt man nach Korolar 28.

31] **Satz.** Für die Funktionen $g_{n,i}(t)$, $i = 0, 1, \ldots, n-1$, gilt eine Analogie des *Moivre*-schen Satzes:

$$\sum_{i=0}^{n-1} g_{n,i}(rt) = \left[\sum_{i=0}^{n-1} g_{n,i}(t)\right]^r \tag{27}$$

wo r, t, beliebige reelle Zahlen sind.

Beweis. Multiplizieren wir die Gleichung (12) mit dem Faktor $\exp\left\{-\left(\cos\dfrac{2\pi}{n}\right).\, rt\right\}$. Nach kleiner Umformung mit Hinsicht auf (23), bekommen wir sofort die zu beweisende Relation.

32] **Satz.** Wenn r in der Gl. (27) eine ganze Zahl ist, dann zerfällt — nach der Substitution $t = \varepsilon_1 \,.\, z$ und nach der folgenden Durchführung der angezeigten Operationen auf der rechten Seite. — diese Gleichung in n Teilgleichungen, von denen die $(n-k)$-te aus allen Gliedern der summarischen Gleichung besteht, die den Faktor ε_k enthalten; $k = 0, 1, \ldots, n-1$.

Beweis. Von dem Satz 16 ausgehend, vefahren wir ebenso wie beim Beweis des Satzes 31.

33] **Satz.** Es sei $n \geqq 3$ und $k \in \{0, 1, \ldots, n-1\}$ sei ein fest gewählter Index. Dann gelten für die Funktionen $g_{n,j}(t)$, $j = 0, 1, \ldots, n-1$, folgende Analogien der additiven Relationen bei den goniometrischen Funktionen:

$$g_{n,k}(x + y) + \sum_{j=0}^{n-1} g_{n,j}(x) \,.\, g_{n,k-j}(y) \tag{28}$$

wo x, y, beliebige reelle Zahlen sind und wo wir im Falle $k - j < 0$ den Index $n + k - j$ anstatt $k - j$ einsetzen.

Beweis. Multiplizieren wir die Gleichnug (13) mit dem Faktor $\exp\left\{-\left(\cos\frac{2\pi}{n}\right)(x+y)\right\}$. Nach kleiner Umformung bekommen wir, mit Hinsicht auf (23), die zu beweisende Relation.

D. Die Übertragung der Relationen aus dem Koordinatensystem von 2n Halbachsen auf dasjenige von n Achsen.

Setzen wir voraus, dass wir eine Ebene α mit $2n$ Halbachsen haben; $n \geqq 2$. Wir wollen nun immer die zwei Halbachsen vereinigen, die in derselber Gerade liegen, d.h. wir wollen zugleich auch die entsprechenden Paare der Koordinaten vereinigen.

34] **Bemerkung.** *Erst nach den analytischen, bzw. metrischen Reduktionen, die dieser Vereinigung entsprechen, erreichen wir die engsten Analogien der goniometrischen Funktionen, während die bisherigen als die Analogien der hyperbolischen Funktionen betrachtet werden müssen: vergleiche z. B.* (10) *mit* (34) *und siehe auch die Bemerkung* 9.

35] **Definition.** Eine Ebene mit dem Koordinatensystem von n Achsen, bezeichnen wir Knappheitshalber „*die Ebene* β".

In der Ebene γ, siehe Def. 18, seien bezeichnet einzelne Wurzeln $\varepsilon_i = \varepsilon_{2n,i}$ der Gleichung $\lambda^{2n} = 1$, gemäss der Vereinbarung 2. Ausserdem seien hier auch die zugehörigen Radiusvektoren $\bar{\varepsilon}_i = \overrightarrow{0\varepsilon_i}$ bezeichnet. Führen wir die Deckung der Ebene β mit γ durch:

Das Koordinatensystem von n Achsen in der Ebene β ist das System der n orientierten Geraden $X_0, X_1, \ldots, X_{n-1}$, die aus einem gemeinsamen Anfangspunkt 0 in solcher Richtung führen, dass die Achse $X_i \in \beta$ mit der Gerade $\varepsilon_i\varepsilon_{n+i} \in \gamma$ zusammenfällt; dabei haben X_i, ε_i auch dieselbe Orientierung.

Ein beliebiger Punkt $P \in \beta$ wird durch einen Koordinatensatz $(b_0, b_1, \ldots, b_{n-1})$ so bestimmt, dass der durch die Deckung entsprechende Punkt $P \in \gamma$ den Endpunkt des orientierten Vektorpolygons $b_0\bar{\varepsilon}_0 \wedge b_1\bar{\varepsilon}_1 \wedge \ldots \wedge b_{n-1}\bar{\varepsilon}_{n-1}$ bedeutet, das in dem Anfangspunkt 0 beginnt.

Mit dem Symbol r bezeichnen wir (auch hier) den Radius OP.

36] **Bemerkung.** Nach der eben angeführten Definition, ist die positiv orientierte Achse X_{n+i} zugleich auch die negativ orientirete Achse X_i. Aus denselben Gründen gilt: $b_{n+i} = -b_i$; $i = 0, 1, \ldots, n-1$.

37] **Lemma.** Es sei ein beliebiger Punkt $P \in \alpha$ durch seinen beliebigen Koordinatensatz $(a_0, a_1, \ldots, a_{2n-1})$ gegeben; dann wird derselbe Punkt $P \in \beta$ durch den Koordinatensatz $(b_0, b_1, \ldots, b_{n-1})$ bestimmt, wobei gilt:

$$b_i = a_i - a_{n+i}; \qquad i = 0, 1, \ldots, n-1 \tag{29}$$

Beweis — ist ersichtlich aus den Definitionen 18 und 35.

38] **Vereinbarung.** In der Ebene β werden wir (wieder) das Koordinaten-$(n+1)$-eck benützen, das einem Punkt $P \in \beta$ angehört. Seine Seiten sind: $r, b_0, b_1, \ldots, b_{n-1}$. Den Radius $r = OP$, bzw. die Seite b_i bezeichnen wir (wieder) als Hypothenuse, bzw. die i-te Kathete.

39] **Satz.** Es sei ein Punkt $P \in \beta$ gegeben, durch seinen beliebigen Koordinatensazt $(b_0, b_1, \ldots, b_{n-1})$; dann gilt:

$$r^2 = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} + \sum_{j=1}^{n-1} \right) b_i \, . \, b_{i+j} \, . \cos \frac{\pi j}{n} \tag{30}$$

Beweis. Die Formel (16), in der wir hier n anstatt k schrieben, kann man folgenderweise umformen:

$$r^2 = \left\{ 2 \, . \sum_{i=0}^{2n-1} \sum_{j=1}^{n-1} a_i \, . \, a_{i+j} \, . \cos \frac{2\pi j}{2n} \right\} + \left\{ \sum_{i=0}^{2n-1} (a_i^2 - a_i \, . \, a_{i+n}) \right\} \tag{31}$$

Als Folgerung der Vereiningung der $2n$ Halbachsen zu n Achsen, entstehen in dieser Formel folgende Reduktionen:

In dem Ausdruck in den ersten geschweiften Klammern vereinigen sich immer vier Glieder zu einem neuen:

$$2a_i a_{i+j} \cos \frac{\pi}{n} j + 2a_{n+i} a_{n+i+j} \cos \frac{\pi}{n} j + 2a_i a_{n+i+j} \cos \left(\pi + \frac{\pi}{n} j \right) +$$

$$+ 2a_{n+i} a_{i+j} \cos \left(\pi + \frac{\pi}{n} j \right) = 2 \, . \left(\cos \frac{\pi}{n} j \right) (a_i a_{i+j} + a_{n+i} a_{n+i+j} - a_i a_{n+i+j} -$$

$$- a_{n+i} a_{i+j}) = 2 \left(\cos \frac{\pi}{n} j \right) (a_i - a_{n+i}) (a_{i+j} - a_{n+i+j}) = 2 \left(\cos \frac{\pi}{n} j \right) . \, b_i b_{i+j} .$$

Ebenso in dem Ausdruck in den zweiten geschweiften Klammern: $a_i^2 - 2a_i a_{i+n} + a_{i+n}^2 = (a_i - a_{i+n})^2 = b_i^2$. (Das Glied $a_i a_{i+n}$ wird hier zweimal enthalten, siehe (17) für $n = 2k$, $j = k$).

Aus dieser Analyse geht hervor, dass die Gl. (31) nach den angeführten Reduktionen lautet:

$$r^2 = 2 \, . \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} b_i b_{i+j} \cos \frac{\pi}{n} j + \sum_{i=0}^{n-1} b_i^2$$

Diese Gleichung wird aber leicht in die Gl. (30) umgeformt.

40] **Definition.** Es gelte: $n \geqq 2$; $i = 0, 1, 2, \ldots$. Die Funktion $F_{n,i}(t)$ wird folgendermassen definiert:

$$F_{n,i}(t) = f_{2n,i}(t) - f_{2n,i+n}(t) \tag{32}$$

wo die Funktionen $f_{2n,s}(t)$ nach (9) definiert werden.

41] **Satz.** Für $F_{n,i}(t)$, $i = 0, 1, \ldots, n - 1$; $n \geqq 2$, gilt:

$$F_{n,i}(t) = \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon_{2k+1}^i \, . \, e^{\varepsilon_{2k+1} \cdot t} ; \qquad \varepsilon_j = \varepsilon_{2n,j} \tag{33}$$

Beweis. Nach (9) gilt: $f_{2n,i}(t) - f_{2n,i+n}(t) = \frac{1}{2n} \cdot \sum_{k=0}^{2n-1} (\varepsilon_k^i - \varepsilon_k^{i+n}) \, . \, e^{\varepsilon_k \cdot t}$; $i = 0, 1, \ldots, n - 1$. Dabei betrachten wir zwei Fälle:

1) $k = 2s$, $s = 0, 1, 2, \ldots$. Dann gilt: $\varepsilon_{2s}^i - \varepsilon_{2s}^{i+n} = \varepsilon_{2s}^i (1 - \varepsilon_{2s}^n) = \varepsilon_{2s}^i [1 - (\varepsilon_s)^{2n}] = \varepsilon_{2s}^i (1 - 1) = 0$.

2) $k = 2s + 1$, $s = 0, 1, 2, \ldots$ Dann gilt: $\varepsilon_k^i - \varepsilon_k^{i+n} = \varepsilon_k^i (1 - \varepsilon_k^n)$, wo $\varepsilon_k^n =$

$= \varepsilon_1^{n(2s+1)} = \varepsilon_1^{2ns} . \varepsilon_1^n = \varepsilon_s^{2n} . \varepsilon_1^n = 1 . \varepsilon_n = -1$, so dass gilt: $\varepsilon_k^i - \varepsilon_k^{i+n} =$ $= \varepsilon_k^i[1-(-1)] = 2\varepsilon_k^i$. Es gilt also summarisch:

$$\frac{1}{2n} \cdot \sum_{k=0}^{2n-1} (\varepsilon_k^i - \varepsilon_k^{i+n}) . e^{\varepsilon_k . t} = \frac{1}{n} \cdot \sum_{k=0}^{n-1} \varepsilon_{2k+1}^i . e^{\varepsilon_{2k+1} . t}; \qquad \varepsilon_j = \varepsilon_{2n,j}$$

42] **Satz.** Die Funktionen $F_{n,i}(t)$; $i = 0, 1, \ldots, n-1$; $n \geqq 2$, kann man in der Form folgender Potenzreihen ausdrücken, die ersichtlich überall konvergieren:

$$F_{n,0}(t) = \sum_{s=0}^{\infty} (-1)^s . \frac{t^{ns}}{(ns)!}$$

(34)

$$F_{n,j}(t) = \sum_{s=1}^{\infty} (-1)^s . \frac{t^{ns-j}}{(ns-j)!}; \qquad j = 1, 2, \ldots, n-1$$

Beweis. Nach (33) gilt: $F_{n,j}(t) = \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon_{2k+1}^j . e^{\varepsilon_{2k+1} . t} = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \varepsilon_{2k+1}^j \frac{(\varepsilon_{2k+1} . t)^i}{i!} =$ $= \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k=0}^{n-1} \varepsilon_{2k+1}^{i+j} . \frac{1}{n}$. Dabei gilt: $\sum_{k=0}^{n-1} \varepsilon_{2k+1}^r = \varepsilon_1^r + \varepsilon_3^r + \ldots + \varepsilon_{2n-1}^r = (\varepsilon_0^r + \varepsilon_1^r +$ $+ \varepsilon_2^r + \ldots + \varepsilon_{2n-1}^r) - (\varepsilon_0^r + \varepsilon_2^r + \varepsilon_4^r + \ldots + \varepsilon_{2(n-1)}^r) = s_{2n,r} - (\varepsilon_0^{2r} + \varepsilon_1^{2r} + \ldots +$ $+ \varepsilon_{n-1}^{2r}) = s_{2n,r} - s_{n,2r}$, siehe das Lemma 5. Also gilt: $F_{n,j}(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} .$ $. \frac{1}{n} (s_{2n,i+j} - s_{n,2(i+j)})$. Hier aber gilt nach 5.:

$$\frac{1}{n}(s_{2n,r} - s_{n,2r}) = \begin{cases} = \frac{1}{n}(2n - n) = 1 \text{ für } r = n . 2p; \quad p = 0, 1, 2, \ldots \\ = \frac{1}{n}(0 - n) = -1 \text{ für } r = n(2p+1); \quad p = 0, 1, 2, \ldots \\ = 0 \text{ für die übrigen natürlichen Zahlen } r. \end{cases}$$

Summarisch: Die von der Null verschiedenen Koeffizienten, d. h. $+1$, -1, erscheinen nur für $r = i + j = n . s$, $(s = 0, 1, 2, \ldots) \Rightarrow i = ns - j$; dabei entspricht einem ungeraden (geraden) s das Zeichen minus (plus). (Hier gilt $i \geqq 0$, wie aus der Gl. $i = ns - j$ ersichtlich ist, so dass die untere Grenze für s erst von der Einser an genommen wird, wenn $j > 0$ ist. Dabei gilt nach (32) und (4): $F_{n,0}(0) = 1$, so dass für $j = 0$ eine selbständige Formel eingeführt werden muss).

43] **Lemma.** Für $j = 0, 1, \ldots, n-1$; $n \geqq 2$, gilt:

(35)
$$F_{n,n+j}(t) = -F_{n,j}(t).$$

Beweis. Nach (32) gilt: $F_{n,n+j}(t) = f_{2n,n+j}(t) - f_{2n,2n+j}(t)$, wobei nach (9) gilt: $f_{2n,2n+j}(t) = \frac{1}{2n} \cdot \sum_{s=0}^{2n-1} \varepsilon_s^{2n+j} . e^{\varepsilon_s . t} = \frac{1}{2n} \sum_{s=0}^{2n-1} \varepsilon_s^{2n} . \varepsilon_s^j e^{\varepsilon_s . t} = \frac{1}{2n} \sum_{s=0}^{2n-1} \varepsilon_s^j . e^{\varepsilon_s . t} = f_{2n,j}(t)$, so dass gilt: $F_{n,n+j}(t) = f_{2n,n+j}(t) - f_{2n,j}(t) = -F_{n,j}(t)$.

44] **Satz.** Zwischen den Funktionen $F_{n,i}(t)$; $i = 0, 1, \ldots, n-1$; $n \geqq 2$, gilt folgende Relation:

(36)
$$e^{2\left(\cos\frac{\pi}{n}\right).t} = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} + \sum_{j=1}^{n-1}\right) F_{n,i}(t) . F_{n,i+j}(t) . \cos\frac{\pi}{n} j$$

wo immer $F_{n,n+s}(t) = -F_{n,s}(t)$; $s = 0, 1, \ldots, n-1$.

Beweis. In der Relation (22), in der wir n anstatt k schreiben, kann man, nach (32) und (35), immer vier Glieder zu einem einzigen vereinigen. Vor allem formen wir die G (22) folgenderweise um:

$$\mathrm{e}^{2\left(\cos\frac{\pi}{n}\right).t} = \left\{2\sum_{i=0}^{2n-1}\sum_{j=0}^{n-1} f_i \cdot f_{i+j} \,.\, \cos\frac{\pi}{n} j\right\} + \left\{\sum_{i=0}^{2n-1}(f_i^2 - f_i \cdot f_{n+i})\right\} \tag{37}$$

wo $f_i = f_{2n,i}(t)$, usw. Das weitere Verfahren ist ganz analogisch mit der Umformung der Gleichung (31), d. h. wir erreichen folgende Umformung der Gl. (37):

$$\mathrm{e}^{2\left(\cos\frac{\pi}{n}\right).t} = 2\sum_{i=0}^{n-1}\sum_{j=1}^{n-1} F_{n,i}(t) \,.\, F_{n,i+j}(t) \,.\, \cos\frac{\pi}{n} j + \sum_{i=0}^{n-1} F_{n,i}^2(t)$$

Diese Gleichung wird aber leicht in die Gl. (36) umgeformt.

45] **Definition.** Mit dem Symbol $G_{n,i}(t)$; $i = 0, 1, \ldots, n-1$; $n \geqq 2$, wird folgende Funktion bezeichnet:

$$G_{n,i}(t) = F_{n,i}(t) \,.\, \mathrm{e}^{-\left(\cos\frac{\pi}{n}\right).t} \tag{38}$$

wo die Funktionen $F_{n,i}(t)$ nach (33) festgesetzt werden.

46] Satz. Zwischen den Funktionen $G_{n,i}(t)$; $i = 0, 1, \ldots, n-1$; $n \geqq 2$, gilt folgende Relation:

$$1 = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} + \sum_{j=1}^{n-1}\right) G_{n,i}(t) \,.\, G_{n,i+j}(t) \,.\, \cos\frac{\pi}{n} j \tag{39}$$

wo gilt: $G_{n,n+s}(t) = -G_{n,s}(t)$, $\quad s = 0, 1, \ldots, n-1$.

Beweis. Wenn wir die Gl. (36) mit dem Faktor $\exp\left\{-2\left(\cos\frac{\pi}{n}\right)t\right\}$ multiplizieren, erhalten wir, nach kurzer Umformung mit Hinsicht auf (38), sofort die Relation (39).

47] **Definition.** Es sei ein Punkt $P \in \mathfrak{P}$ durch seinen beliebigen Koordinatensatz $(b_0, b_1, \ldots, b_{n-1})$ gegeben; $n \geqq 2$, $\mathrm{OP} = r \neq 0$.

Wir definieren in dem zugehörigen Koordinaten-$(n + 1)$-eck folgende metrische Funktionen:

$$\Phi_{n,i} = \frac{b_i}{r}; \qquad i = 0, 1, \ldots, n-1 \tag{40}$$

Mit Worten: Die metrische Funktion $\Phi_{n,i}$ ist das Verhältnis der i-ten Kathete zur Hypothenuse, in einem beliebigen Koordinaten-$(n + 1)$-eck eines beliebegen Punktes $P \in \mathfrak{P}$, $P \neq 0$; $n \geqq 2$.

48] **Satz.** Zwischen den metrischen Funktionen $\Phi_{n,i}$, die einem Punkt $P \in \mathfrak{P}$ im Sinn der Def. 47 gehören, gilt folgende Relation:

$$1 = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} + \sum_{j=1}^{n-1}\right) \Phi_{n,i} \,.\, \Phi_{n,i+j} \,.\, \cos\frac{\pi}{n} j \tag{41}$$

wo gilt: $\Phi_{n,n+s} = -\Phi_{n,s}$, $s = 0, 1, \ldots, n-1$.

Beweis. Dividieren wir die Gl. (30) durch r^2; dann bekommen wir, nach kleiner Umformung mit Hinsicht auf (40), sofort die zu beweisende Relation.

Beachten wir nun die Übereinstimmung der Form der Gleichungen (39) und (41). Auf Grund dieser Übereinstimmung finden wir einen reellen Zusammenhang zwischen den Werten der analytischen Funktion $G_{n,i}(t)$ und den Werten der entsprechenden metrischen Funktion $\Phi_{n,i}$; man kann es folgenderweise ausdrücken:

49] **Korolar.** *Es sei $n \geqq 2$. Einem reellen Radius $r > 0$ und einem reellen Argument t, kann man gerade einen Punkt $P \in \beta$, $OP = r$, zuordnen, u. z. mit jenem Koordinatensatz dieses Punktes, für den es gilt:*

$$b_i = r \,.\, \Phi_{n,i} = r \,.\, G_{n,i}(t); \qquad i = 0, 1, \ldots, n-1. \tag{42}$$

50] **Vereinbarung.** Für die Grössen aus dem Korolar 49. führen wir die Bezeichnung $P(b_0, b_1, \ldots, b_{n-1})$, bzw. $P(r, t)$ ein, und benennen wir sie „*der metrische* —" bzw. „*der polare Koordinatensatz des Punktes $P \in \beta$.*"

51] **Satz.** Es gilt die Identität für $n \geqq 3$, $t \in (-\infty, \infty)$:

$$\sum_{k=0}^{n-1} \left(\cos \frac{\pi k}{n}\right) . \, G_{n,k}(t) = \cos \left[\left(\sin \frac{\pi}{n}\right) . \, t\right] \tag{43}$$

Beweis. Multiplizieren wir die Funktion $F_{n,k}(t)$ mit der Einheitswurzel a) $\varepsilon_{2n,k}$, b) $\varepsilon_{2n,2n-k}$ und führen wir dann die Summation für $k = 0, 1, \ldots, n-1$ durch. Mit Hinsicht auf die Gleichungen: 1) $(-1) \,.\, \varepsilon_{2n,k} = \varepsilon_{2n,n+k}$, 2) $\varepsilon_{2n,k} = \varepsilon_{2n,k+2ns} = (\varepsilon_{2n,1})^{k+2ns}$, $s = 0, 1, 2, \ldots$, bekommen wir nach (34):

ad a) $\sum\limits_{k=0}^{\infty} \dfrac{(t \,.\, \varepsilon_{2n,1})^k}{k!} = e^{t \,.\, \varepsilon_{2n,1}}$, ad b) $\sum\limits_{k=0}^{\infty} \dfrac{(t \,.\, \varepsilon_{2n,2n-1})^k}{k!} = e^{t \,.\, \varepsilon_{2n,2n-1}}$. Durch Summation dieser zwei Gleichungen bekommen wir: $\sum\limits_{k=0}^{n-1} (\varepsilon_{2n,k} + \varepsilon_{2n,2n-k}) \,.\, F_{n,k}(t) = e^{t \,.\, \varepsilon_{2n,1}} + e^{t \,.\, \varepsilon_{2n,2n-1}}$, und weiter, nach (2): $\sum\limits_{k=0}^{n-1} 2 \,.\, \left(\cos \dfrac{\pi k}{n}\right) . \, F_{n,k}(t) = e^{t \,.\, \cos \frac{\pi}{n}} \,.\, 2 \,.\, \cos \left(t \,.\, \sin \dfrac{\pi}{n}\right)$ Multiplizieren wir diese Gleichung mit $\dfrac{1}{2} \,.\, e^{-t \,.\, \cos \frac{\pi}{n}}$, so bekommen wir nach (38) sofort die zu beweisende Gleichung.

52] **Lemma.** Die Gleichung $\bar{r} = \sum\limits_{i=1}^{m} \bar{a}_i$ bedeute eine willkürliche Zerlegung eines Vektors $\bar{r}$ in m Komponenten, $m \geqq 2$. Es sei $\bar{s}$ eine willkürliche Richtung. Dann gilt:

Der Absolutwert der Summe der lotrechten Projektionen der Komponenten $\bar{a}_i$, $i = 1, 2, \ldots, m$, in die Richtung $\bar{s}$, ist minder oder höchstens gleich dem Absolutwert $|\,\bar{r}\,|$.

Die Gültigkeit des Lemmas ist elementar bekannt.

53] **Folgerung.** Für einen beliebigen Koordinaten $(n+1)$-eck aus der Ebene β, $n \geq 3$, wenn wir die lotrechte Projektion seiner Katheten b_i in die Achse X_0

voraussetzen, gilt die Ungleichung: $-r \leqq \sum_{k=0}^{n-1} b_k \cdot \cos \frac{\pi k}{n} \leqq r$, wo r die Hypothenuse ist; d. h. es gilt auch

$$(44) \qquad -1 \leqq \frac{1}{r} \cdot \sum_{k=0}^{n-1} b_k \cdot \cos \frac{\pi k}{n} \leqq 1$$

54] **Vereinbarung.** Wenn ein Argument einer Funktion (38) aus dem Intervall $\left[0, \pi \left(\sin \frac{\pi}{n}\right)^{-1}\right]$ ist, dann und nur dann bezeichnen wir es mit einem Streifen; also z. B. $\bar{t}$ anstatt t.

55] **Korolar.** *Es sei $n \geqq 3$. Jedem metrischen Koordinatensatz $P(b_0, b_1, \ldots, b_{n-1})$, der einem bestimmten Punkt $P \in \beta$ gehört, können wir gerade einen polaren Koordinatensatz $P(r, \bar{t})$ zuordnen, und umgekehrt.*

Beweis. Den Radius r ordnen wir nach (30) zu. Dann gilt es nach (43) und (42): $\bar{t} = \left(\sin \frac{\pi}{n}\right)^{-1} \cdot \arccos \left[\sum_{k=1}^{n-1} \left(\cos \frac{\pi k}{n}\right) \cdot G_{n,k}(t)\right] = \left(\sin \frac{\pi}{n}\right)^{-1} \cdot \arccos \left[\frac{1}{r} \cdot \sum_{k=0}^{n-1} \cdot \left(\cos \frac{\pi k}{n}\right) \cdot b_k\right]$, wobei für den Ausdruck in den eckigen Klammern die Ungleichheit (44) gilt. Das rückwertige Verfahren führen wir nach dem Korolar 49. durch.

56] **Satz.** Es sei $n \geq 3$ und $k \in \{0, 1, \ldots, n-1\}$ sei ein fest gewählter Index. Dann gelten für die Funktionen $G_{n,k}(x)$, $k = 0, 1, \ldots, n-1$, folgende Analogien der additiven Relationen bei den goniometrischen Funktionen:

$$(45) \qquad G_{n,k}(x + y) = \sum_{j=0}^{n-1} G_{n,j}(x) \cdot G_{n,k-j}(y)$$

wo x, y, beliebige reellen Zahlen sind und wo wir im Falle, dass $k - j < 0$ ist, den Index $n + k - j$ anstatt $k - j$, zugleich mit dem negativen Vorzechen für das zugehörige Glied einsetzen.

Beweis. Nach (13) gilt a) $f_{2n,k}(x + y) = \sum_{j=0}^{2n-1} f_{2n,j}(x) \cdot f_{2n,k-j}(y)$, b) $f_{2n,k+n}(x + y) = \sum_{j=0}^{2n-1} f_{2n,j}(x) \cdot f_{2n,k+n-j}(y)$. Bilden wir nun die Differenz b) — a), so gilt es nach (32) und (35): $F_{n,k}(x + y) = \sum_{j=0}^{2n-1} f_{2n,j}(x) \cdot F_{n,k-j}(y) = \sum_{j=0}^{n-1} [f_{2n,j}(x) \cdot F_{n,k-j}(y) + f_{2n,j+n}(x) \cdot F_{n,k-j-n}(y)] = \sum_{j=0}^{n-1} [f_{2n,j}(x) - f_{2n,j+n}(x)] \cdot F_{n,k-j}(y) = \sum_{j=0}^{n-1} F_{n,j}(x) \cdot F_{n,k-j}(y)$. Wenn wir nun diese Gleichung mit dem Faktor $\exp\left\{-\left(\cos \frac{\pi}{n}\right) \cdot (x + y)\right\}$ multiplizieren, so bekommen wir, nach kleiner Umformung mit Hinsicht auf (38), die zu beweisende Gleichung. Die Regel für das Vorzeichen ist direkte Folgerung von (35).

57] **Satz.** Es sei $n \geqq 3$ und r, x, seien reelle Zahlen. Dann gilt folgende Analogie des *Moivre*-schen Satzes:

$$\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot G_{n,k}(rx) = \left[\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot G_{n,k}(x)\right]^r \tag{46}$$

Beweis. Benützen wir die Gleichung ad a), aus dem Beweis des Satzes 51:

$$\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot F_{n,k}(t) = e^{\varepsilon_{2n,1} \cdot t} \tag{47}$$

Potenzieren wir diese Gleichung zum r-ten Grad und legen wir x anstatt t:

$$\left[\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot F_{n,k}(x)\right]^r = [e^{\varepsilon_{2n,1} \cdot x}]^r \tag{48}$$

Durch die Subst. $t = rx$ bekommen wir aus (47):

$$\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot F_{n,k}(rx) = e^{\varepsilon_{2n,1} \cdot rx} \tag{49}$$

Da die rechten Seiten von (48), (49) übereinstimmen, wird hiemit die Gleichung, die aus den zugehörigen linken Seiten besteht, erwiesen. Multiplizeiren wir nun diese Gl. mit $\exp\left\{-\left(\cos\frac{\pi}{n}\right) \cdot rx\right\}$, so bekommen wir, mit Hinsicht auf (38), den zu beweisenden Satz.

58] **Satz.** Wenn r in der Gleichung (46) eine natürliche Zahl ist, dann zerfällt diese Gleichung in n Teilgleichungen, von denen die k-te Gleichung aus allen Gliedern der summarischen Gl. besteht, die den Faktor $\varepsilon_{2n,k}$ enthalten; $k = 0, 1, \ldots, n - 1$.

Beweis. Setzen wir in die Gl. (45) $y = (r - 1) \cdot x$ ein und multiplizieren wir zugleich diese Gl. mit der Wurzel $\varepsilon_{2n,k}$. Dann, durch die Summation solcher Gleichungen, von $k = 0$ bis $n - 1$, bekommen wir:

$$\sum_{k=0}^{n-1} \varepsilon_{2n,k} \cdot F_{n,k}(rx) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \varepsilon_{2n,k} \cdot F_{n,j}(x) \cdot F_{n,k-j}((r-1) \cdot x)$$

Eine Seite von dieser Gl. und von der Gl., die zwischen den liknen Seiten der Gleichungen (48), (49) existiert, stimmt also überein. Die weiteren Erwägungen sind dann ganz analogisch mit jenen im Beweis des Satzes 16. Der nachfolgende Übertrag, d. h. mit Hinsicht auf (38), wurde schon oben gezeigt.

59] **Die Funktionskurven** $y = G_{n,i}(x)$, $i = 0,1, \ldots, n - 1$, $n \geqq 2$.

Diese Kurven wurden gründlich in den Rechenmaschinenlaboratorien der *FS-VUT* in Brno untersucht: diese Arbeit vollzog *Ing. Ivan Direr.* Diese Rechnungen bestätigen die engsten analogischen Eigenshaften mit den Kurven $y = \cos x = G_{2,0}(x)$, $y = -\sin x = G_{2,1}(x)$:

Wenn der Absolutwert des Argumentes x wächst, nähert sich der Verlauf dieser Funktionen folgenden Kurven:

$$y = H_{n,i}(x) = A_n \cdot \sin(B_n \cdot x + C_{n,i}), \tag{50}$$

wo A_n, B_n, bzw. $C_{n,i}$ für einen festen Index n, bzw. feseten Indexenpaar (n, i), bestimmte Konstanten sind.

Die Näherung der Kurve $G_{n,i}(x)$ zur $H_{n,i}(x)$, $i = 0, 1, \ldots, n-1$, verläuft sehr progressiv. Die Kurve $y = H_{n,i}(x)$ verzögert sich gegenüber der $y = H_{n,i+1}(x)$. Die Phasenverzögerung macht $\frac{1}{2n}$ der Wellenlänge aus.

60] **Die Angaben über die Funktionskurven $G_{n,j}(x)$ für $n = 3, 4, \ldots, 7$.** *Zusammengestellt mittels* (34), (38). Die Periode, d. h. die Wellenlänge, und die Amplitüde dieser Kurven, bei $x \in (10, 30)$:

n	*Periode*		*Amplitüde*	
3	7,254	$\doteq 2\pi \,.\, \operatorname{cosec} \frac{\pi}{n}$	0,667	$\equiv \frac{2}{n}$ (Vermutung von Ing. Dírer)
4	8,464		0,501	
5	10,688		0,401	
6	12,851		0,333	
7	14,374		0,286	

E. Die Transformationen der Koordinaten eines allgemeinen Punktes einer Ebene mit Bezug auf ihre beiden Koordinatensysteme.

61] **Verabredung.** Man vähle in der Ebene den Punkt 0 als den (gemeinsamen) Anfangspunkt. Man führe durch denselben

a) zwei gegeneinander senkrechte Achsen eines Cartesischen Systems; man bezeichne sie als Y, Z; ihre Orientierung wählt man so, dass die Halbachse $+Y$ durch die Drehung um $\frac{\pi}{2}$ um den Anfangspunkt im positiven Sinn in die Lage $+Z$ übergehe.

b) die Achsen $X_0, X_1, \ldots, X_{n-1}$ des Koordinatensystems β_n für $n > 2$, nach der Definition 35.

Dabei mögen die Achsen X_0, Y den Winkel α einschliessen, den wir im positiven Sinn von $+Y$ zu $+X_0$ messen.

62] **Lemma.** Es sei $a + bi = \cos 2n\alpha + i \,.\, \sin 2n\alpha$ eine komplexe Zahl, wo $n \geq 2$ eine natürliche Zahl und α ein beliebig gewählter Winkel ist. Dann gilt die Gleichung:

$$\sum_{j=0}^{n-1} \cos\left(2\alpha + \frac{2\pi}{n} j\right) = \sum_{j=0}^{n-1} \sin\left(2\alpha + \frac{2\pi}{n} j\right) = 0 \tag{53}$$

Beweis. Es gilt für die Wurzeln $\varepsilon_{n,j}$; $j = 0, 1, \ldots, n-1$, der binomischen Gleichung $\lambda^n = a + bi$: $\sum_{j=0}^{n-1} \varepsilon_{n,j} = 0$, siehe die symmetrischen Funktionen der Wurzeln, so dass sowohl die reellen als auch die imaginären Komponenten dieser Wurzeln ebenso die Summe Null ergeben.

63] **Lemma.** Für den Index $n \geq 2$ und für den beliebigen Winkel α gilt die Gleichung:

$$(54) \qquad \sum_{j=0}^{n-1} \cos^2\left(\alpha + \frac{\pi j}{n}\right) = \sum_{j=0}^{n-1} \sin^2\left(\alpha + \frac{\pi j}{n}\right) = \frac{n}{2}$$

Beweis. Es gilt nach der bekannten Formel: $\sum_{j=0}^{n-1} \cos^2\left(\alpha + \frac{\pi j}{n}\right) = \sum_{j=0}^{n-1} \frac{1}{2}\left[1 + \right.$ $\left. + \cos\left(2\alpha + \frac{2\pi}{n} j\right)\right] = \frac{n}{2} + \frac{1}{2} \sum_{j=0}^{n-1} \cos\left(2\alpha + \frac{2\pi}{n} j\right) = \frac{n}{2}$, siehe (53). Analog beweist man den übrigbleibenden Teil des Lemmas.

64] **Lemma.** Für den Index $n \geq 2$ und für beliebige Winkel $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, gelten die Gleichungen:

$$(55) \qquad \sum_{j=0}^{n-1} \cos\left(\alpha + \frac{\pi}{n} j\right) . \cos\left(\alpha + \beta + \frac{\pi}{n} j\right) = \frac{n}{2} . \cos\beta$$

$$(56) \qquad \sum_{j=0}^{n-1} \cos\left(\alpha + \frac{\pi}{n} j\right) . \sin\left(\alpha + \beta + \frac{\pi}{n} j\right) = \frac{n}{2} . \sin\beta$$

Beweis ad (55). $\sum_{j=0}^{n-1} \cos\left(\alpha + \frac{\pi}{n} j\right) . \cos\left(\alpha + \boldsymbol{\beta} + \frac{\pi}{n} j\right) = \sum_{j=0}^{n-1} \cos\left(\alpha + \frac{\pi}{n} j\right) \cdot$ $\cdot \left[\cos\left(\alpha + \frac{\pi}{n} j\right) . \cos\boldsymbol{\beta} - \sin\left(\alpha + \frac{\pi}{n} j\right) . \sin\boldsymbol{\beta}\right] = \cos\boldsymbol{\beta} . \sum_{j=0}^{n-1} \cos^2\left(\alpha + \frac{\pi}{n} j\right) -$ $- \frac{1}{2} . \sin\boldsymbol{\beta} . \sum_{j=0}^{n-1} \sin\left(2\alpha + \frac{2\pi}{n} j\right) = \frac{n}{2} . \cos\beta$, siehe (54) und (53). Analog beweist man die Gleichung (56).

65] **Verabredung.** Wir bezeichnen die Koordinaten eines allgemeinen Punktes P in Bezug auf das Koordinatensystem $\boldsymbol{\beta}_2$, bzw. $\boldsymbol{\beta}_n$ $(n > 2)$, $P(y, z)$, bzw. $P(x_0, x_1, \ldots, x_{n-1})$.

66] **Satz.** Es mögen die Verabredungen 61. und 65. gelten. Zur Festsetzung des Satzes $P(x_0, x_1, \ldots, x_{n-1})$ durch den gegebenen (bekannten) Satz $P(y, z)$ benützt man das folgende Gleichungssystem mit einer symmetrischen Systemsdeterminante:

$$(57) \qquad \begin{array}{l} a_0 . x_0 + a_1 . x_1 + a_2 . x_2 + \ldots + a_{n-1} . x_{n-1} = b_0 . y \quad + c_0 . z \\ a_1 . x_0 + a_0 . x_1 + a_1 . x_2 + \ldots + a_{n-2} . x_{n-1} = b_1 . y \quad + c_1 . z \\ a_2 . x_0 + a_1 . x_1 + a_0 . x_2 + \ldots + a_{n-3} . x_{n-1} = b_2 . y \quad + c_2 . z \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1} x_0 + a_{n-2} x_1 + a_{n-3} x_2 + \ldots + a_0 \quad . x_{n-1} = b_{n-1} . y + c_{n-1} . z, \end{array}$$

wo $a_k = \cos\left(\frac{\pi}{n} k\right)$, $b_k = \cos\left(\alpha + \frac{\pi}{n} k\right)$, $c_k = \sin\left(\alpha + \frac{\pi}{n} k\right)$.

Beweis. Die k-te Gleichung dieses Systems, $(k = 0, 1, \ldots, n-1)$, drückt (mit Rücksicht auf die Achsenorientierung nach der Def. 35.) die Gleichheit zwischen den ortogonalen Projektionen beider Polygonalen des Punktes P, d. h. $(\bar{x}_0 \wedge \bar{x}_1 \wedge \ldots \wedge \bar{x}_{n-1})$, $(\bar{y} \wedge \bar{z})$, auf die Achse X_k aus.

67] **Lemma.** *Die Determinante des Systems (57) ist gleich Null.*

Beweis. Nach der Def. 35. hat der Anfangspunkt 0 ersichtlich auch Nicht-Null Koordinatensätze in Bezug auf das β_n-System; er hat dabei aber bloss einen Null-Koordinatensatz in Bezug auf das System β_2, d. h. $y = z = 0$. Projizieren wir also die dem Nicht-Nullsatz des Anfanges 0 zugehörige Polygonale, so wird das Gleichungssystem (57) ein homogenes und es wird eine Nicht-Null-Lösung haben. Nach einem bekannten Satz muss also seine Systemsdeterminante gleich Null sein. Nach der Def. 35 ist aber diese Determinante immer dieselbe für einen jeden Punkt $P \in \beta_n$.

68] **Satz.** Das Gleichungssystem (57) hat (im Einklang mit dem Lemma 67) unendlich viele Lösungen folgender Form:

$$(58)\quad x_j = \frac{2}{n}\cdot\left[y\,.\cos\left(\alpha + \frac{\pi}{n}j\right) + z\,.\sin\left(\alpha + \frac{\pi}{n}j\right)\right]; \qquad j = 0, 1, \ldots, n-1$$

Jeder Grösse des Winkels α entspricht also gerade ein Satz $P(x_1, x_2, \ldots, x_{n-1})$.

Beweis. Mit Rücksicht darauf, dass der Cosinus eine gerade Funktion ist, kann man die k-te Gleichung des Systems (57), $k = 0, 1, \ldots, n-1$, so scheriben:

$$(59)\qquad \sum_{j=0}^{n-1} x_j\,.\cos\left(-\frac{\pi k}{n} + \frac{\pi}{n}j\right) = y\,.\cos\left(\alpha + \frac{\pi k}{n}\right) + z\,.\sin\left(\alpha + \frac{\pi k}{n}\right)$$

Durch das Einsetzen der vorausgesetzten Lösung (58) in die linke Seite dieser Gleichung bekommt man: $\sum_{j=0}^{n-1}\frac{2}{n}\left[y\,.\cos\left(\alpha + \frac{\pi}{n}j\right) + z\,.\sin\left(\alpha + \frac{\pi}{n}j\right)\right]$.
$.\cos\left(-\frac{\pi}{n}k + \frac{\pi}{n}j\right) = \frac{2}{n}\,.\,y\,.\sum_{j=0}^{n-1}\cos\left(\alpha + \frac{\pi}{n}j\right).\cos\left(-\frac{\pi}{n}k + \frac{\pi}{n}j\right) + \frac{2}{n}\,.\,z\,.$
$.\sum_{j=0}^{n-1}\sin\left(\alpha + \frac{\pi}{n}j\right).\cos\left(-\frac{\pi}{n}k + \frac{\pi}{n}j\right) = |\text{ siehe } 64\,| = y\,.\cos\left(\alpha + \frac{\pi}{n}k\right) + z\,.$
$.\sin\left(\alpha + \frac{\pi}{n}k\right)$; dies ist aber identisch gleich der rechten Seite von (59).

69] **Satz.** Für die Bestimmung des Satzes $P(y, z)$ mittels des gegebenen (bekannten) Satzes $P(x_0, x_1, \ldots, x_{n-1})$ benützen wir die Formel:

$$(60)\qquad \text{a)}\ y = \sum_{j=0}^{n-1} x_j\,.\cos\left(\alpha + \frac{\pi}{n}j\right), \qquad \text{b)}\ z = \sum_{j=0}^{n-1} x_j\,.\sin\left(\alpha + \frac{\pi}{n}j\right)$$

Beweis. Nach der Verabredung 61 stellt die erste, bzw. die zweite dieser Gleichungen ersichtlich die senkrechte Projektion der Polygonale $(\bar{x}_0 \wedge \bar{x}_1 \wedge \ldots \ldots \wedge \bar{x}_{n-1})$ in die Y- bzw. Z-Achse dar.

70] **Bemerkung.** Aus den Gleichungen (58) kann man die Gleichungen (60-a, -b) folgendermassen ableiten:

ad a) Man multipliziere die Gl. (58) mit dem Faktor cos $\left(\alpha + \frac{\pi}{n}j\right)$:

Durch die Summierung solcher Gleichungen vom Index j $= 0$ bis n $- 1$ bekommt

man mit Rücksicht auf (54) und (53): $\sum_{j=0}^{n-1} x_j \,.\, \cos\left(\alpha + \frac{\pi}{n} j\right) = \frac{2}{n} \,.\, y \,.\, \sum_{j=0}^{n-1} \cos^2\left(\alpha + \right.$ $\left. + \frac{\pi}{n} j\right) + \frac{1}{n} \,.\, z \,.\, \sum_{j=0}^{n-1} \sin\left(2\alpha + \frac{2\pi}{n} j\right) = y$, was die Gleichung (60-a) ist.

ad b) Wir multiplizieren die Gl. (58) mit dem Faktor $\sin\left(\alpha + \frac{\pi}{n} j\right)$:

Durch die Summierung solcher Gleichungen vom Index $j = 0$ bis $n - 1$ bekommen wir analog (60-b).

71] **Bemerkung.** Den Zusammenhang gegenseitig rez'proker Transformationen (58) und (60) kann man folgendermassen veranschaulichen:

$$(61) \qquad \begin{array}{c|cccc} & x_0 & x_1 & \dots & x_{n-1} \\ \hline y & a_0 & a_1 & \dots & a_{n-1} \\ z & b_0 & b_1 & \dots & b_{n-1} \end{array} \quad \Big\downarrow \frac{2}{n}$$

wo $a_j = \cos\left(\alpha + \frac{\pi}{n} j\right)$, $b_j = \sin\left(\alpha + \frac{\pi}{n} j\right)$.

Es gilt dabei: Stellt man (nach dem Schema) die Gleichung in der vertikalen Richtung zusammen, so multipliziert man ihre rechte Seite mit dem Koeffizienten $\frac{2}{n}$.

72] **Verabredung.** Den Winkel des Radiusvektors $r = OP$ mit der Halbachse $+Y$, im positiven Sinne von $+Y$ gemessen, bezeichnen wir ω.

73] **Satz.** Es sei der Index $n \geqq 2$. Für die metrischen Funktionen $\Phi_{n,j}$, siehe (40), gilt die Beziehung:

$$(62) \qquad \Phi_{n,j} = \frac{2}{n} \,.\, \cos\left(\omega - \alpha - \frac{\pi}{n} j\right); \qquad j = 0, 1, \dots, n - 1$$

Beweis. Dividieren wir die Gleichung (58) durch die Zahl $r = OP$, $(P \neq O)$; wir bekommen:

$$\frac{x_j}{r} = \frac{2}{n} \cdot \left[\frac{y}{r} \,.\, \cos\left(\alpha + \frac{\pi}{n} j\right) + \frac{z}{r} \,.\, \sin\left(\alpha + \frac{\pi}{n} j\right)\right]$$

oder nach 47, 61, 72 und nach der bekannten Formel:

$$\Phi_{n,j} = \frac{2}{n} \cdot \left[\cos\omega \,.\, \cos\left(\alpha + \frac{\pi}{n} j\right) + \sin\omega \,.\, \sin\left(\alpha + \frac{\pi}{n} j\right)\right] = \frac{2}{n} \,.\, \cos\left(\omega - \alpha - \right.$$
$$\left. - \frac{\pi}{n} j\right), \text{ w. z. b. w.}$$

74] **Bemerkung.** Durch den Satz 73 wird — mit Rücksicht auf Korolar 49 — analytisch die Vermutung über die genaue Amplitudengrösse der Funktionen $G_{n,j}(t)$, $(j = 0, 1, \dots, n - 1)$, bestätigt, siehe 60.

Wir wissen jedoch aus den nummerischen Untersuchungen, dass es in einem kleinen Intervall um den Punkt $t = 0$ die Ungleichung $1 \geqq |G_{n,j}(t)| > \frac{2}{n}$ gilt. Wir sollten also diese Erscheinung im Hinblick auf den Satz 73 ausklären:

Nach 35. gehört einem Punkt auf der X_0-Achse auch der Koordinatensatz $P(a, 0, 0, \ldots, 0)$, $a \neq 0$. Die entsprechende Koordinatenpolygonale reduziert sich hier auf die einzige Komponente x_0. Diese Reduktion betrifft also auch das System (57), so dass der für eine n-gliedrige Polygonale abgeleitete Satz 73. für diesen Fall nicht gilt.

75] **Bemerkung.** Durch den Satz 73. wird weiter die im Absatz 60. angeführte nummerische Feststellung bestätigt, dass die Phasenverspätung der Funktion $G_{n,j}(t)$ gegenüber der Funktion $G_{n,j+1}(t)$ ein $\frac{1}{2n}$ — tel der Wellenlänge beträgt.

76] **Bemerkung.** Die übrigbleibende, aus der nummerischen Untersuchung folgende Vermutung, dass nämlich die Wellenlänge bei den Funktionen $G_{n,j}(t)$ zur Grösse $2\pi \, . \operatorname{cosec} \frac{\pi}{n}$, konvergiert, sollte im Zusammenhang mit 51., 54. und 55. untersucht werden.

77] **Folgerung.** Überlegen wir nun die Möglichkeit die analytische Geometrie (weiter AG) ind der Ebene β für den Index $n \geqq 3$ zu entwickeln. Überlegen wir, dass hier eine (1,1)-Korrespondenz zwischen den metrischen und polaren Koordinatensätzen existiert. Dies bedeutet also, dass es hier möglich wäre, zwei Arten der Gleichungen parallel zu entwickeln: die allgemeinen und die parametrischen.

Als eine Illustration dieser Idee führe ich eine Kurve an, die in der Ebene β für $n = 3$ durch folgende parametrischen Gleichungen definiert wird:

$$x_0 = a \, . \, G_{30}(t), \qquad x_1 = b \, . \, G_{31}(t), \qquad x_2 = c \, . \, G_{32}(t), \tag{51}$$

wo a, b, c reelle Konstanten sind.

Setzen wir diese Gleichungen in die Gleichung (39) für $n = 3$ ein, so bekommen wir die entsprechende allgemeine Gleichung:

$$\frac{x_0^2}{a^2} + \frac{x_1^2}{b^2} + \frac{x_2^2}{c^2} - \frac{x_0 x_1}{a \, . \, b} - \frac{x_0 x_2}{a \, . \, c} - \frac{x_1 x_2}{b \, . \, c} = 1 \tag{52}$$

Es handelt sich wahrscheinlich um eine Analogie einer Ellipse aus der Kartesischen $A.\,G.$

Man kann also voraussetzen, dass die $A.\,G.$ in der Ebene β für $n > 2$ die einfachen Gleichungen für verschiedene Kurven bieten würde, die zu ihren Vorbildern aus der kartesischen $A.\,G.$ analog sind.

Dies gilt insbesondere für jene Kurven, die man als die Analogien der goniometrisch definierbaren Kurven einführen kann.

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EMBEDDINGS OF LATTICES IN THE LATTICE OF TOPOLOGIES

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R. Duda put the problem (Coll. Math. XXIII, 2 (1971), Problem 749) whether any lattice can be realized as a sublattice of the lattice of all topologies (or even of all T_1-topologies) on a certain set. We even prove that for any lattice L there exists a set E and an embedding ψ of L in the lattice of all topologies on E such that ψx is a completely Hausdorff topology for every $x \in L$. This embedding we get in two steps. Firstly, there exists a set E and a sublattice L' of the lattice of all topologies on E isomorphic to L, which follows from the well-known Whitman's result that any lattice is isomorphic to a sublattice of the lattice of all partitions on a certain set. Secondly, we construct a completely Hausdorff topology $\mathfrak{T}$ on E such that $\psi_2(\mathfrak{S}) = \mathfrak{S} \vee \mathfrak{T}$ for $\mathfrak{S} \in L'$ defines an embedding of L' in the lattice of all topologies on E finer then $\mathfrak{T}$.

This construction is given in § 3. In § 3. it is also shown that there exists a lattice L for which no embedding ψ of L in the lattice of all topologies on a set exists such that ψx is a metrizible topology for every $x \in L$. In addition we give in § 2. another but far simpler proof that any lattice can be embedded in the lattice of all $\mathfrak{T}_1$-topologies on some set.

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§ 1. BASIC NOTIONS

Definitions concerning lattices can be found in [12]. We recall some of them. A mapping φ from a lattice L into a lattice L' is defined to be a $\vee$-homomorphism if $\varphi(a \vee b) = \varphi a \vee \varphi b$ for every $a, b \in L$. Dually we define a $\wedge$-homomorphism. An embedding is an injective homomorphism. A lattice L is called simple if any homomorphism of L onto a lattice L' is either an isomorphism or L' consists of a single element. Let L be a lattice. We put $[a) = \{x \in L / x \geqq a\}$, $(a] = \{x \in L / x \leqq a\}$. The set-theoretic union (intersection) will be denoted by $\cup(\cap)$, a lattice join (meet) by $\bigvee(\bigwedge)$. All necessary topological definitions are given in [4]. We identify a topology with the system of its open sets. The closure of a set X in a topology $\mathfrak{T}$, we denote by $Cl_{\mathfrak{T}}(X)$. A topology $\mathfrak{T}$ on E is called completely Hausdorff if for any two distinct points $a, b \in E$ there exists a continuous function f from $\mathfrak{T}$ to the real line with $fa \neq fb$. Any completely Hausdorff topology is Hausdorff.

We shall give some results concerning lattices of topologies. Let $\mathscr{B}(E)$ be the system of all topologies on a set E ordered by the set-inclusion. $\mathscr{B}(E)$ is a complete lattice. The least element is the indiscrete topology $\{\varnothing, E\}$ and the greatest element is the discrete topology exp E. Meets coincide with set intersections and the join of two topologies $\mathfrak{T}_1$, $\mathfrak{T}_2$ is the topology with the basis $\{V \cap W / V \in \mathfrak{T}_1,\ W \in \mathfrak{T}_2\}$. $\mathscr{B}(E)$ is atomic and any topology is a join of atoms. Atoms are precisely topologies $\{\varnothing, X, E\}$, where $\varnothing \neq X \subsetneqq E$ (see Vaidyanathaswamy [13]). $\mathscr{B}(E)$ is dually atomic and any topology is a meet of dual atoms. Dual atoms are precisely topologies $\mathfrak{G} \cup \cup\, exp\, (E - \{a\})$, where $a \in E$ and $\mathfrak{G}$ is an ultrafilter on E different from the principal ultrafilter generated by a (see Fröhlich [1] or Sekanina [10]). Let $\mathscr{K}(E)$ be the lattice of all $\mathfrak{T}_1$-topologies on E. $\mathscr{K}(E)$ is a complete sublattice of $\mathscr{B}(E)$. The least element in $\mathscr{K}(E)$ is the cofinite topology $\mathfrak{K}(E) = \{X \subseteq E / E - X \text{ is finite}\} \cup \{\varnothing\}$. It holds $(E) = [\mathfrak{K}(E))$. Hence $\mathscr{K}(E)$ is dually atomic. The dual atoms of $\mathscr{K}(E)$ are free ultraspaces, i.e. ultraspaces for which $\mathfrak{G}$ is a free ultrafilter. A topology is called principal if the union of an arbitrary family of its closed sets is closed. Principal topologies form a sublattice of the lattice of topologies (Steiner [11]). More detailed information on lattices of topologies can be found in Larson, Zimmerman [6].

§2. ONE CONSTRUCTION OF EMBEDDINGS OF LATTICES IN THE LATTICE OF $\mathfrak{T}_1$-TOPOLOGIES

It was already mentioned that the starting point of our investigation is the following well-known Whitman's result.

2.1 Theorem. (see [14]): *Any lattice is isomorphic to a sublattice of the lattice of all partitions on a certain set.*

The lattice of all partitions on a set E will be denoted by $\mathscr{P}(E)$. We recall that $\mathfrak{R}_1 \leqq \mathfrak{R}_2$ for $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(E)$ iff for every $X \in \mathfrak{R}_1$ there exists $Y \in \mathfrak{R}_2$ such that $X \subseteq Y$.

From this Whitman's result it follows that any lattice can be embedded in the lattice of topologies. A topology is called a partition topology if every its open set is closed. Let $\mathscr{P}^\circ(E)$ be the system of all partition topologies on E.

2.2. Theorem (see [13]): *$\mathscr{P}^\circ(E)$ is a sublattice of $\mathscr{B}(E)$.*

Proof: Evidently the intersection of two partition topologies is a partition topology. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}^\circ(E)$. It is easy to show that $V \cap W$ is open-closed in $\mathfrak{T}_1 \vee \mathfrak{T}_2$ for every $V \in \mathfrak{T}_1$ and $W \in \mathfrak{T}_2$. Any partition topology is a principal topology. Thus $\mathfrak{T}_1 \vee \mathfrak{T}_2$ is a principal topology for principal topologies form a sublattice of $\mathscr{B}(E)$. $\mathfrak{T}_1 \vee \mathfrak{T}_2$ has a basis $\{V \cap W / V \in \mathfrak{T}_1,\ W \in \mathfrak{T}_2\}$ composed of open-closed sets and therefore it is a principal topology.

But $\mathscr{P}^\circ(E)$ is not a complete sublattice of $\mathscr{B}(E)$ as it is stated in [13]. Even the following theorem holds.

2.3. Theorem: *Let E be an infinite set. Then the smallest complete sublattice of $\mathscr{B}(E)$ containing $\mathscr{P}^\circ(E)$ is $\mathscr{B}(E)$ itself.*

Proof: Let $\mathscr{L}$ be the smallest complete sublattice of $\mathscr{B}(E)$ containing $\mathscr{P}^\circ(E)$. At first we prove that any $\mathfrak{T}_1$-topology belongs to $\mathscr{L}$. It is sufficient to show that any free ultratopology belongs to $\mathscr{L}$. Let $\mathfrak{T} = \mathfrak{G} \cup exp\, (E - \{a\})$ be a free ultratopology. $\mathfrak{G} \cup \{E - X / X \in \mathfrak{G}\}$ is a base of $\mathfrak{T}$ composed of open-closed sets. Hence $\mathfrak{T} = \bigvee_{X \in \mathfrak{G}} \{\varnothing, X, E - X, E\}$ and $\{\varnothing, X, E - X, E\} \in \mathscr{P}^\circ(E)$ for every $X \in \mathfrak{G}$. Therefore $\mathfrak{T} \in \mathscr{L}$.

Now we prove that any atom of $\mathscr{B}(E)$ belongs to $\mathscr{L}$. Let $\varnothing \neq X \subsetneqq E$. If $E - X$ is finite, then $\{\varnothing, X, E\} = \{\varnothing, X, E - X, E\} \cap \mathfrak{K}(E) \in \mathscr{L}$. If X and $E - X$ are infinite, then $\{\varnothing, X, E\} = \{\varnothing, X, E - X, E\} \cap (\mathfrak{K}(E) \vee \{\varnothing, X, E\}) \in \mathscr{L}$ because $\mathfrak{K}(E) \vee \{\varnothing, X, E\}$ is a $\mathfrak{T}_1$-topology. Let X be finite. There exist infinite sets $X_1, X_2 \subseteq E$ such that

$E - X_1$, $E - X_2$ are infinite and $X = X_1 \cap X_2$, $E = X_1 \cup X_2$. Thus $\{\varnothing, X, X_1, X_2, E\} = \{\varnothing, X_1, E\} \vee \{\varnothing, X_2, E\} \in \mathscr{L}$. Hence $\{\varnothing, X, E\} = \{\varnothing, X, X_1, X_2, E\} \wedge (\mathfrak{K}(E) \vee \vee \{\varnothing, X, E\}) \in \mathscr{L}$.

Since any topology is a join of atoms, $\mathscr{L} = \mathscr{B}(E)$ holds.

2.4. Theorem (see [9]): *The lattice $\mathscr{P}^\circ(E)$ of all partition topologies on E is isomorphic to the dual of the lattice $\mathscr{P}(E)$ of all partitions on E. This isomorphism α is defined by this way: $\alpha\mathfrak{R} = \{\bigcup X_i / X_i \in \mathfrak{R}\}$ for every $\mathfrak{R} \in \mathscr{P}(E)$.*

2.5. Corollary (see [6]): *Any lattice is isomorphic to a sublattice of the lattice of all topologies on a certain set.*

Proof follows from 2.1., 2.2. and 2.4.

2.6. Lemma: *Let E, F be sets, $\mathfrak{R}$ a partition on F, ξ: $E \to \mathfrak{R}$ an injective mapping. Let $\xi_{\mathfrak{R}}(\mathfrak{T}) = \{\bigcup_{x \in X} \xi(x) / X \in \mathfrak{T}\}$ for every $\mathfrak{T} \in \mathscr{B}(E)$. Then the mapping $\xi_{\mathfrak{R}}: \mathscr{B}(E) \to \mathscr{B}(F)$ is an embedding.*

Proof is evident.

Let E be a set and $\mathfrak{m}$ an infinite cardinal number. Put $\mathfrak{K}(E, \mathfrak{m}) = \{X \subseteq E / \mathit{card}\, (E - X) < \mathfrak{m}\} \cup \{\varnothing\}$. It is $\mathfrak{K}(E, \mathfrak{m}) \in \mathscr{B}(E)$. It holds $\mathfrak{K}(E, \mathfrak{m}) \subseteq \mathfrak{K}(E, \mathfrak{n})$ for $\mathfrak{m} \leq \mathfrak{n}$. It is $\mathfrak{K}(E, \aleph_0) = \mathfrak{K}(E)$. Larson in [5] proved that $\mathfrak{K}(E, \mathfrak{m})$ and the indiscrete topology are exactly topologies which are the least or the greatest element with respect to some topological property.

2.7. Lemma: *Let E, F be sets, $\mathit{card}\, E = \mathfrak{m}$ and $\mathit{card}\, F = \mathfrak{n}$. Let $\mathfrak{n}$ be regular, $\mathfrak{n} \geq \aleph_0$, $\mathfrak{n} > 2^{\mathfrak{m}}$. Let $\mathfrak{R}$ be a partition on E such that $\mathit{card}\, \mathfrak{R} = \mathfrak{n}$ and $\mathit{card}\, X = \mathfrak{n}$ for every $X \in \mathfrak{R}$. Let ξ: $E \to \mathfrak{R}$ be an injective mapping. Let $\psi\mathfrak{T} = \xi_{\mathfrak{R}}(\mathfrak{T}) - \mathfrak{K}(F, \mathfrak{n})$ for every $\mathfrak{T} \in \mathscr{B}(E)$. Then ψ: $\mathscr{B}(E) \to [\mathfrak{K}(F, \mathfrak{n}))$ is an embedding.*

Proof: It follows from 2.6. that ψ is a $\vee$-homomorphism. For verifying that ψ is a homormophism it is sufficient to show that $\psi\mathfrak{T}_1 \wedge \psi\mathfrak{T}_2 \leq \psi(\mathfrak{T}_1 \wedge \mathfrak{T}_2)$ for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$. At first we prove some property of a topology $\psi\mathfrak{T}$.

Let $\mathfrak{T} \in \mathscr{B}(E)$, $\varnothing \neq X \in \psi\mathfrak{T}$. It is $X = \bigcup_{i \in I} V_i \cap W_i$, where $\varnothing \neq V_i \in \xi_{\mathfrak{R}}(\mathfrak{T})$, $\varnothing \neq W_i \in \mathfrak{K}(F, \mathfrak{n})$ for every $i \in I$ and further $V_i \neq V_j$ for $i \neq j$. It holds $W_i = F - X_i$, where $\mathit{card}\, X_i < \mathfrak{n}$ for every $i \in I$. Hence $X = \bigcup_{i \in I} (V_i - X_i)$. Since $V_i \neq V_j$ for $i \neq j$ and $\mathit{card}\, E = \mathfrak{m}$, it is $\mathit{card}\, I \leq \mathit{card}\, \xi_{\mathfrak{R}}(\mathfrak{T}) = \mathit{card}\, \mathfrak{T} \leq 2^{\mathfrak{m}} < \mathfrak{n}$. Hence card $\bigcup_{i \in I} X_i < \mathfrak{n}$ for $\mathfrak{n}$ is regular. From $\bigcup_{i \in I} V_i - \bigcup_{i \in I} X_i \subseteq X \subseteq \bigcup_{i \in I} V_i$ it follows that there exists $V \in \xi_{\mathfrak{R}}(\mathfrak{T})$ and $Y \subseteq F$ with $\mathit{card}\, Y < \mathfrak{n}$ such that $X = V - Y$.

Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $X \in \psi\mathfrak{T}_1 \cap \psi\mathfrak{T}_2$. There exist $V_k \in \mathfrak{T}_k, Y_k \subseteq F$ with $\mathit{card}\, Y_k < \mathfrak{n}$ for $k = 1, 2$ such that $X = V_1 - Y_1 = V_2 - Y_2$. Since the symmetric difference $V_1 \div V_2$ is contained in $Y_1 \cup Y_2$, it holds $\mathit{card}\, (V_1 \div V_2) < \mathfrak{n}$. It is $V_k = \bigcup_{x \in U_k} \xi(x)$, where $U_k \in \mathfrak{T}_k$ for $k = 1, 2$. Hence $V_1 = V_2$ because $\mathit{card}\, \xi(x) = \mathfrak{n}$ for every $x \in E$. Thus $V_1 = V_2 \in \mathfrak{T}_1 \wedge \mathfrak{T}_2$ and from $X = V_1 - Y_1$ it follows that $X \in \psi(\mathfrak{T}_1 \wedge \mathfrak{T}_2)$.

It remains to prove that ψ is injective. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $\psi\mathfrak{T}_1 = \psi\mathfrak{T}_2$. Let $X \in \mathfrak{T}_1$. Then $\bigcup_{x \in X} \xi(x) \in \xi_{\mathfrak{R}}(\mathfrak{T}_1) \subseteq \psi\mathfrak{T}_1 = \psi\mathfrak{T}_2$. There exists $V \in \psi_{\mathfrak{R}}(\mathfrak{T}_2)$ and $Y \subseteq F$ with $\mathit{card}\, Y < \mathfrak{n}$ such that $\bigcup_{x \in X} \xi(x) = V - Y$. Further $V - Y = \bigcup_{x \in U} \xi(x) - Y$ for a certain $U \in \mathfrak{T}_2$. Since $\mathit{card}\, \xi(x) = \mathfrak{n}$ for every $x \in E$, it holds $X = U$. Therefore $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Analogously we can prove $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$.

2.8. Theorem: *Let $\mathfrak{n}$ be an infinite cardinal number. Any lattice can be embedded in the lattice $[\mathfrak{K}(F, \mathfrak{n}))$ for a certain set F.*

Proof: Let L be a lattice. According to 2.5. there exists a set E such that L can be embedded in $\mathscr{B}(E)$. Let $\mathfrak{m} = card\ E$, $\mathfrak{p} = max\ \{\mathfrak{n}, 2^{\mathfrak{m}}\}$. Let $\mathfrak{p}^+$ be the successor of $\mathfrak{p}$. Since $\mathfrak{p}^+$ is regular, it follows from 2.7. that there exists an embedding ψ: $\mathscr{B}(E) \to [\mathfrak{K}(F, \mathfrak{p}^+))$, where F is a set of cardinality $\mathfrak{p}^+$. Since $[\mathfrak{K}(F, \mathfrak{p}^+))$ is a sublattice of $[\mathfrak{K}(F, \mathfrak{n}))$, the proof is accomplished.

2.9. Corollary: *Any lattice is isomorphic to a sublattice of the lattice of all $\mathfrak{T}_1$-topologies on a certain set.*

The constructed embedding ψ maps elements of a lattice L to topologies structure of which is to be easily clearyfied. For instance they are locally connected and disconnected $\mathfrak{T}_1$-topologies.

§ 3. REPRESENTATIONS OF LATTICES BY MORE SPECIAL TOPOLOGIES

Let $\mathfrak{R}$ be a partition on a set E, $\alpha\mathfrak{R}$ the partition topology from 2.4. Let $\mathscr{P}^0_{\mathfrak{R}}(E) = \mathscr{P}^\circ(E) \cap (\alpha\mathfrak{R}]$. Evidently $\mathscr{P}^0_{\mathfrak{R}}(E)$ is a sublattice of $\mathscr{B}(E)$.

3.1. Lemma: *Let E, F be sets and $\mathfrak{R}$ a partition on F with card E = card $\mathfrak{R}$. Then the lattices $\mathscr{P}^\circ[E)$ and $\mathscr{P}^0_{\mathfrak{R}}(F)$ are isomorphic.*

Proof: There exists a bijective mapping $\xi : E \to \mathfrak{R}$. Let $\xi_{\mathfrak{R}} : \mathscr{B}(E) \to \mathscr{B}(F)$ be the embedding from 2.6. Evidently $\xi_{\mathfrak{R}}(\mathscr{B}(E)) = (\alpha\mathfrak{R}]$ holds. Since $\xi_{\mathfrak{R}}(T)$ is a partition topology iff $\mathfrak{T}$ is, $\xi_{\mathfrak{R}}/\mathscr{P}^\circ(E) : \mathscr{P}^\circ(E) \to \mathscr{P}^0_{\mathfrak{R}}(F)$ is an isomorphism.

3.2. Lemma: *Let E be a set, $\mathfrak{S} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$ with card $\mathfrak{R} > 1$. Let φ : $:\mathscr{P}^0_{\mathfrak{R}}(E) \to \mathscr{B}(E)$, $\varphi\mathfrak{T} = \mathfrak{S} \vee \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{P}^0_{\mathfrak{R}}(E)$, be a homomorphism. Then φ is injective iff $\alpha\mathfrak{R} \nsubseteq \mathfrak{S}$.*

Proof: Supposing $\alpha\mathfrak{R} \subseteq \mathfrak{S}$, $\varphi\mathfrak{T} = \mathfrak{S}$ holds for every $\mathfrak{T} \in \mathscr{P}^0_{\mathfrak{R}}(E)$. Since *card* $\mathfrak{R} > 1$, φ is not injective.

Assume that $\alpha\mathfrak{R} \nsubseteq \mathfrak{S}$. Then $\varphi\{\varnothing, E\} = \mathfrak{S} \neq \mathfrak{S} \vee \alpha\mathfrak{R} = \varphi(\alpha\mathfrak{R})$ and $\{\varnothing, E\}, \alpha\mathfrak{R} \in \mathscr{P}^0_{\mathfrak{R}}(E)$. Ore proved in [7] that the lattice of all partitions on a set is simple. Hence it follows from 2.4. and 3.1. that the lattice $\mathscr{P}^0_{\mathfrak{R}}(E)$ is simple. Thus φ is injective.

3.3. Definition: Let E be a set, $\mathfrak{S} \in \mathscr{B}(E)$, $\mathfrak{R} \in \mathscr{P}(E)$.

Let $M \in \alpha\mathfrak{R}$. Let $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M)$, $\mathfrak{R}/M \leqq \mathfrak{R}_1 \wedge \mathfrak{R}_2$; $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. Let $\mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2) = \{\langle\{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2}\rangle / \{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2} \subseteq \mathfrak{S}, \bigcup_{X \in \mathfrak{R}_1} (Z^1_X \cap X) = \bigcup_{X \in \mathfrak{R}_2} (Z^2_X \cap X)\}$ be a set of pairs of subsystems of $\mathfrak{S}$. Let $\pi = \langle\{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2}\rangle \in \mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)$. Put $A_1(\pi) = \bigcup_{X \in \mathfrak{R}_1} (Z^1_X \cap X) = \bigcup_{X \in \mathfrak{R}_2} (Z^2_X \cap X)$, $A_2(\pi) = \bigcup_{X \in \mathfrak{R}_1} Z^1_X \cup \bigcup_{X \in \mathfrak{R}_2} Z^2_X$, $A(\pi) = A_1(\pi) \cup (A_2(\pi) - M)$. Let $\mathfrak{A}(\mathfrak{S}, \mathfrak{R}, M) = \{A(\pi) / \mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M), \mathfrak{R}/M \subseteq \mathfrak{R}_1 \wedge \mathfrak{R}_2, \mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}, \pi \in \mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)\}$.

Let $\mathfrak{A}(\mathfrak{S}, \mathfrak{R}) = \bigcup_{M \in \alpha\mathfrak{R}} \mathfrak{A}(\mathfrak{S}, \mathfrak{R}, M)$.

3.4. Definition: Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. Put $\mathfrak{T}^0_{\mathfrak{R}} = \mathfrak{T}$. Suppose that the topologies $\mathfrak{T}^\xi_{\mathfrak{R}}$ are defined for every ordinal $\xi < \alpha$. For an isolated α let $\mathfrak{T}^\alpha_{\mathfrak{R}}$ be the topology generated by the system $\mathfrak{T}^{\alpha-1}_{\mathfrak{R}} \cup \mathfrak{A}(\mathfrak{T}^{\alpha-1}_{\mathfrak{R}}, \mathfrak{R})$. For a limit α let $\mathfrak{T}^\alpha_{\mathfrak{R}} = \bigvee_{\xi < \alpha} \mathfrak{T}^\xi_{\mathfrak{R}}$. We have constructed the transfinite sequence $\mathfrak{T}^0_{\mathfrak{R}} \subseteq \ldots \subseteq \mathfrak{T}^\xi_{\mathfrak{R}} \subseteq \ldots$ of topologies on E. Evidently there exists an ordinal γ such that $\mathfrak{T}_{\mathfrak{R}}^{\xi} = \mathfrak{T}_{\mathfrak{R}}^{\gamma}$ for any $\xi > \gamma$. Let $\overset{*}{\mathfrak{T}}_{\mathfrak{R}} = \mathfrak{T}_{\mathfrak{R}}^{\gamma}$.

Let $\varphi\mathfrak{S} = \mathfrak{S} \vee \overset{*}{\mathfrak{T}}_{\mathfrak{R}}$ for every $\mathfrak{S} \in \mathscr{P}_{\mathfrak{R}}^{0}(E)$. We get a mapping $\varphi = \varphi(\mathfrak{T}. \mathfrak{R}) : \mathscr{P}_{\mathfrak{R}}^{0}(E) \to \to \mathscr{B}(E)$.

3.5. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. The mapping $\varphi(\mathfrak{T}, \mathfrak{R}) : \mathscr{P}_{\mathfrak{R}}^{0}(E) \to \mathscr{B}(E)$ is a homomorphism.*

Proof: We shall prove that for every ordinal β and for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}_{\mathfrak{R}}^{0}(E)$ it holds $(\mathfrak{T}_1 \vee \mathfrak{T}_{\mathfrak{R}}^{\beta}) \cap (\mathfrak{T}_2 \vee \mathfrak{T}_{\mathfrak{R}}^{\beta}) \subseteq (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \mathfrak{T}_{\mathfrak{R}}^{\beta+1}$.

Let β be an ordinal and $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}_{\mathfrak{R}}^{0}(E)$. Let $V \in (\mathfrak{T}_1 \vee \mathfrak{T}_{\mathfrak{R}}^{\beta}) \cap (\mathfrak{T}_2 \vee \mathfrak{T}_{\mathfrak{R}}^{\beta})$. From 2.4. it follows that there exist partitions $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ on E such that $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2 \geqq \mathfrak{R}$ and $\mathfrak{T}_i = \alpha\overline{\mathfrak{R}_i}$ for $i = 1, 2$. Evidently $V = \bigcup_{X \in \overline{\mathfrak{R}}_1} (Z_X^1 \cap X) = \bigcup_{X \in \overline{\mathfrak{R}}_2} (Z_X^2 \cap X)$, where $Z_X^i \in \mathfrak{T}_{\mathfrak{R}}^{\beta}$ for every $X \in \overline{\mathfrak{R}}_i$, $i = 1, 2$. Let $M \in \overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2$. It is $M \in \alpha\mathfrak{R}$. Let $\mathfrak{R}_1 = \overline{\mathfrak{R}}_1/M$, $\mathfrak{R}_2 = \overline{\mathfrak{R}}_2/M$ be partitions induced by $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ on M. It holds $\mathfrak{R}/M \leq \mathfrak{R}_1 \wedge \mathfrak{R}_2$ and $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. Further $\bigcup_{X \in \mathfrak{R}_1} (Z_X^1 \cap X) = V \cap M = \bigcup_{X \in \mathfrak{R}_2} (Z_X^2 \cap X)$. Hence $\pi_M = \langle \{Z_X^1\}_{X \in \mathfrak{R}_1}, \{Z_X^2\}_{X \in \mathfrak{R}_2} \rangle \in \mathfrak{B}(\mathfrak{T}_{\mathfrak{R}}^{\beta}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)$. Thus $A(\pi_M) \in \mathfrak{A}(\mathfrak{T}_{\mathfrak{R}}^{\beta}, \mathfrak{R}) \subseteq \mathfrak{T}_{\mathfrak{R}}^{\beta+1}$. It holds $A(\pi_M) \cap M = A_1(\pi_M) \cap M = V \cap M$. Therefore $V = \bigcup_{M \in \overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2} (A(\pi_M) \cap M)$. Since $\mathfrak{T}_1 \cap \mathfrak{T}_2 = \alpha(\overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2)$, it holds $V \in (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \mathfrak{T}_{\mathfrak{R}}^{\beta+1}$.

Since $\overset{*}{\mathfrak{T}}_{\mathfrak{R}} = \mathfrak{T}_{\mathfrak{R}}^{\gamma} = \mathfrak{T}_{\mathfrak{R}}^{\gamma+1}$ for a certain ordinal γ, it holds $\varphi\mathfrak{T}_1 \cap \varphi\mathfrak{T}_2 = (\mathfrak{T}_1 \vee \overset{*}{\mathfrak{T}}_{\mathfrak{R}}) \cap (\mathfrak{T}_2 \vee \overset{*}{\mathfrak{T}}_{\mathfrak{R}}) \subseteq (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \overset{*}{\mathfrak{T}}_{\mathfrak{R}} = \varphi(\mathfrak{T}_1 \cap \mathfrak{T}_2)$. Therefore φ is a homomorphism because according to the definition φ is a $\vee$-homomorphism.

3.6. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R}$ a partition on E such that every element of $\mathfrak{R}$ is dense in $\mathfrak{T}$. Let $M \in \alpha\mathfrak{R}$ and $A(\pi) \in \mathfrak{A}(\mathfrak{T}, \mathfrak{R}, M)$. Let $V \in \mathfrak{T}$, $T \in \mathfrak{R}$ and $V \cap A(\pi) \cap T = \varnothing$. Then $V \cap A_2(\pi) = \varnothing$ holds.*

Proof: Let $T \cap M = \varnothing$. Then $V \cap A_2(\pi) \cap T = V \wedge A(\pi) \cap T = \varnothing$ because $A_1(\pi) \subseteq M$. Since T is dense in $\mathfrak{T}$ and $V \cap A_2(\pi) \in \mathfrak{T}$, it holds $V \cap A_2(\pi) = \varnothing$.

Let $T \cap M \neq \varnothing$. Then $T \subseteq M$. It is $\pi = \langle \{Z_X^1\}_{X \in \mathfrak{R}_1}, \{Z_X^2\}_{X \in X_2} \rangle \in \mathfrak{B}(\mathfrak{T}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)$ for suitable $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M)$ with $\mathfrak{R}/M \leqq \mathfrak{R}_1 \wedge \mathfrak{R}_2$ and $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. There exist $X_1 \in \mathfrak{R}_1$, $X_2 \in \mathfrak{R}_2$ such that $T \subseteq X_1 \cap X_2$.

Let $X \in \mathfrak{R}_1 \cup \mathfrak{R}_2$. According to the construction of joins in the lattice $\mathscr{P}(E)$ there exist $T_i \in \mathfrak{R}_1 \cup \mathfrak{R}_2$ for $i = 1, \ldots, n$ such that $T_1 = X_1$, $T_n = X$ and $T_i \cap T_{i+1} \neq \varnothing$ for $i = 1, \ldots, n-1$. It holds $\varnothing = V \cap A(\pi) \cap T \supseteq V \cap A_1(\pi) \cap T \supseteq V \cap (Z_{T_1}^1 \cap T_1) \cap T = V \cap Z_{T_1}^1 \cap T$. Since T is dense in $\mathfrak{T}$ and $V \cap Z_{T_1}^1 \in \mathfrak{T}$, it holds $V \cap Z_{T_1}^1 = \varnothing$. Suppose that $V \cap Z_{T_k}^s = \varnothing$, where $k < n$, $T_k \in \mathfrak{R}_s$, $s = 1, 2$. Let $T' = T_k \cap T_{k+1}$. It is $Z_{T_k}^s \cap T' = \bigcup_{X \in \mathfrak{R}_s} (Z_X^s \cap X) \cap T' = \bigcup_{X \in \mathfrak{R}_r} (Z_X^r \cap X) \cap T' = Z_{T_{k+1}}^r \cap T'$, where $r \in \{1, 2\}$, $T_{k+1} \in \mathfrak{R}_r$. Hence $\varnothing = V \cap Z_{T_k}^s \cap T_k \supseteq V \cap Z_{T_k}^s \cap T' = V \cap Z_{T_{k+1}}^r \cap T'$. Since T' is dense in $\mathfrak{T}$, it holds $V \cap Z_{T_{k+1}}^r = \varnothing$. It can be concluded that $V \cap Z_X^t = \varnothing$, where $X \in \mathfrak{R}_t$.

Therefore $V \cap A_2(\pi) = \varnothing$ and the proof is accomplished.

3.7. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. Let every element of $\mathfrak{R}$ be dense in $\mathfrak{T}$. Then every element of $\mathfrak{R}$ is dense in $\overset{*}{\mathfrak{T}}_{\mathfrak{R}}$.*

Proof: We shall use the transfinite induction. Suppose that every element of $\mathfrak{R}$

is dense in $\mathfrak{T}^{\xi}_{\mathfrak{R}}$ for every ordinal $\xi < \beta$. If β is limit, every element of $\mathfrak{R}$ is evidently dense in $\mathfrak{T}^{\beta}_{\mathfrak{R}}$. Let β be isolated. The system of all finite intersections of elements of $\mathfrak{T}^{\beta-1}_{\mathfrak{R}} \cup \mathfrak{A}(\mathfrak{T}^{\beta-1}_{\mathfrak{R}}, \mathfrak{R})$ forms a basis of $\mathfrak{T}^{\beta}_{\mathfrak{R}}$. Let Y be such an intersection. We shall show that $Y \neq \varnothing$ implies $Y \cap T \neq \varnothing$ for every $T \in \mathfrak{R}$. Thereby the proof will be accomplished.

It is $Y = W \cap \bigcap_{i=1}^{n} A(\pi_i)$, where $W \in \mathfrak{T}^{\beta-1}_{\mathfrak{R}}$ and $A(\pi_i) \in \mathfrak{A}(\mathfrak{T}^{\beta-1}_{\mathfrak{R}}, \mathfrak{R})$ for $i = 1, \ldots, n$. There exist $M_i \in \alpha\mathfrak{R}$ such that $A(\pi_i) \in \mathfrak{A}(\mathfrak{T}^{\beta-1}_{\mathfrak{R}}, \mathfrak{R}, M_i)$ for $i = 1, \ldots, n$. Suppose that $T \in \mathfrak{R}$ exists with $Y \cap T = \varnothing$. There exist $X_i \in \mathfrak{T}^{\beta-1}_{\mathfrak{R}}$ with $X_i \cap T = A(\pi_i) \cap T$ for every $i = 1, \ldots, n$. It is $\varnothing = Y \cap T = W \cap \bigcap_{i=1}^{n} A(\pi_i) \cap T = (W \cap \bigcap_{i=1}^{n-1} X_i) \cap$ $\cap A(\pi_n) \cap T$. Since $W \cap \bigcap_{i=1}^{n-1} X_i \in \mathfrak{T}^{\beta-1}_{\mathfrak{R}}$, it follows from 3.6. that $V \cap \bigcap_{i=1}^{n-1} X_i \cap$ $\cap A_2(\pi_n) = \varnothing$. Suppose that $W \cap \bigcap_{i=1}^{n-k} X_i \cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) = \varnothing$. Then $\varnothing = W \cap$ $\cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) \cap \bigcap_{i=1}^{n-k} X_i \cap T = (W \cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) \cap \bigcap_{i=1}^{n-k-1} X_i) \cap A(\pi_{n-k}) \cap T$. 3.6. implies $W \cap \bigcap_{i=1}^{n-k-1} X_i \cap \bigcap_{i=n-k}^{n} A_2(\pi_i) = \varnothing$. We can conclude that $W \cap \bigcap_{i=1}^{n} A_2(\pi_i) = \varnothing$. Since $A(\pi_i) \subseteq A_2(\pi_i)$ for every i, it holds $Y = \varnothing$.

Let $\mathfrak{m}$ be a cardinal number. A topology $\mathfrak{T}$ is called $\mathfrak{m}$-resolvable if it contains $\mathfrak{m}$ pairwise disjoint dense sets. A 2-resolvable topology is called briefly resolvable. The concept of resolvable topologies was introduced by Hewitt ([3]). He proved that every metrizible topology devoid of isolated points is resolvable.

3.8. Lemma: *There exists an $\mathfrak{m}$-resolvable completely Hausdorff topology for any cardinal number $\mathfrak{m}$.*

Proof: Let I be a set, *card* $I = \mathfrak{m}$. Let Q be the set of all rational numbers and $\mathfrak{S}$ the usual topology of Q. Put $E = \prod_{i \in I} Q_i$, $\mathfrak{T} = \prod_{i \in I} \mathfrak{S}_i$, where $Q_i = Q$ and $\mathfrak{S}_i = \mathfrak{S}$ for every $i \in I$. Evidently $\mathfrak{T}$ is completely Hausdorff. Since $\mathfrak{S}$ is resolvable, there exist sets A_i, $B_i = Q_i - A_i$ dense in $\mathfrak{S}_i$ for every $i \in I$. Let $\mathfrak{B} = \{\prod_{i \in I} X_i / X_i = A_i$ or $X_i = B_i\}$. Every element of $\mathfrak{B}$ is dense in $\mathfrak{T}$ and elements of $\mathfrak{B}$ are pairwise disjoint. Since card $\mathfrak{B} = 2^{\mathfrak{m}}$, the topology $\mathfrak{T}$ is $2^{\mathfrak{m}}$-resolvable and therefore it is $\mathfrak{m}$-resolvable.

3.9. Theorem: *For every lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψx is a completely Hausdorff topology for every $x \in L$.*

Proof: Let L be a lattice. According to 2.1 and 2.4. L can be embedded in the lattice of all partition topologies on some set F. According to 3.8. there exists a *card* F-resolvable completely Hausdorff topology $\mathfrak{T}$. Let E be the underlying set of $\mathfrak{T}$. There exists a partition $\mathfrak{R}$ on E every element of which is dense in $\mathfrak{T}$ and *card* $\mathfrak{R}$ = *card* F. From 3.1. it follows that L can be embedded in $\mathscr{P}^{0}_{\mathfrak{R}}(E)$. Let $\varphi = \varphi(\mathfrak{T}, \mathfrak{R})$: $: \mathscr{P}^{0}_{\mathfrak{R}}(E) \to \mathscr{B}(E)$ be the mapping from 3.4. According to 3.5. φ is a homomorphism. 3.7. implies that every element of $\mathfrak{R}$ is dense in $\mathfrak{T}^{*}_{\mathfrak{R}}$. It follows from 3.2. that φ is injective. Since $\varphi\mathscr{P}^{0}_{\mathfrak{R}}(E) \subseteq [\mathfrak{T}^{*}_{\mathfrak{R}}) \subseteq [\mathfrak{T})$, every topology from $\varphi\mathscr{P}^{0}_{\mathfrak{R}}(E)$ is completely Hausdorff. The proof is ready.

Since the topology of the rationals numbers is O-dimensional, every topology ψx is totally disconnected. Even in the same way as the previous theorem we can prove the following one.

3.10. Theorem: *Let $\mathscr{C}$ be a class of topologies with the properties:* $1° \mathfrak{T} \in \mathscr{C} \cap \mathscr{B}(F)$, $\mathfrak{T}' \in \mathscr{B}(F)$, $\mathfrak{T} \subseteq \mathfrak{T}' \Rightarrow \mathfrak{T}' \in \mathscr{C}$
$2°$ *$\mathscr{C}$ contains an $\mathfrak{m}$-resolvable topology for any cardinal number $\mathfrak{m}$.*
Then for any lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that $\psi L \subseteq \mathscr{C}$.

Analogously as in 3.8. we can show that $\mathscr{C}$ fulfils $2°$ whenever it is closed under products and contains a resolvable topology.

A question arises whether any lattice can be represented by topologies more special than completely Hausdorff. We shall show that for metrizible topologies it is not true.

3.11. Lemma: *Let E be a set and $\mathscr{L}$ a sublattice of $\mathscr{B}(E)$. Let $A \subseteq E$ with $E - A \in \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{L}$. Then a mapping $\psi_A : \mathscr{L} \to \mathscr{B}(A)$, $\psi_A \mathfrak{T} = \mathfrak{T}/A$ is the relative topology for every $\mathfrak{T} \in L$, is a homomorphism.*

Proof: Evidently ψ_A is isotone. Hence $\psi_A \mathfrak{T}_1 \vee \psi_A \mathfrak{T}_2 \subseteq \psi_A(\mathfrak{T}_1 \vee \mathfrak{T}_2)$ holds for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$ and $X \in \psi_A(\mathfrak{T}_1 \vee \mathfrak{T}_2)$. There exist $V_i \in \mathfrak{T}_1$, $W_i \in \mathfrak{T}_2$ for $i \in I$ such that $X = \bigcup_{i \in I} (V_i \cap W_i) \cap A$. Hence $X = \bigcup_{i \in I} [(V_i \cap A) \cap (W_i \cap A) \in \in \psi_A \mathfrak{T}_1 \vee \psi_A \mathfrak{T}_2$.

It holds $\psi_A(\mathfrak{T}_1 \cap \mathfrak{T}_2) \leq \psi_A \mathfrak{T}_1 \cap \psi_A \mathfrak{T}_2$. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$, $X \in \psi_A \mathfrak{T}_1 \cap \psi_A \mathfrak{T}_2$. There exists $V \in \mathfrak{T}_1$ and $W \in \mathfrak{T}_2$ with $X = V \cap A = W \cap A$. It is $(E - A) \cup V \in \mathfrak{T}_1$ and $(E - A) \cup W \in \mathfrak{T}_2$. Since $(E - A) \cup V = (E - A) \cup (V \cap A) = (E - A) \cup \cup (W \cap A) = (E - A) \cup W$, it holds $X \in \psi_A(\mathfrak{T}_1 \cap \mathfrak{T}_2)$.

Let $\mathfrak{m}$ be an infinite cardinal number. A topology $\mathfrak{T}$ on a set E is called $\mathfrak{m}$-generated if it has the following property: $X \in \mathfrak{T}$ iff $X \cap A \in \mathfrak{T}/A$ for every $A \subseteq E$ with *card* $A < < \mathfrak{m}$ (see Herrlich [2]).

3.12. Theorem: *Let $\mathfrak{m}$ be an infinite cardinal number and L be a simple lattice with the least element a. Let there exist a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψa is an $\mathfrak{m}$-generated Hausdorff topology. Then card $L \leq 2^{2^{\mathfrak{n}}}$, where $\mathfrak{n} = 2^{2^{\mathfrak{m}}}$.*

Proof: In the case *card* $L = 1$ the theorem holds. Let *card* $L > 1$. Then there exists $b \in L$ with $a < b$. Thus $\psi a \subset \psi b$. There exists $X \subseteq E$ with $X \in \psi b$ and $X \notin \psi a$. Since ψa is $\mathfrak{m}$-generated, there exists $B \subseteq E$ with *card* $B < \mathfrak{m}$ such that $X \cap B \notin \notin \psi a/B$. It is *card* $(B - X) < \mathfrak{m}$ and $Cl_{\psi b}(B - X) \subsetneqq Cl_{\psi a}(B - X)$. Let $C = Cl_{\psi a}(B - - X)$. Since ψa is Hausdorff, every filter on E has at most one limit point in ψa. It implies *card* $C \subseteq 2^{2^{\mathfrak{m}}} = \mathfrak{n}$. Since $E - C \in \psi a \subseteq \psi x$ for every $x \in L$, it follows from 3.11. that $\psi_C : \psi L \to \mathscr{B}(C)$, $\psi_C \mathfrak{T} = \mathfrak{T}/C$ for every $\mathfrak{T} \in \psi L$, is a homomorphism. Since $Cl_{\psi b}(B - X) \subsetneqq C$, it holds $\psi_C \psi b \neq \psi_C \psi a$. Since L is simple, the mapping $\psi_C \psi$ is injective. Therefore *card* $L \leq$ *card* $\mathscr{B}(C)$. Pospíšil proved in [8] that *card* $\mathscr{B}(C) = 2^{2^{\text{card } C}}$ whenever C is infinite. We have obtained that *card* $L \leq 2^{2^{\mathfrak{n}}}$.

3.13. Corollary: *There exists a lattice L for which no set E exists such that there exists an embedding $\psi : L \to \mathscr{B}(E)$ having the property that ψx is a metrizible topology for every $x \in L$.*

Proof: Evidently any metrizible topology is $\aleph_0$-generated. The result follows from 3.12. and from the existence of simple lattices of an arbitrary cardinality (e.g. the lattice of partitions is always simple).

There is a problem whether for any lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψx is a (completely) regular $\mathfrak{T}_1$-topology.

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ON A CERTAIN ANALOGY OF STIRLING'S NUMBERS OF THE 2ND KIND

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Stirling's number of the second kind (which is usually noted as $S(n, k)$) gives the number of ways in which it is possible to carry out the partition of a set of n mutually different elements into k nonempty sets. To compare with the results that will follow, let us return to the recurrent formula, see [1], p. 168:

$$S(n, k) = k \,.\, S(n - 1, k) + S(n - 1, k - 1) \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} S(n, k) &= 1 \text{ for } n = k,\ k = 0, 1, 2, \dots \\ &= 0 \text{ for } k = 0,\ n = 1, 2, 3, \dots \\ &= 0 \text{ for } n < k \end{aligned} \tag{2}$$

This formula was the base for the deduction of a generating function, see [1], p. 170:

$$y_k(x) = \prod_{j=1}^{k} \frac{1}{1 - jx}, \qquad k = 1, 2, 3, \dots \tag{3}$$

whose expansion into a power series runs as follows:

$$y_k(x) = S(k, k) + S(k + 1, k) \,.\, x + S(k + 2, k) \,.\, x^2 + \dots \tag{4}$$

Definition 1. A finite set will be called *pair* (*odd*) if it contains a pair (odd) number of elements. A *pair* (*odd*) partition on a finite set is a partition whose every set contains a pair (odd) number of elements.

Convention 1. We are going to use the following symbols: $N = \{x_1, x_2, \dots, x_n\}$ is a finite set, containing $n = 2\nu$ mutually different elements, where ν is a natural number. $L \subset N$; $L = \{x_1, x_2, \dots, x_{n-2}\}$. Let $p \geqq q$ be natural numbers. By the symbol $S_2(2p, q)$, resp. $S_2[2p, q]$ we shall denote the numbers, resp. the family of all pair partitions of the set of $2p$ different elements into q nonempty sets.

Theorem 1. For the numbers $S_2(n, k)$ the recurrent formula holds good:

$$S_2(n, k) = k^2 \,.\, S_2(n - 2, k) + (2k - 1) \,.\, S_2(n - 2, k - 1) \tag{5}$$

with boundary conditions:

$$\begin{aligned} S_2(n, k) &= \frac{(2k)!}{(2!)^k \,.\, k!} \text{ for } n = 2k,\ k = 0, 1, 2, \dots \\ &= 0 \text{ for } k = 0,\ n = 2, 4, 6, \dots \\ &= 0 \text{ for } n < 2k \end{aligned} \tag{6}$$

Proof. Let $\mathfrak{R}(X, s)$, resp. $R(X, s)$ denote a pair, resp. non-pair partition of the set X into s nonempty sets. Let us choose a certain partition $\mathfrak{R}(L, k)$. If we let the elements x_{n-1}, x_n belong to the sets of this partition (there are k^2 ways of doing so), two cases may occur:

The elements x_{n-1}, x_n belong: 1) both to the same set of the partition, 2) each to a different set of the partition.

In the first case the resulting partition is pair. In the second case it is possible to form a (1,1) correspondence between the resulting and its respective pair partition, through the following transformation:

Let $x_{n-1} \in M_i$, $x_n \in M_j$, so that M_i, M_j (the sets of the resulting partition $R(N, k)$) are odd. Let x_p be the element with the lowest index in the set $M_i \cup M_j$. Then if $x_p \in M_i$, resp. $x_p \in M_j$, we shall class this element into the set M_j, resp. M_i.

As all the partitions $\mathfrak{R}(L, k)$ form the group $S_2[n - 2, k]$, we shall derive from this group by means of the mentioned proceeding (i.e. either directly or by means of the transformation) on the whole $k^2 \,.\, S_2(n - 2, k)$ mutually different partitions $\mathfrak{R}(N, k)$.

Let us consider further that none of the partitions $\mathfrak{R}(N, k)$ we have formed doesn't turn through the transformation to a partition $R(N, k)$ having the following quality: one of its odd sets is formed by the element x_{n-1} or x_n itself. Besides none from the partitions $\mathfrak{R}(N, k)$ we have formed as yet contains $\{x_{n-1}, x_n\}$ as an independent set of the partition. That's why to form the remaining partitions $\mathfrak{R}(N, k) \in S_2[n, k]$ we shall take into consideration individual partitions $\mathfrak{R}(L, k - 1) \in S_2[n - 2, k - 1]$:

a) To each of these partitions we shall add one of the elements x_{n-1}, x_n as a k-th independent set of the partition, while the other of these elements will be included in one of the original sets of the partition. Thus we shall form $2(k - 1) \,.\, S_2(n - 2, k - 1)$ different partitions $R(N, k)$, each of which contains two odd and $k - 2$ even sets. We shall then form a correspondence between these partitions and the respective even partitions by means of the quoted transformation.

b) We shall join to each partition $\mathfrak{R}(L, k - 1) \in S_2[n - 2, k - 1]$ a set $\{x_{n-1}, x_n\}$ as a k-th set of the partition. Thus we shall form the remaining partitions of the family $S_2[n, k]$.

The validity of the recurrent formula (5) is now evident from what has been said.

Let us go on and consider how many ways there are of performing the partition of a set containing $2k$ different elements into k even sets: Evidently each set of the partition must have just two elements. Let us choose one of those partitions and let us choose arbitrarily the order of its sets as well as the order of the elements in individual sets: We shall get a certain sequence of $2k$ different elements. Forming successively all possible permutations of the chosen order of sets and elements belonging to each set, we shall make the chosen partition correspond to $(2!)^k \,.\, k!$ different sequences containing $2k$ elements each. But the number of all sequences, formed from $2k$ different elements, is $(2k)!$. Thus we have $S_2(2k, k) = \dfrac{(2k)!}{(2!)^k \,.\, k!}$; let us notice that this expression is different from zero even for $k = 0$, i.e. $S_2(0, 0) = 1$.

The validity of the remaining boundary conditions in (6) is natural. That was to be proved.

Remark 1. From the given formula for $S_2(2k, k)$ there follows evidently the following recurrent formula:

$$S_2(2k, k) = (2k - 1) \,.\, S_2(2k - 2, k - 1); \qquad k = 1, 2, 3, \ldots \tag{7}$$

Further we are going to treat the generating function for the numbers $S_2(n, k)$.

Definition 2. By the symbol $y_k(x)$ we shall denote the generating function for the numbers $S_2(n, k)$, that is the function which in the form of a power series runs as follows:

$$(8) \qquad y_k(x) = S_2(2k, k) + S_2(2k + 2, k) \cdot x^2 + S_2(2k + 4, k) \cdot x^4 + \ldots$$

Theorem 2. The generating function of the numbers $S_2(n, k)$ runs in its final form as follows:

$$(9) \qquad y_k(x) = \prod_{j=1}^{k} \frac{2j - 1}{1 - (jx)^2}; \qquad k = 1, 2, 3, \ldots$$

Proof. If we subtract from the equation (8) its multiple by the factor k^2x^2, then if we note briefly $S_2(a, b) = S_a^b$ we get: $(1 - k^2x^2) \cdot y_k(x) = S_{2k}^k + (S_{2k+2}^k - k^2S_{2k}^k) \cdot x^2 + (S_{2k+4}^k - k^2S_{2k+2}^k) \cdot x^4 + \ldots$, so that considering (5) and (7) we have: $(1 - k^2x^2) \cdot y_k(x) = (2k - 1) \cdot (S_{2k-2}^{k-1} + S_{2k}^{k-1} \cdot x^2 + S_{2k+2}^{k-1} \cdot x^4 + \ldots) = (2k - 1) \cdot y_{k-1}(x)$. At the same time we have: $S_2(2n, 1) = 1, n = 1, 2, 3, \ldots$ so that $y_1(x) = 1 + x^2 + x^4 + \ldots = \frac{1}{1 - x^2}$ for $|x| < 1$, which was to be proved.

Convention 2. Analogously to [2], p. 27 let us denote by the Symbol $P_2(r)$, resp. $P_2[r]$ the number, resp. the family of all even partitions on a set of r different elements $(r = 0, 2, 4, \ldots)$ independently on the number of the sets forming the partition.

Evidently the following relation holds good:

$$(10) \qquad P_2(r) = \sum_{j=1}^{r/2} S_2(r, j); \qquad r = 2, 4, 6, \ldots$$

Theorem 3. For the numbers $P_2(r)$ this recurrent formula holds good:

$$(11) \qquad P_2(n) = \sum_{j=1}^{n/2} \binom{n-1}{2j-1} \cdot P_2(n - 2j); \qquad P_2(0) = 1.$$

Proof. Let us choose an arbitrary but constant element of the set N. Let us denote it x_t. Let us consider that it is possible to form just $\binom{n-1}{k-1}$ sets by k elements containing the element x_t each. From the remaining $(n - k)$ elements of the set N it is always possible to form just $P_2(n - k)$ even partitions. Let us then divide the family $P_2[n]$ into partial families according to the number of elements which are contained by that set of partitions in which the chosen element is contained. If we denote $k = 2j$, then $j = 1, 2, \ldots, \frac{n}{2}$, for n is an even number, see the def. 1.

Remark 2. See also the author's article [3].

Finally I give a few lines of the system of numbers $S_2(n, k)$, factors included by which the slipped numbers are supposed to be multiplied: see the numbers in the middle of the arrows.

n	$k = 0$	1	2	3	4	5	$P_2(n)$
0	**1**						1
	↓0 ↘1						
2	**0**	**1**					1
	↓0 ↘1	↓1 ↘3					
4	**0**	**1**	**3**				4
	↓0 ↘1	↓1 ↘3	↓4 ↘5				
6	**0**	**1**	**15**	**15**			31
	↓0 ↘1	↓1 ↘3	↓4 ↘5	↓9 ↘7			
8	**0**	**1**	**63**	**210**	**105**		379
	↓0 ↘1	↓1 ↘3	↓4 ↘5	↓9 ↘7	↓16 ↘9		
10	**0**	**1**	**255**	**2205**	**3150**	**945**	6556

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DIRECT PRODUCTS OF HOMOMORPHIC MAPPINGS

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It is well-known in the theory of abstract algebras that for arbitrary class of algebras $\mathfrak{A}$ the direct product $\prod_{\tau\in T} h_\tau$ of homomorphic mappings h_τ of algebras $A_\tau \in \mathfrak{A}$ onto $B_\tau \in \mathfrak{A}$ is a homomorphic mapping of the direct product $\prod_{\tau\in T} A_\tau$ onto $\prod_{\tau\in T} B_\tau$. The prupose of this paper is to give sufficient conditions for the converse of this theorem. It will be shown that the class of algebras, for which the converse of the theorem is valid, is enough extensive. It contains for example atomic Boolean algebras, discrete direct products of completely ordered groups or rings and lattices which are direct products of chains with the least or greatest element.

1.

BASIC CONCEPTS

Let $\mathfrak{A}$ be a class of algebras with the zero element 0 and the binary operation $\oplus$ and a set Ω of n-ary operations ($n \geqq 1$) fulfilling for each algebra $A \in \mathfrak{A}$ and each element $a \in A$ identities:

(i) $a \oplus 0 = a = 0 \oplus a$

(ii) for each $\omega \in \Omega$ is $00 \dots 0\, \omega = 0$

The operations in all algebras of $\mathfrak{A}$ will be denoted by the same symbols.

Definition 1. An algebra $A \in \mathfrak{A}$ is said to be *without zero-divisors* iff there exists $\Omega' \subseteq \Omega$, $\Omega' \neq \varnothing$ with following properties:

(a) the arity of each $\omega \in \Omega'$ is greater than 1
(b) for each $\omega \in \Omega'$ the identity $a_1 a_2 \dots a_n \omega = 0$ holds
iff $a_i = 0$ for at least one i ($1 \leqq i \leqq n$).

The set Ω' is called *the set of regular operations.*

Definition 2. An algebra $A \in \mathfrak{A}$ is called *N-algebra* iff there exist algebras $A_\tau \in \mathfrak{A}$, $\tau \in T$ without zero-divisors such that A is equal to the direct product of A_τ, i.e. $A = \prod_{\tau\in T} A_\tau$ and at least one of the following conditions is satisfied:

(iii) for each $\tau \in T$ in A_τ there exists "sum" (in the sense of $\oplus$) of arbitrary set $\{a^\gamma;\ a^\gamma \in A_\tau,\ \gamma \in G,\ a^\gamma = 0 \text{ for } \gamma \neq \gamma_0 \in G,\ \textit{card}\ G \leqq \textit{card}\ T\}$ and it is equal to a^{γ_0}.

(iv) $\prod_{\tau\in T} A_\tau$ is the discrete direct product.

Remark. If T is a finite index set, then the conditions (iii) and (iv) of the definition 2 can be omitted because for a finite index set is each direct product discrete one and (iii) follows directly from (i). From (i) it follows that the "sum" of the set from the condition (iii) does not depend on a bracketing.

For some algebras the conditions for direct decomposition to algebras without zero-divisors are known. For example an atomic Boolean algebra is direct decomposable to two-element Boolean algebras (they do not contain zero-divisors), see [10] and [3]. The conditions for Ω-algebras and Ω-groups are given recently in [8] and [9]. These algebras are N-algebras.

Notation. Let $\mathfrak{A}$ be a fixed class of algebras with 0, operation $\oplus$ and a set Ω of operations fulfilling (i), (ii), let T be an index set. The direct product of algebras $A_\tau \in \mathfrak{A}$ for $\tau \in T$ will be denoted by $A = \prod_{\tau \in T} A_\tau$, the zero of A is denoted by O_A. Let $a \in A$, the projection of a into A_τ is denoted by $pr_\tau(a) = a(\tau)$. It is easy to show that $pr_\tau(O_A) = 0$ for each $\tau \in T$. For $T' \subseteq T$ there is $\overline{\prod_{\tau \in T'} A_\tau} = \{a;\ a \in A,\ pr_\tau(a) = 0$ for $\tau \in T - T'\}$. Specially for $T' = \{\tau_0\}$ is $\overline{\prod_{\tau_0} A_\tau}$ denoted by $\overline{A_{\tau_0}}$. An element of $\overline{A_\tau}$ is denoted by $\overline{a_\tau}$. Let A_τ, $B_\sigma \in \mathfrak{A}$. By the symbol $H(A_\tau, B_\sigma)$ we denote the set of all homomorphic mappings of A_τ into B_σ, by $\overline{H}(A_\tau, B_\sigma)$ the set of all homomorphic mappings of A_τ onto B_σ.

Definition 3. Let $A, B \in \mathfrak{B}$, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$, $\varphi_\tau : A_\tau \to B_\tau$ for each $\tau \in T$, $\mathfrak{B}$ beeing an arbitrary class of algebras. The mapping $\varphi : A \to B$ defined by the rule:

$$pr_\tau \varphi(a) = \varphi_\tau(pr_\tau(a)) \text{ for each } \tau \in T,\ a \in A$$

is called the *direct product of mappings* φ_τ and it is denoted by $\varphi = \prod_{\tau \in T} \varphi_\tau$ (see [12], p. 127, Lemma 3).

Lemma. *Let A, B be N-algebras, $\varphi \in \overline{H}(A, B)$ and O_B be a zero of the algebra B. Then $\varphi(O_A) = O_B$.*

Proof. Let ω be a direct product of n-ary regular operations ($n \geqq 2$), let $\varphi^{-1}(O_B) = V$. Then for each $v \in V$ it holds

$$\varphi(O_A) = \varphi(vO_A \dots O_A\omega) = \varphi(v)\,\varphi(O_A) \dots \varphi(O_A)\,\omega = O_B\varphi(O_A) \dots \varphi(O_A)\,\omega = O_B.$$

Theorem 1. Let $\mathfrak{L}$ be a class of algebras with a set of operations Ω, let A_τ, $B_\tau \in \mathfrak{L}$ for $\tau \in T$ and $\varphi_\tau \in \overline{H}(A_\tau, B_\tau)$. Then $\prod_{\tau \in T} \varphi_\tau \in \overline{H}(\prod_{\tau \in T} A_\tau, \prod_{\tau \in T} B_\tau)$ (see [12]).

Proof. For each $\tau \in T$ and arbitrary n-ary operation $\omega \in \Omega$ there is $pr_\tau\varphi(a_1a_2 \dots a_n\omega) = \varphi_\tau(pr_\tau(a_1a_2 \dots a_n\omega)) = \varphi_\tau(a_1(\tau)\,a_2(\tau) \dots a_n(\tau)\,\omega) = \varphi_\tau(a_1(\tau))\,\varphi_\tau(a_2(\tau)) \dots \varphi_\tau(a_n(\tau))\,\omega$, i.e. $\varphi(a_1a_2 \dots a_n\omega) = \varphi(a_1)\,\varphi(a_2) \dots \varphi(a_n)\,\omega$. This implies that $\varphi = \prod_{\tau \in T} \varphi_\tau \in \overline{H}(\prod_{\tau \in T} A_\tau, \prod_{\tau \in T} B_\tau)$.

Definition 4. A mapping φ of an N-algebra A into an N-algebra B is said to be *trivial* iff card $\varphi(A) = 1$. If $\varphi(A) = \{O_B\}$ and $\varphi \in H(A, B)$, φ is called a *zero-homomorphism* and it is denoted by o.

2.

THEOREMS ON HOMOMORPHIC MAPPINGS OF N-ALGEBRAS

Theorem 2. Let A_τ, $B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors for $\tau \in T$, $\sigma \in S$ and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$ be N-algebras. Let φ be a homomorphic mapping of A onto B which is not trivial. Then for each $\sigma \in S$ there exists just one $\tau_\sigma \in T$ such that $\overline{B}_\sigma \subseteq \varphi(\overline{A}_{\tau_\sigma})$.

Proof. Let the assumption of the theorem be valid and let there exists $\sigma_0 \in S$ such that the assertion of the theorem is not true. Let T' be an arbitrary subset of T such that $\varphi\ (\overline{\prod_{\tau \in T'} A_\tau}) \supseteq \overline{B}_{\sigma_0}$ (such T' exists, for example $T' = T$). Evidently $card\ T' \geqq 1$. Denote $A' = \overline{\prod_{\tau \in T'} A_\tau}$. As the assertation of the theorem is not valid, it is $card\ T' > 1$, so that there exist $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$.

(a) Let there exist $\bar{a}_{\tau_1} \in \overline{A}_{\tau_1}$, $\bar{a}_{\tau_2} \in \overline{A}_\tau$, $\bar{a}_{\tau_1}, \bar{a}_{\tau_2} \in A'$ such that $\varphi(\bar{a}_{\tau_1}) \neq O_B \neq \varphi(\bar{a}_{\tau_2})$. For each n-ary operation $\omega \in \Omega'$ which is the direct product of regular operations the relation

$$\varphi(\bar{a}_{\tau_1}\bar{a}_{\tau_2} \dots \bar{a}_{\tau_2}\omega) = \varphi(O_A) = O_B$$

holds by the lemma, but by the assumption (a):

$$\varphi(\bar{a}_{\tau_1})\ \varphi(\bar{a}_{\tau_2}) \dots \varphi(\bar{a}_{\tau_2})\ \omega \neq O_B$$

which is a contradiction.

(b) Let (a) does not hold. Thus there exists $\tau_0 \in T'$ such that $\varphi(\bar{a}_\tau) = O_B$ for each $\bar{a}_\tau \in \overline{A}_\tau$, $\tau \in T'$ $\tau \neq \tau_0$. Let $\bar{b}_{\sigma_0} \in \overline{B}_{\sigma_0}$, $\bar{b}_{\sigma_0} \neq O_B$. Choose $a \in \varphi^{-1}(\bar{b}_{\sigma_0})$, $a \in A'$ arbitrary; according to lemma we have $a \neq O_A$. We can write $a = \overline{a(\tau_0)} \oplus c$, where $c(\tau_0) = 0$. Then $\varphi(a) = \varphi(\overline{a(\tau_0)}) \oplus \varphi(c) = \varphi(\overline{a(\tau_0)})$, $\varphi(c)$ being O_B according to the assumption (b) and (iii) of the definition 2. Thus $\overline{B}_{\sigma_0} \subseteq \varphi(\overline{A}_{\tau_0})$, in contradiction with the assumption of the proof.

The proof of the theorem 2 is complete.

Definition 5. An algebra $A \in \mathfrak{A}$ without zero-divisors is said to be *pseudo-ordered*, if there exists a set $\Omega'' \subseteq \Omega'$, $\Omega'' \neq \varnothing$ such that for each n-ary $\omega \in \Omega''$ there is $a_1a_2 \dots a_n\omega = a_i\alpha$ where $i \in \{1, 2, \dots, n\}$ and α is the identity operation (i.e. $a\alpha = a$) or $\alpha \in \Omega$ is a unary operation with $a\alpha = 0$ iff $a = 0$.

From the inclusion $\Omega'' \subseteq \Omega'$ it holds that the arity of $\omega \in \Omega''$ is greater than 1. Let us denote $T^* = \{\tau_\sigma;\ \sigma \in S\}$ where τ_σ is corresponding to $\sigma \in S$ by the theorem 2, evidently $T^* \subseteq T$.

Theorem 3. Let A_τ, $B_\sigma \in \mathfrak{A}$ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$ and φ be a non trivial homomorphic mapping of A onto B. Then there exists an algebra $C = \prod_{\tau \in T} C_\tau$ (isomorphic with B), where $C_\tau = B_\sigma$ for $\tau = \tau_\sigma \in T^*$ and $C_\tau = \{0\}$ for $\tau \in T - T^*$ such that $i.\varphi = \prod_{\tau \in T} \varphi_\tau$ where φ_τ is a homomorphic mapping of A_τ onto C_τ and i is a natural isomorphism of B onto C.

Proof. It is clear that C is isomorphic with B. By the theorem 2 for each $\sigma \in S$ there exist just one $\tau_\sigma \in T$ for which $\overline{C}_{\tau_\sigma} = \overline{B}_\sigma \subseteq \varphi(\overline{A}_{\tau_\sigma})$.

(a) Let $\overline{C}_{\tau_\sigma} = \overline{B}_\sigma = \varphi(\overline{A}_{\tau_\sigma})$ for each $\sigma \in S$, then $pr_{\tau_\sigma} \varphi \in \overline{H}(A_{\tau_\sigma}, B_\sigma) = \overline{H}(A_{\tau_\sigma}, C_{\tau_\sigma})$. Let $\varphi_\tau = pr_{\tau_\sigma} \varphi$ for $\tau = \tau_\sigma \in T^*$ and $\varphi_\tau = o$ for $\tau \in T - T^*$, then $i \, . \, \varphi = \prod_{\tau \in T} \varphi_\tau$ and $\varphi_\tau \in \overline{H}(A_\tau, C_\tau)$.

(b) Let there be $\overline{B}_{\sigma_0} \neq \varphi(\overline{A}_{\tau_{\sigma_0}})$, $\overline{B}_{\sigma_0} \subseteq \varphi(\overline{A}_{\tau_{\sigma_0}})$ for some $\sigma_0 \in S$. Because φ is the mapping of the type "onto", there exists a that set $S' \subseteq S$, card $S' > 1$, such $\varphi(\overline{A}_{\tau_{\sigma_0}}) \supseteq \overline{B}_\sigma$ for $\sigma \in S'$. Let $\sigma_1 \neq \sigma_2$, $\sigma_1, \sigma_2 \in S'$ and $b_1 \in \overline{B}_{\sigma_1}$, $b_2 \in \overline{B}_{\sigma_2}$, $b_1 \neq O_B \neq b_2$. Let $a_1, a_2 \in \overline{A}_{\tau_{\sigma_0}}$ and $\varphi(a_1) = b_1$ $\varphi(a_2) = b_2$. Then for each ω which is the direct product of operations from Ω'' we have:

$$O_B = b_1 b_2 \dots b_2 \omega = \varphi(a_1)\, \varphi(a_2) \dots \varphi(a_2)\, \omega = \varphi(a_1 a_2 \dots a_2 \omega) =$$
$$= \varphi(a_i \alpha) = \varphi(a_i)\, \alpha = b_i \alpha \neq O_B, \text{ where } i = 1 \text{ or } 2,$$

which is a contradiction. The proof is complete.

Theorem 4. Each chain with the least element 0 or the greatest element 1 is a pseudo-ordered algebra. Each completely ordered group is a pseudo-ordered algebra.

Proof. Let A be a chain with the least element 0. Put: $a \oplus b = \max\{a, b\}$, $a \, . \, b = \min\{a, b\}$, $0 = \{0\}$, $\Omega' = \Omega'' = \{.\}$. Dually for a chain with the greatest element.

Let A be a completely ordered group. Then $\oplus$ be the group composition, 0 the unit element of A and $\Omega' = \Omega'' = \{.\}$, where $a \, . \, b = \min(\max(a, a^{-1}), \max(b, b^{-1}))$.

Corollary 5. Let $A_\tau, B_\sigma \in \mathfrak{A}$ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in S} B_\sigma$ be N-algebras and φ be a non trivial homomorphic mapping of A onto B. Then card $S \leqq$ card T.

It follows directly from the theorems 2 and 3.

Corollary 6. Let A_τ, A^*_γ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$ and $A = \prod_{\gamma \in G} A^*_\gamma$ be N-algebras. Then card G = card T and $A^*_\gamma = A_{\pi(\tau)}$, where π is a permutation of the set T.

It follows directly from the theorem 3 and corollary 5.

Theorem 7. Let $A_\tau, B_\tau \in \mathfrak{A}$ be pseudo-ordered algebras and φ be a homomorphic mapping of an N-algebra $A = \prod_{\tau \in T} A_\tau$ onto N-algebra $B = \prod_{\tau \in T} B_\tau$. Then there exists a permutation π of the set T and the natural isomorphism p of $\prod_{\tau \in T} B_\tau$ onto $\prod_{\tau \in T} B_{\pi(\tau)}$ such that

$$p \, . \, \varphi = \prod_{\tau \in T} \varphi_\tau$$

where φ_τ is a homomorphic mapping of A_τ onto $B_{\pi(\tau)}$.

It follows directly from the theorem 3 and corollary 6. The theorem 7 is the converse of the theorem 1 for pseudo-ordered algebras. From theorems 7 and 4 we obtain:

Corollary 8. For atomic Boolean algebras, for l-groups discretely directly decomposable into completely ordered groups, for lattices which are direct products of chains with the least (or the greatest) element and for ordered rings which are discrete direct products of completely ordered rings is the converse of the theorem 1 valid.

Remark. For l-groups and lattice-ordered rings is by a "homomorphism" in the sense of this paper understood the homomorphic mapping preserving lattice operation (because it must preserve the direct product of operations introduced in the proof of theorem 4). It is easy to show that this homomorphism is also o-homomorphism in the sense of [11].

The conditions for discreie direct decompositions of l-groups and ordered rings into completely ordered groups and rings are given in [11].

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REMARKS ON QUASIVARIETIES OF ALGEBRAS

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A class $\mathscr{K}$ of Ω-algebras (for arbitrarily fixed type or species Ω) is called a quasivariety if $\mathscr{K}$ can be defined by a set of identical implications,

$$\forall x_1, \ldots, x_n(w_1 = w_1' \wedge \ldots \wedge w_n = w_m' \rightarrow w = w')$$

where w's are Ω-words in $x_1, \ldots, x_n$. Every variety is an example of quasivariety. But there are other examples, too, the class of cancellative semigroups being a familiar one. Many other examples can be constructed by the following simple general method. Let S be a finite set of finite Ω-algebras. Then the class $Q(S)$ of algebras embeddable in cartesian products of families of algebras from S is a locally finite quasivariety. This follows from [2]. (A class $\mathscr{K}$ of Ω-algebras is locally finite if finitely generated subalgebras of algebras of $\mathscr{K}$ are finite.)

In general $\mathscr{K}$ is a quasivariety if and only if [3] $\mathscr{K}$ is closed under the formation of (i) subalgebras (ii) cartesian products (iii) direct limits of mono direct systems (direct systems in which all morphisms are mono) (iv) direct limits of epi direct systems. (We remark in passing that (iii) is the categorical way of saying that $\mathscr{K}$ is of local character, i.e., $\mathscr{K}$ contains an algebra A if every finitely generated subalgebra of A is in $\mathscr{K}$.) Under a special circumstance the most awkward of the above closure properties, namely (iv), can be omitted. Let us say that $\mathscr{K}$ *has finite basis property for equations* if within $\mathscr{K}$ every system of equations in finite number of variables is equivalent to a finite system in those variables. This is equivalent to saying that every congruence over a finitely generated algebra $A \in \mathscr{K}$ is finitely generated as a subalgebra of $A \times A$. In still other terms our finite basis property can be expressed by saying that every finitely generated subalgebra of an algebra of $\mathscr{K}$ is finitely presented. Clearly locally finite classes have the finite basis property for equations. But there are other examples, too, the class of abelian groups being a familiar one. It follows from Theorem 70 of [1] that the class of commutative monoids also has the finite basis property for equations. Subquasivarieties of a quasivariety with the finite basis property for equations have a simple characterization given by.

Theorem 1. *Let $\mathscr{K}$ be a quasivariety with the finite basis property for equations. Then a subclass $\mathscr{K}'$ of $\mathscr{K}$ is a quasivariety if and only if $\mathscr{K}'$ is closed under the formation of subalgebras and cartesian products and is of local character.*

The above is a generalization of Lemma 1 of [2] and is proved by an essentially the same argument. We omit the proof. The result applies to commutative monoids because of Theorem 70 of [1]. We state this observation in the form of

Corollary 1. *A class of commutative monoids is a quasivariety if and only if it is closed with respect to submonoids and cartesian products and is of local character.*

The rest of this note concerns the situation in which a quasivariety has only finitely many subquasivarieties.

Theorem 2. *Let a quasivariety $\mathscr{K}$ be generated by finitely many finite algebras. Let all the subdirectly irreducible algebras of $\mathscr{K}$ be projective in $\mathscr{K}$. Then the lattice $\mathscr{L}_{qv}(\mathscr{K})$ of subquasivarieties of $\mathscr{K}$ is a finite, distributive lattice.*

Proof. Within isomorphism let S be the set of all subdirectly irreducible algebras of $\mathscr{K}$. Then, by the first assumption of the theorem, S is a finite set of finite algebras. Let $\mathscr{K}'$ be a subquasivariety of $\mathscr{K}$. Clearly every algebra of $\mathscr{K}'$ can be represented as a subcartesian product of algebras from S; let $S(\mathscr{K}')$ be the set of all algebras in S that occur in such representations of algebras of $\mathscr{K}'$. We show that $S(\mathscr{K}') \subseteq \mathscr{K}'$. Let $A \in S(K')$. Then there is a subcartesian product B with A as a factor such that $B \in \mathscr{K}'$. Since A is projective the diagram

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur] \arrow[r] & A \end{tikzcd}$$

commutes for some homomorphism $A \to B$, where $B \to A$ is the usual projection map and $A \to A$ is the identity map. It follows that A is embeddable in B. Since $\mathscr{K}'$ is a quasivariety and $B \in \mathscr{K}'$ we conclude that $A \in \mathscr{K}'$. This proves $S(\mathscr{K}') \subseteq \mathscr{K}'$. It follows from this, in view of the definition of $S(\mathscr{K}')$, that $\mathscr{K}' = Q(S(\mathscr{K}'))$. The function $S(\mathscr{K}')$ from the lattice $\mathscr{L}_{qv}(\mathscr{K}')$ into the ring of subsets of $S(= S(\mathscr{K}))$ is, therefore one-to-one. In fact $S(\mathscr{K}')$ is a lattice homomorphism. This follows fairly easily from the fact, mentioned in the beginning of this note, that $Q(S')$ is a quasivariety for every finite set S' of finite algebras. We leave the very easy details and conclude the proof of the teorem.

Remark 1. In the notation of the above proof it is clear that if $A_1, A_2 \in S(\mathscr{K})$, $A_2 \in S(\mathscr{K}')$ and A_1 is embeddable in A_2 then $A_1 \in S(\mathscr{K}')$. We can express this by saying that $S(\mathscr{K}')$ is closed under embeddability. Since $Q(S') \in \mathscr{L}_{qv}(\mathscr{K})$ for all $S' \subseteq$ $\subseteq S(\mathscr{K})$ it follows from the proof of the last theorem that $\mathscr{L}_{qv}(\mathscr{K})$ is isomorphic to the ring of subsets of $S(\mathscr{K})$ that are closed under embeddability.

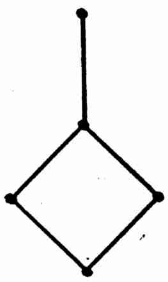
Our next theorem obtains the conclusion of Theorem 2 under somewhat different assumptions.

Theorem 3. *Let a quasivariety $\mathscr{K}$ have only finitely many subquasivarieties and let all subdirectly irreducible algebras in $\mathscr{K}$ be projective in $\mathscr{K}$. Then $\mathscr{L}_{qv}(\mathscr{K})$ is distributive.*

Proof. The theorem is proved on lines of the proof of Theorem 2 except that we need proof of the following crucial point on the basis of our present assumptions: For every $S' \subseteq S(\mathscr{K})$ the class $Q(S')$ is a quasivariety, where $S(\mathscr{K})$ is, as before, the set of subdirectly irreducible algebras of $\mathscr{K}$. This follows from Theorem 2 of [2] which states that if $\mathscr{K}$ is a quasivariety with finitely many subquasivarieties then every subclass $Q(\mathscr{K}'), \mathscr{K}' \subseteq \mathscr{K}$, is a quasivariety.

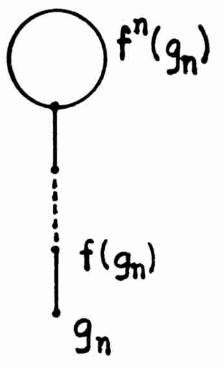
Remark 2. The converse of each of the last two theorems is false. More specifically, there are quasivarieties $\mathscr{K}$ such that $\mathscr{L}_{qv}(\mathscr{K})$ is finite and distributive but not all the subdirectly irreducible algebras of $\mathscr{K}$ are projective. We give an example. Let $\mathscr{V}$ be the variety of left normal semigroups, i.e., semigroups satisfying the identities $x^2 = x$, $xyz = xzy$. The variety $\mathscr{V}$ was shown in [4] to be $Q(\{\Sigma_2^-, \Sigma_2^\circ, \Sigma_3^-\})$, where

Σ_2^-, Σ_2°, Σ_3^- are semigroups defined as follows: Σ_2^- is the two-element semigroup satisfying the identity $xy = x$, Σ_3^- is obtained from Σ_2^- by adding a zero and Σ_2° is the two element semilattice. It was further shown in [4] that $\mathscr{L}_{qv}(\mathscr{V})$ has the graph



The lattice $\mathscr{L}_{qv}(\mathscr{V})$ is thus finite and distributive. We complete this remark by showing that Σ_3^- is subdirectly irreducible but not projective. Clearly a proper subcartesian factor of Σ_3^- must have two elements and hence should be Σ_2^- or Σ_2°. Since Σ_3^- has a zero while Σ_2^- does not Σ_2^- cannot be a subcartesian factor of Σ_3^-. Hence if Σ_3^- is subdirectly reducible, then Σ_3^- must be a subcartesian product of Σ_2^-. Since Σ_3^- is not a semilattice this is impossible to shat Σ_3^- is subdirectly irreducible. To show that Σ_3^- is not projective let Σ_2^- have elements a, b and let Σ_3^- in addition have 0 as the zero element. The set $\{\langle a, a\rangle, \langle b, b\rangle, \langle o, a\rangle, \langle o, b\rangle\}$ forms a subcartesian product of Σ_3^- and Σ_2^-; call it Σ. The projectivity of Σ_3^- would imply the embeddability of Σ_3^- into Σ. However Σ_3^- can be easily seen not to be embeddable in Σ.

Remark 3. In the last part of Remark 2 we used the fact (also used and proved in the proof of Theorem 2) that if A is projective in a class $\mathscr{K}$ of algebras then (*) A is embeddable in every subcartesian product in $\mathscr{K}$ of which A is a factor. Let us call A semiprojective in $\mathscr{K}$ if A satisfies (*). Equivalently, A is semiprojective in $\mathscr{K}$ if for every epimorphism $B \to A$, $B \in \mathscr{K}$, A is embeddable in B. It is easy to see that semiprojectivity is indeed a weaker property than projectivity. We give an example which we shall find of use later. Let ω_n be the variety of unary algebras $\langle A, f\rangle$ satisfying $f^{n+1}(x) = f^n(x) = f^n(y)$ identically. Let F_n be the free algebra with one generator g_n. We show that F_m is semiprojective in ω_n but not projective if $2 \leq \leq m < n - 2$. First note that if $m \leqq n$ then F_m is isomorphic to a subalgebra of F_n, namely, the subalgebra generated by $f^{n-m}(g_n)$. For this and other easy assertions which we will make without proof it may be helpful to refer to the graph



of F_n. Let Θ: $B \to F_m$ be an epimorphism with $B \in \omega_n$, $m \leq n$. Let $b \in B$ be such that $\Theta(b) = g_m$. Then b generate in B an lagebra isomorphic to F_l for some l, $n \geq \geq l \geqq m$. Since F_m is embeddable in F_l we see that F_m is embeddable in B. This proves that F_m is semiprojective in ω_n for $m < n$. Assume now that $2 \leq m < n-2$. Let α: $F_n \to F_2$, β: $F_m \to F_2$ be epimorphisms; the (unique) existence of such epimorphisms is clear. Let γ: $F_m \to F_n$ be any homomorphism. Then $\gamma(g_m) = f^r(g_n)$ for some integer $r > n - m > 2$. Since $\alpha(f^r(g_n)) = f(g_2)$ we see that $\alpha\gamma(F_m) = \{f(g_2)\} \neq \beta(F_m) = \{g_2, f(g_2)\}$. Hence for given epimorphisms $F_n \to F_2$, $F_m \to F_2$ there exists no homomorphism $F_m \to F_n$ which makes the diagram
commute. Thus F_m is not projective in ω_n. It is easy to see that F_m is subdirectly

$$\begin{tikzcd}
 & F_n \arrow[d] \\
F_m \arrow[ur] \arrow[r] & F
\end{tikzcd}$$

irreducible in ω_n for $m \leqq n$. We have thus shown that semiprojectivity of A in $\mathscr{K}$ does not imply projectivity of A in $\mathscr{K}$ even when $\mathscr{K}$ is a variety and A is subdirectly irreducible in $\mathscr{K}$.

Remark 4. Theorem 2, Theorem 3, Remark 1 and Remark 2 hold if the condition of projectivity is replaced by that of semiprojectivity.

Theorem 4. *Let all subclasses of a variety $\mathscr{K}$ that are closed under the formation of subalgebras and cartesian products be subvarieties. Then all subdirectly irreducible algebras of $\mathscr{K}$ are semiprojective.*

Proof. Let A be subdirectly irreducible in $\mathscr{K}$. Let $B \to A$ be an epimorphism with $B \in \mathscr{K}$. Consider $Q(\{B\})$. By assumption $Q(\{B\})$ is a variety. Hence $A \in Q(\{B\})$. Since A is subdirectly irreducible this implies that A is embeddable in B. This proves the theorem.

Remark 5. In the last theorem "semiprojective" cannot be replaced by "projective". To show this consider the variety ω_n of Remark 4. It follows from [5] that an algebra in ω_n is subdirectly irreducible if and only if it is isomorphic to F_m, $m \leqq n$. From this and the fact that F_m is embeddable in F_n for $m \leq n$ we see that $\omega_n = Q(\{F_n\})$. Let $\mathscr{K} \subseteq \omega_n$ be closed under the formation of subalgebras and cartesian products. From $\omega_n = Q(\{F_n\})$ and the semiprojectivity of F_m for $m \leq n$ it follows that $\mathscr{K} = \omega_m$ for some $m \leq n$ and hence is a variety. However, as noted in Remark 4, if $n > 4$, then not all the subdirectly irreducible algebras of ω_n are projective. Hence in Theorem 4 "semiprojective" cannot be improved to 'projective'. Nor can "variety" be replaced by quasivariety in the last theorem. Thus the variety $\mathscr{V}$ of Remark 2 has [4] the property that if $\mathscr{K} \subseteq \mathscr{V}$ is closed in $\mathscr{V}$ under the formation of subalgebras and cartesian products, then $\mathscr{K}$ is a quasivariety. Yet, as shown in Remark 2, not all subdirectly irreducible algebras of $\mathscr{V}$ are semiprojective.

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DISTINGUISHING SUBSETS IN SEMILATTICES

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1. INTRODUCTORY DEFINITIONS AND LEMMAS

1.1 Definition. A semilattice is a set G with an idempotent, commutative, and associative binary operation $\circ$ which assigns to each pair $(x, y) \in G$ a single element $x \circ y \in G$.

1.2 Lemma. *Let G be a join-semilattice (a semilattice under $\cup$). Then G is partially ordered set (poset) where the partial ordering $\leqq$ is defined by the following condition: $x \leqq y$ iff $x \circ y = y$. For all $x, y \in G$, we have $x \cup y = x \circ y$.* (Proof for lattices see [1], Theorem 2.)

1.3 Definition. Let G be a poset, $E \subseteq G$. The set E is called an *end of G* if, for all elements $x \in E$ and $y \in G$, the condition $x \leqq y$ implies $y \in E$.

1.4 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end. Then E is a join-subsemilattice in G.*

Proof. Let $x, y \in E$. Then $x \circ y \geqq x$ which implies $x \circ y \in E$.

1.5 Definition. Let G be a semigroup, Θ a equivalence relation on G. The relation Θ is called a congruence relation if for all $a, b, c, d \in G$ the conditions $a\Theta b$, $c\Theta d$ imply $a \circ c\Theta b \circ d$.

1.6 Agreement. Let Θ be a congruence relation on a semigroup G. We denote the elements of G/Θ by capital letters $X, Y, \ldots, W$.

1.7 Remark. Let Θ be a congruence relation on a semigroup G. For each $X \in G/\Theta$ and each $Y \in G/\Theta$ there exists such a $Z \in G/\Theta$ that $X \circ Y = \{x \circ y; x \in X, y \in Y\} \subseteq Z$. We put $Y \bullet Y = Z$. (See [4] page 188.)

1.8 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G. The set G/Θ is a join-semilattice.* (See [4] page 189.)

1.9 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G, $X, Y \in G/\Theta$. Let $\leqq$ be an ordering on G/Θ generated by the join-semilattice operation $\bullet$. Then $X \leqq Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leqq y$.*

Proof. Let $X \leqq Y$. Then $X \bullet Y = Y$ and hence $X \circ Y \subseteq Y$. For arbitrary elements $x \in X$, $y \in Y$, we have $x \circ y \in Y$ and $x \leqq x \circ y$. Now we suppose that for each $x \in X$ there exists an element $y \in Y$ such that $x \leqq y$. Hence $x \circ y = y$ and therefore $X \circ Y \subseteq Y$. The last inclusion is, by 1.7, equivalent to the equation $X \bullet Y = Y$ and $X \leqq Y$.

1.10 Lemma *Let G be a join-semilattice, Θ a congruence relation on G. Then each Θ-class is a join-subsemilattice in G.*

Proof. For lattices see [4] Theorem 75.

1.11 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G. Let $X, Y \in G/\Theta$ be such that $X \leqq Y$. Then $y \circ X \subseteq Y$ holds for each $y \in Y$.*

Proof. Let $x \in X$ be arbitrary. Then there exists an element $z \in Y$ such that $x \leqq z$.

It holds $x \circ z = z \in Y$. Simultaneously $x \circ y \Theta x \circ z$ and we have $x \circ y \in Y$. Hence $y \circ X \subseteq Y$.

1.12 Definition. Let G be either a jon-semilattice or a monoid, $L \subseteq G$ its subset. For $x, y \in G$ we put $(x, y) \in \Xi_{(G, L)}$ if, for each $u, v \in G$, the condition $u \circ x \circ v \in L$ is equivalent to $u \circ y \circ v \in L$.

Some well known results concerning monoids can be formulated for join-semilattices.

1.13 Lemma. *A relation $\Xi_{(G, L)}$ is a congruence relation on the join-semilattice G.*

Proof. See [5] page 386. (The proof is given for monoids).

1.14 Remark. Let G be a join-semilattice, Θ a congruence relation on G. Then Θ is called *principal* if there is a set $L \subseteq G$ such that $\Theta = \Xi_{(G, L)}$. (The definition of principal congruences on semigroups see [6] page 530.)

1.15 Lemma. *Let G be a join-semilattice, $L \subseteq G$ its subset and $X \in G/\Xi_{(G,L)}$.* If $X \cap L \neq \emptyset$, then $X \subseteq L$.

Proof. Let $x \in X$. There exists $y \in X \cap L$. It is $x \Xi_{(G,L)} y$ and $y = y \circ y \in L$ hence $x \circ y \in L$ and also $x = x \circ x \in L$. Thus $X \subseteq L$.

1.16 Corollary. *Let G be a join-semilattice, $L \subseteq G$. Then $L = \bigcup \{X;\ X \in G/\Xi_{(G,L)}$ $X \cap L \neq \emptyset\}$*

1.17 Definition. Let G be a semigroup, $L \subseteq G$ a set, $u \in G$. We say that the elements $x, y \in G$, $x \neq y$, *are distinguished by u with respect to L* if the conditions $u \circ x \in L$, $u \circ y \notin L$ are equivalent. We say that *L distinguishes G* and we write $L\delta G$ if, for each $x, y \in G$, $x \neq y$, there is $u \in G$ such that x, y are distinguished by u with respect to L.

It is easy to prove the following two Theorems. The proofs are similar to the proof of the Theorem 2.6 in [7].

1.18 Theorem. *Let G be a monoid, $L \subseteq G$, Θ a congruence relation on G. Then the following two assertions are equivalent:*

(A) $\Theta = \Xi_{(G,L)}$.

(B) *There exists a subset $\mathbf{L}$ in G/Θ such that $L = \bigcup\limits_{X \in \mathbf{L}} X$ and $\mathbf{L}$ distinguishes G/Θ.*

1.19 Theorem. *Let G be a join-semilattice, $L \subseteq G$, a congruence relation on G. Then the following two assertions are equivalent:*

(A) $\Theta = \Xi_{(G,L)}$.

(B) *There exists a subset $\mathbf{L}$ in G/Θ such that $L = \bigcup\limits_{X \in \mathbf{L}} X$ and $\mathbf{L}$ distinguishes G/Θ.*

1.20 Remark. It is not possible to formulate previous Theorems as one Theorem for semigroups.

1.21 Example. Let B be a semigroup with two elements 0 and a with the following operation: $a \circ a = 0$, $a \circ 0 = 0$, $0 \circ a = 0$, $0 \circ 0 = 0$. Let us put $L = \{a\}$. For all $u,v \in B$ $u \circ a \circ v = 0 \in B - L$, $u \circ 0 \circ v = 0 \in B - L$ and hence $a\ \Xi_{(B,L)}\ 0$. The congruence relation has only one class which is equal to B. Hence the equation $L = \bigcup \{X;\ X \in G/\Xi_{(G,L)},\ X \cap L \neq \emptyset\}$ does not hold.

2. JOIN-SEMILATTICES WITH THE PROPERTY (β)

2.1 Definition. Let G be a join-semilattice. We say that G *has the property* (β) or that G *is of the type* (β) if it has the greatest element i and for each pair $x, y \in G$, $x \neq y$, for which $x \circ y < i$ there exists an element $z \in G$ such that either $x < z$ and simultaneously $z \parallel y$ or $y < z$ and simultaneously $z \parallel x$.

2.2 Lemma. *Let G be a join-semilattice of the type (β) satisfying the maximum condition. Then for each pair $x, y \in G$, $x \neq y$ there exists an element $u \in G$ such that either $x \circ u = i$, $y \circ u \neq i$ or $x \circ u \neq i$, $y \circ u = i$ holds.*

Proof. Let $x, y \in G$.

I. Let $x \circ y = i$. For $x \neq y$, it is $x \neq i$ or $y \neq i$; let us suppose the first case. Then it is sufficient to put $u = x$.

II. Let $x \circ y < i$. Let us denote by the letter a that of the elements x, y to which there exists an element $z_0 \in G$ such that $a < z_0$, and such that it is incomparable with the other of the elements x, y. We denote the other element by b. It is obvious that $z_0 < i$.

α) Let $z_0 \circ b = i$. We put then $u = z_0$ and we get $a \circ u = a \circ z_0 = z_0 < i$, $b \circ u = b \circ z_0 = i$.

β) Let $z_0 \circ b \neq i$. We consider the pair z_0, $b \circ z_0$. To this pair there exists an element $z_1 < i$ for which $z_0 < z_1$, $z_1 \parallel b \circ z_0$. If $b \circ z_1 = i$, then we put $u = z_1$. In the reverse case we construct an element z_2 by similar way as element z_1 with the property $a < z_0 < z_1 < z_2 < i$, $b \parallel z_2$. As G satisfies the maximum condition we attain, in a finite number of steps, an element z_n such that $a \circ z_n < i$, $b \circ z_n = i$.

2.3 Corollary. Let G be a join-semilattice with the property (β) satisfying the maximum condition. *Then $\{i\}$ distinguishes G.*

2.4 Lemma. *Let G be a join-semilattice with the greatest element i. Suppose $\{i\}\delta G$. Then G has the property (β).*

Proof. Let us admit that G has not the property (β). Then there exist x, y, $x \neq y$, $x \circ y < i$ such that every $z > x$ is comparable with y and every $z > y$ is comparable with x. There are two possibilities.

I. The elements x, y are comparable, for instance $x < y$. Then for each $z > x$ either $z \leqq y$ or $z > y$ holds. Let $u \in G$ be arbitrary. If $u \circ x = i$, then it is obvious $u \circ y = i$, too. Let $u \circ y = i$, $u \circ x < i$. If $u \circ x = x$, then $u \leqq x$ hence $u \leqq y$ and $u \circ y = y = x \circ y < i$. It is a contradiction. Therefore $x < u \circ x$. Hence $u \circ x \leqq y$ or $u \circ x \geqq y$. In the first case $u \leqq u \circ x \leqq y$ and then $u \circ y = y = x \circ y < i$. It is a contradiction, too. In the second case $i = u \circ y \leqq (u \circ x) \circ y = u \circ x < i$ and it is again a contradiction. We get that $u \circ y = i$ implies $u \circ x = i$.

II. The elements x, y are incomparable. Then the element $z > x$ is comparable with y, it is $z \leqq y$ or $z > y$. The first case implies $x < z \leqq y$ and it is impossible. Therefore $z > x$ implies $z > y$ and conversely. Let $u \in G$ be arbitrary. Let $u \circ x = i$, $u \circ y < i$. Then $u \circ y \geqq y$. If $u \circ y = y$, then it is $u \leqq y$ and hence $i = u \circ x \leqq y \circ x < i$ and this is a contradiction. Therefore $u \circ y > y$ and it implies $u \circ y > x$. Hence we get $i = u \circ x \leqq u \circ (u \circ y) = u \circ y < i$ and it is again a contradiction.

We get that for each $u \in G$ the relation $u \circ x = i$ implies $u \circ y = i$ and conversely $u \circ y = i$ implies $u \circ x = i$. This is a contradiction with the asumption that $\{i\}\delta G$. We have proved that G has the property (β).

2.5 Theorem. *Let G be a join-semilattice satisfying the maximum condition with the greatest element i. Then the following statements are equivalent:*

(A) $\{i\}\delta G$.

(B) *G has the property (β).*

2.6 Theorem. *Let G be a dually atomic join-semilattice with the greatest element i. Let M be a set of all dual atoms in G. Suppose $\{i\}\delta G$. Then $M\delta G$.*

Proof. Let $x, y \in G$, $x \neq y$. Since $\{i\}\delta G$, there is an element $u \in G$ such that $u \circ x = i$, $u \circ y \neq i$ or $u \circ x \neq i$, $u \circ y = i$. Let us denote by the letter a that element of

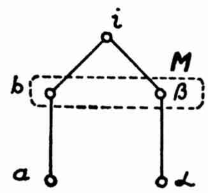
x, y for which the join with the element u is equal i and by the letter b the other element.

Let $u \circ a = i$, $u \circ b \in M$. Then the proof is finished.

Let $u \circ a = i$, $u \circ b \notin M$. G is a dually atomic semillattice and simultaneously $u \circ b < i$. There exists $p \in M$ for which $u \circ b < p$ and hence $(p \circ u) \circ b = p \circ (u \circ b) = p \in M$ and $(p \circ u) \circ a = p \circ (u \circ a) = p \circ i = i \notin M$. We have found $u' \in G$, $u' = u \circ p$ such that $u' \circ a \notin M$ and $u' \circ b \in M$. Thus $M \delta G$.

2.7 Remark. We cannot formulate theorem 2.6 as an equivalence.

2.8 Example. Let G be a join-semilattice with the following diagram:



Then $M = \{b, \beta\}$, $M \delta G$ but $\{i\}$ does not distinguish G.

2.9 Theorem. *Every Boolean algebra has the property* (β).

Proof. In the proof of this theorem we denote the operation $\circ$ by $\cup$.

Let B be Boolean algebra, $x, y \in B$, $x \neq y$. Let us choose the notation in such a way that $y \nleqq x$. If $x \cup y' = i$, then $y = y \cap i = y \cap (x \cup y') = y \cap x$ which implies $y \leqq x$ and we have a contradiction. Therefore $x \cup y' < i$, $y \cup y' = i$ and $\{i\} \delta B$. The statement follows from Lemma 2.4.

3. DISTINGUISHING SUBSETS IN JOIN-SEMILATTICES

3.1 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end and $M \delta E$. Let $x \in G - E$ and suppose the existence of at least one element $s \in E$ such that, for each $u \in E$, M contains either both elements $u \circ x$, $u \circ s$ or none of them. Then there is precisely one element s with this property.*

Proof. Suppose the existence of $s_1, s_2 \in E$, $s_1 \neq s_2$ with this property. Then, for each $u \in E$, the condition $u \circ s_1 \in M$ implies $u \circ x \in M$ which implies $u \circ s_2 \in M$ and conversely, $u \circ s_2 \in M$ implies $u \circ s_1 \in M$.

It is a contradiction to the hypothesis $M \delta E$.

3.2 Definition. Let G be a join-semilattice, $E \subseteq G$ its end, $M \subseteq E$, $M \delta E$, $x \in G - E$.

We put

$$\mathscr{L}(E, M, x) = \begin{cases} \{M, M \cup \{x\}\} & \text{if, for each } t \in E, \text{ there is } u \in E \text{ such that } M \text{ contains} \\ & \text{precisely one of the elements } u \circ x,\ u \circ t. \\ \{M\} & \text{if there is } t \in E \text{ such that } t \circ x \in M \text{ and, for each} \\ & u \in E,\ M \text{ contains either both elements } u \circ x,\ u \circ t \text{ or} \\ & \text{none of them.} \\ \{M \cup \{x\}\} & \text{if there is } t \in E \text{ such that } t \circ x \notin M \text{ and, for each} \\ & u \in E,\ M \text{ contais either both elements } u \circ x,\ u \circ t \text{ or} \\ & \text{none of them.} \end{cases}$$

3.3 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end, $M \subseteq E$, $M\delta E$, $x \in G - E$. Then $\mathscr{L}(E, M, x)$ is the system of all sets L distinguishing $E \cup \{x\}$ such that $L \cap E = M$.*

Proof. We denote by $\mathscr{D}(E, M, x)$ the system of all sets L distinguishing $E \cup \{x\}$ such that $L \cap E = M$.

Clearly, $L \in \mathscr{D}(E, M, x)$ implies either $L = M$ or $L = M \cup \{x\}$.

(i) If $t, z \in E$, $t \neq z$, then there is $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. The following cases can occur:

(1) For each $t \in E$, there is $u \in E$ such that M contains precisely one of the ele.-ents $u \circ x$, $u \circ t$.

We have $\mathscr{L}(E, M, x) = \{M, M \cup \{x\}\} \supseteq \mathscr{D}(E, M, x)$.

We prove $M\delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following two possibilities: (a) $t \neq x \neq z$ (b) $t \neq x = z$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), the condition (1) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ z = u \circ x$, $u \circ t$.

We prove $(M \cup \{x\})\ \delta(E \cup \{x\})$.

Indeed, if $t,z \in \{E \cup \{x\}\}$, $x \neq t \neq z$, then we have the following two possibilites: (a) $t \neq x \neq z$ (b) $t \neq x = z$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. Since $u \circ t \neq$ $\neq x \neq u \circ z$ the set $M \cup \{x\}$ contains precisely one of the element $u \circ t$, $u \circ z$. In the case (b) the condition (1) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ z = u \circ x$, $u \circ t$. Since $u \circ z = u \circ x \neq x \neq u \circ t$ the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$.

We have proved $\mathscr{L}(E, M, x) = \{M, M \cup \{x\}\} \subseteq \mathscr{D}(E, M, x)$.

Thus, $\mathscr{L}(E, M, x) = \mathscr{D}(E, M, x)$.

(2) There is precisely one element $s \in E$ such that $s \circ x \in M$ and, for each $u \in E$, M contains either both elements $u \circ x$, $u \circ s$ or none of them.

We have $\mathscr{L}(E, M, x) = \{M\}$.

We prove $M\delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following possibilities: (a) $t \neq x \neq z$ (b) $t \neq s$, $z = x$ (c) $t = s$, $z = x$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), Lemma 3.1 implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z = u \circ x$. In the case (c), we have $x \circ t = x \circ s \in M$, $x \circ z = x \circ x = x \notin M$.

We prove that $(M \cup \{x\})\ \delta(E \cup \{x\})$ does not hold. Indeed, $s \neq x$. If $u \in E$, then M contains either both elements $u \circ s$, $u \circ x$ or none of them by (2). Since $u \circ s \neq x \neq$ $\neq u \circ x$ for each $u \in E$ the set $M \cup \{x\}$ contains both elements $u \circ s$, $u \circ x$ or none of them.

Finally, $x \circ s \in M \subseteq M \cup \{x\}$, $x \circ x = x \in M \cup \{x\}$.

We have proved $\mathscr{D}(E, M, x) = \{M\}$.

It follows $\mathscr{L}(E, M, x) = \{M\} = \mathscr{D}(E, M, x)$.

(3) There is precisely one element $s \in E$ such that $s \circ x \notin M$ and, for each $u \in E$, the set M contains either both elements $u \circ x$, $u \circ s$ or none of them.

We have $\mathscr{L}(E, M, x) = \{M \cup \{x\}\}$.

We prove $(M \cup \{x\})\ \delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following possibilities: (a) $t \neq x \neq z$ (b) $t \neq s$, $z = x$ (c) $t = s$, $z = x$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. Since $u \circ t \neq x \neq u \circ z$ the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), Lemma 3.1 implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z = u \circ x$. Since $u \circ t \neq x \neq$ $\neq u \circ x = u \circ z$, the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (c), we have $x \circ s \notin M$, $x \circ s \neq x$ which implies $x \circ s \notin M \cup \{x\}$, $x \circ x =$ $= x \in M \cup \{x\}$.

We prove that $M\delta(E \cup \{x\})$ does not hold.

Indeed, $s \neq x$. If $u \in E$, then M contains either both elements $u \circ s$, $u \circ x$ or none of them.

Finally, $x \circ s \notin M$, $x \circ x = x \notin M$.

We have proved $\mathscr{D}(E, M, x) = \{M \cup \{x\}\}$.

Thus, $\mathscr{L}(E, M, x) = \{M \cup \{x\}\} = \mathscr{D}(E, M, x)$.

The cases (1), (2), (3) represent all possibilities by 3.1. Thus, we have proved $\mathscr{L}(E, M, x) = \mathscr{D}(E, M, x)$ which is the assertion of the Lemma.

3.4 Definition. Let G be a join-semilattice, $L \subseteq G$. Then L is called *hereditary in G* if, for each end E of G, the condition $(E \cap L)\ \delta E$ is satisfied.

3.5 Remark. If G is a join-semilattice, E its end and L is hereditary subset then $E \cap L$ is hereditary in E.

Proof. Indeed, if F is an end of E, then it is an end *of* G which implies $(F \cap L)\ \delta F$. Since $F \subseteq E$ we have $F \cap (E \cap L) = F \cap L$. Thus $(F \cap (E \cap L))\ \delta F$.

3.6 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end, L a hereditary subset in E, x a maximal element in $G - E$, $M \subseteq E \cup \{x\}$ a subset such that $M\delta(E \cup \{x\})$, $E \cap M =$ $= L$. Then M is hereditary in $E \cup \{x\}$.*

Proof. Let $N \subseteq E \cup \{x\}$ be an end, $t, s \in N$, $t \neq s$. Since $t, s \in E \cup \{x\}$, there is $u \in E \cup \{x\}$ such that M contains precisely one of the elements $u \circ t$, $u \circ s$. It follows, especially, $u \circ t \neq u \circ s$. Clearly, $u \circ t$, $u \circ s \in N$. We can suppose, without loss of generality, that $u \circ s \neq x$.

(a) If $u \circ t \neq x \neq u \circ s$, then $u \circ t$, $u \circ s \in E$ which implies $u \circ t$, $u \circ s \in E \cap N$, the latter set being an end in E. Since L is hereditary in E, we have $(E \cap N \cap L)$ $\delta(E \cap N)$. Since $L \subseteq E$, we have $E \cap N \cap L = N \cap L$. Thus $(N \cap L)\ \delta(E \cap N)$. It follows the existence of $v \in E \cap N$ such that $N \cap L$ contains precisely one of the elements $v \circ u \circ t$, $v \circ u \circ s$. Clearly, $v \circ u \circ t \neq x \neq v \circ u \circ s$. Since $N \cap L \subseteq N \cap$ $\cap M \subseteq N \cap (L \cup \{x\})$, the set $N \cap M$ contains precisely one of the elements $v \circ u \circ t$, $v \circ u \circ s$. Clearly $v \circ u \in N$.

(b) If $u \circ t = x \neq u \circ s$, we have $u \leqq x$, $t \leqq x$ which implies $u = t = x$. Thus, $x \neq x \circ s$, x, $x \circ s \in N$ and M contains precisely one of the elements $x = x \circ x$, $x \circ s$. Thus, $x \in N$ and $M \cap N$ contains precisely one of the elements $x \circ t = x \circ x$, $x \circ s$.

We have proved $(N \cap M)\ \delta N$ and M is hereditary in $E \cup \{x\}$.

3.7 Corollary. Let G be a join-semilattice, $E \subseteq G$ its end, L a hereditary subset in E, x a maximal element in $G - E$. Then each $M \in \mathscr{L}(E, L, x)$ is a hereditary subset in $E \cup \{x\}$ such that $M \cap E = L$.

Proof. By 3.3, each $M \in \mathscr{L}(E, L, x)$ distinguishes $E \cup \{x\}$ and $M \cap E = L$. Then M is hereditary in $E \cup \{x\}$ by 3.6.

3.8 Lemma. *Let G be a join-semilattice, $\mathscr{E}$ a chain consisting of ends in G which is ordered by inclusion, $\mathscr{L}$ a chain of subsets in G ordered by inclusion. Let f be a surjection of $\mathscr{E}$ onto $\mathscr{L}$ such, that, for each $E \in \mathscr{E}$, the set $L = f(E)$ is a hereditary subset in E. Suppose that f has the following property:*

(α) *If* $E, E' \in \mathscr{E}$, $E \subseteq E'$, *then* $f(E) = E \cap f(E')$. *Then* $\bigcup_{L \in \mathscr{L}} L$ *is a hereditary subset of* $\bigcup_{E \in \mathscr{E}} E$.

Proof. Let $P \subseteq \bigcup_{E \in \mathscr{E}} E$ be an end in $\bigcup_{E \in \mathscr{E}} E$. Suppose $s, t \in P$, $s \neq t$. Then there is $E_0 \in \mathscr{E}$ such that $s, t \in E_0$. We put $L_0 = f(E_0)$. Then $P \cap E_0$ is an end in E_0; it follows that $(P \cap E_0 \cap L)\, \delta(P \cap E_0)$. Thus, there is an element $u \in P \cap E_0$ such that $P \cap E_0 \cap L_0 = P \cap L_0$ contains precisely one of elements $u \circ s$, $u \circ t$. For instance, we can suppose $u \circ s \in P \cap L_0$, $u \circ t \notin P \cap L_0$. Since $P \cap L_0 \subseteq P \cap (\bigcup_{L \in \mathscr{L}} L)$ we have $u \circ s \in P \cap (\bigcup_{L \in \mathscr{L}} L)$.

Let us admit the existence of $E \in \mathscr{E}$ such that $u \circ t \in f(E) \cap P$. Since $t \in E_0$ we have $u \circ t \geqq t$ and $u \circ t \in E_0$. If $E \subseteq E_0$, then $f(E) = E \cap f(E_0) = E \cap L_0$ and $u \circ t \in f(E) \cap P = E \cap L_0 \cap P \subseteq P \cap L_0$ which is a contradiction. Thus, $E_0 \subseteq E$ which implies $f(E_0) = E_0 \cap f(E)$. It follows $u \circ t \in f(E) \cap P \cap E_0 = f(E_0) \cap P = P \cap L_0$ which is a contradiction.

Thus, $u \circ t \notin f(E) \cap P$ for each $E \in \mathscr{E}$ which implies $u \circ t \notin \bigcup_{E \in \mathscr{E}} (f(E) \cap P) = P \cap (\bigcup_{E \in \mathscr{E}} (E)) = P \cap (\bigcup_{L \in \mathscr{L}} L)$.

We have proved $(P \cap (\bigcup_{L \in \mathscr{L}} L))\, \delta P$ which is by Definition 3.3 the assertion of Lemma.

3.9 Lemma. *Let G be an ordered set satisfying the maximum condition. Then there is a set $\mathscr{E}$ of ends in G having the following properties:*

(i) *$\mathscr{E}$ is well ordered by inclusion; thus, there is an ordinal α such that $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$ and, for λ, $\mu < \alpha$, the condition $E_\lambda \subseteq E_\mu$ is equivalent to $\lambda \leqq \mu$.*

(ii) $E_0 = \varnothing$ $E_\alpha = G$

(iii) *for each $\lambda < \alpha$ there is $a_\lambda \in G - E_\lambda$ which is maximal in $G - E_\lambda$ such that $E_{\lambda+1} - E_\lambda = \{a_\lambda\}$.*

(iv) $E_\gamma = \bigcup_{\lambda < \gamma} E_\lambda$ for each limit ordinal $\gamma < \alpha + 1$.

Proof. The assertion is clear if $G = \varnothing$. Thus we can suppose $G \neq \varnothing$. Let $\leqq$ denote the order relation in G. By [4], Theorem 2.3, there is a linear ordering $\preceq$ on G which is an extension of $\leqq$ such that G is well ordered by the dual ordering of $\preceq$. Thus there is an ordinal α and a sequence $(a_\lambda)_{\lambda < \alpha}$ of elements of G such that each element of G appears in this sequence precisely once and that, for λ, $\mu < \alpha$ the condition $a_\lambda \preceq a_\mu$ is equivalent to $\lambda \geqq \mu$. We put $E_\lambda = \{a_\varkappa;\ \varkappa < \lambda\}$ for each $\lambda \leqq \alpha$, $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$. Then, for λ, $\mu < \alpha + 1$, $E_\lambda \subseteq E_\mu$ is equivalent to the condition $\lambda \leqq \mu$. Thus, $\mathscr{E}$ is isomorph to the set $\{\lambda;\ \lambda < \alpha + 1\}$ which implies that $\mathscr{E}$ is well ordered by set inclusion. If $\lambda < \alpha + 1$, $x \in E_\lambda$, $y \in G$, $x \leqq y$, then there are μ, $\nu < \alpha$ such that $x = a_\mu$, $y = a_\nu$. Since $x \in E_\lambda$ we have $\mu < \lambda$. The condition $x \leqq y$ implies $x \preceq y$, i.e. $a_\mu \preceq a_\nu$ which implies $\nu \leqq \mu$. Thus, $\nu < \lambda$ and $y = a_\nu \in E_\lambda$. It follows that E_λ is an end with respect to the order relation $\leqq$ for each $\lambda < \alpha + 1$. We have (i). The condition (ii) holds obviously. Clearly, $E_{\lambda+1} - E_\lambda = \{a_\lambda\}$ for each $\lambda < \alpha$; suppose $x \in G - E_\lambda$, $a_\lambda \leqq x$. Then there is $\mu < \alpha + 1$ such that $x = a_\mu$ and $a_\lambda \preceq a_\mu$ which implies $\lambda \geqq \mu$. Clearly, $G - E_\lambda = \{a_\varkappa;\ \varkappa \geqq \lambda\}$. Thus $\mu = \lambda$ and $x = a_\lambda$ is maximal in $G - E_\lambda$. We have (iii). If $\gamma < \alpha + 1$ is a limit ordinal, then $E_\gamma = \{a_\varkappa;\ \varkappa < \gamma\} = \bigcup_{\lambda < \gamma} \{a_\varkappa;\ \varkappa < \lambda\} = \bigcup_{\lambda < \gamma} E_\lambda$ and we have (iv).

3.10 Definition. Let G be an ordered set satisfying the maximum condition. Then each set of ends in G having the properties (i), (ii), (iii), (iv) of Lemma 3.9 is called a *suitable set of ends in G*.

3.11 Definition. Let G be a join-semilattice satisfying the maximum condition, $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$ its suitable set of ends.

We put $L_0 = \varnothing$.

Let $0 < \beta < \alpha + 1$ and suppose that we have constructed, for any $\lambda < \beta$, a hereditary subset L_λ of E_λ in such a way that $\lambda < \mu < \beta$ implies $L_\lambda = E_\lambda \cap L_\mu$.

If β is an isolated ordinal, we put $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$ and we define $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$.

If β is a limit ordinal, we put $L_\beta = \bigcup\limits_{\lambda<\beta} L_\lambda$.

By induction, we define L_λ for each $\lambda < \alpha + 1$. Especially, we put $L = L_\alpha$ and we say that L *has been constructed by means the suitable set of ends* $\mathscr{E}$.

3.12 Theorem. *Let G be a join-semilattice satisfying the maximum condition, $L \subseteq G$ a subset. Then the following conditions are equivalent:*

(A) *L is a hereditary subset in G.*

(B) *If $\mathscr{E}$ is an arbitrary suitable set of ends in G, then L has been constructed by means of $\mathscr{E}$.*

Proof. Let (A) hold. Let $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$ be an arbitrary suitable set of ends in G. We put $L_\lambda = E_\lambda \cap L$ for each $\lambda < \alpha + 1$.

Then $L_0 = E_0 \cap L = \varnothing$.

Let $0 < \beta < \alpha + 1$. By Remark 3.5, L_λ is a hereditary subset in E_λ for any $\lambda < \beta$ and $\lambda < \mu < \beta$ implies $L_\lambda = L \cap E_\lambda = L \cap E_\lambda \cap E_\mu = E_\lambda \cap L_\mu$.

If β is an isolated ordinal and if $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$, then L_β is hereditary in $E_\beta = E_{\beta-1} \cup \{a_{\beta-1}\}$ by Remark 3.5 which implies $L_\beta \delta (E_{\beta-1} \cup \{a_{\beta-1}\})$. Further, $L_\beta \cap E_{\beta-1} = L_{\beta-1}$ and $L_{\beta-1}\ \delta E_{\beta-1}$. By Lemma 3.3, we have $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$.

If β is a limit ordinal, then

$L_\beta = E_\beta \cap L = (\bigcup\limits_{\lambda<\beta} E_\lambda) \cap L = \bigcup\limits_{\lambda<\beta} (E_\lambda \cap L) = \bigcup\limits_{\lambda<\beta} L_\lambda$.

Finally, $L_\alpha = E_\alpha \cap L = G \cap L = L$.

We have proved that L has been constructed by means of $\mathscr{E}$ which is (B).

Let (B) hold. Then, trivially, L_0 is a hereditary subset in E_0.

Let $0 < \beta < \alpha + 1$ and suppose that L_λ is hereditary in E_λ for each $\lambda < \beta$ and that $\mu < \lambda < \beta$ implies $L_\mu = E_\mu \cap L_\lambda$.
If β is an isolated ordinal, then $L_{\beta-1}$ is hereditary in $E_{\beta-1}$, $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$, $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$, $a_{\beta-1}$ is maximal in $G - E_{\beta-1}$. By Corollary 3.7, L_β is hereditary in $E_{\beta-1} \cup \{_{\beta-1}\} = E_\beta$ and $E_{\beta-1} \cap L_\beta = L_{\beta-1}$. If $\lambda < \beta$, then $\lambda \leqq \beta - 1$ and $E_\lambda \cap L_\beta = E_\lambda \cap E_{\beta-1} \cap L_\beta = E_\lambda \cap L_{\beta-1} = L_\lambda$ by the induction hypothesis.

If β is a limit ordinal, then $L_\beta = \bigcup\limits_{\mu<\beta} L_\mu$ and L_β is hereditary in $\bigcup\limits_{\mu<\beta} E_\mu = E_\beta$ by Lemma 3.8.

If $\lambda < \beta$, then $E_\lambda \cap L_\beta = E_\lambda \cap (\bigcup\limits_{\mu<\beta} L_\mu) = \bigcup\limits_{\mu<\beta} (E_\lambda \cap L_\mu) = \bigcup\limits_{\mu\leqq\lambda} (\tilde{E}_\lambda \cap L_\mu) \cup$
$\cup \bigcup\limits_{\lambda<\mu<\beta} (E_\lambda \cap L_\mu) = \bigcup\limits_{\mu\leqq\lambda} (E_\lambda \cap L_\mu) \cup L_\lambda = L_\lambda$ because $E_\lambda \cap L_\mu \subseteq L_\mu = E_\lambda \cap L_\lambda \subseteq L_\lambda$ for each $\mu \leqq \lambda$.

We have proved that L_β is hereditary in E_β and that $\lambda < \beta$ implies $L_\lambda = E_\lambda \cap L_\beta$.

It follows by transfinite induction that L_λ is hereditary in E_λ for each $\lambda < \alpha + 1$. Especially, $L = L_\alpha$ is hereditary in $E_\alpha = G$, which is (A).

3.13 Corollary. *Let G be a join semilattice satisfying the maximum condition. Then there is a set $L \subseteq G$ such that $(E \cap L)\ \delta E$ for each end E of G.*

3.14 Remark. In [6] following definitions are given: A subset H of a semigroup G is called *indivisible by an equivalence* Θ (by a subset F) if H is contained in some class of Θ ($\Xi_{(G,F)}$). A subset H is called *disjunctive* if the only subsets indivisible by $\Xi_{(G,H)}$ are empty and one-element.

According to these definitions we can formulate the following Corollary:

3.15 Corollary. *Let G be a join-semilattice satisfying the maximum condition. Then there exists a set $L \subseteq G$ such that for each end $E \subseteq G$ the set $L \cap E$ is disjunctive.*

4. SPECIAL CONGRUENCES ON MONOIDS

4.1 Assumption. We shall suppose in the whole fourth paragraph that G is a monoid and Θ a congruence relation on G such that G/Θ is a join-semilattice satisfying the maximum condition. We denote its greatest element by I.

4.2 Definition. Let G/Θ have the property (β). Then we say that the congruence relation Θ *has the property* (β) or that Θ *is of the type* (β).

4.3 Theorem. *Let $L = I \in G/\Theta$. Then the following statements are equivalent:*

(A) $\Theta = \Xi_{(G,I)}$
(B) $\{I\}\,\delta(G/\Theta)$
(C) Θ *has the property* (β).

Proof. The statements (A) and (B) are equivalent according to Theorem 1.18. Simultaneously, by Theorem 2.6 the statements (B) and (C) are equivalent.

4.4 Theorem. *Let Θ be a (β) congruence on G satisfying the assumption* 4.1. *Let M be the set of dual atoms in G/Θ.* (The set of elements which are covered by I). *Then $\Theta = \Xi_{(G,L)}$, where $L = \bigcup_{m \in M} m$.*

Proof. From Theorem 4.3 follows that $\{I\}\,\delta(G/\Theta)$. So the conditions of Theorem 2.7 are satisfied and the set $L\delta(G/\Theta)$. By Theorem 1.18 we have $\Theta = \Xi_{(G,L)}$.

4.5 Main Theorem. *Let Θ be a congruence relation on G satisfying the assumption* 4.1. *Then there exists a subset $L \subseteq G$ such that $\Theta = \Xi_{(G,L)}$.*

Proof. According to Corollary 3.13 there exists a subset $\mathbf{L} \subseteq G/\Theta$ $\mathbf{L} = \{X;\ X \in G/\Theta,\ X \subseteq L\}$ in G/Θ which distinguishes G/Θ. Hence by Theorem 1.18 $\Theta = \Xi_{(G,L)}$ holds.

4.6 Corollary. *Let Θ be a congruence relation on G satisfying* 4.1. *Let $\bar{L} \subseteq G/\Theta$ be constructed by* 3.11. *Then $\Xi_{(G/\Theta,\bar{L})} = idG/\Theta$.*

Proof. We have $\bar{L}\delta G/\Theta$ which is equivalent to $\Xi_{(G/\Theta,\bar{L})} = idG/\Theta$ by [3] Theorem 1.7.

4.7 Theorem. *All congruence relations on a join-semilattice S satisfying the maximum condition are principal congruences.*

Proof. Join-semilattice S satisfies the maximum condition. All factor—join-semilattices on S satisfy also the maximum condition and they are join-semilatices. By Corollary 3.13 we obtain a subset $L \subseteq S/\Theta$ for all congruence relations on S which distinguishes S/Θ. Hence by 1.19 $\Theta = \Xi_{(S,L)}$ holds.

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NOTE ON CERTAIN PARTITIONS OF POINTS IN R^d

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Let $\mathscr{X}$ be a finite set of points in the d-dimensional Euclidean space R^d. In [4] Radon partitions of types $\{r, s\}$ (i.e. $\mathscr{X}$ admits a partition into non-empty subsets $\mathscr{X}_1$ and $\mathscr{X}_2$ such that card $\mathscr{X}_1 = r$, card $\mathscr{X}_2 = s$ and conv $\mathscr{X}_1 \cap$ conv $\mathscr{X}_2 \neq \emptyset$) are studied. In this note a similar question for the cone hulls (with certain singularity) is solved.

Let o be the fixed origin and $\mathscr{X} = (x_1, \ldots, x_f)$ be an f-tuple of (not necessarily different) points in the d-dimensional Euclidean space R^d, $f \geqq d + 2$, for which $o \notin \mathscr{X}$ and $\dim \mathscr{X} = d$. We say that $\mathscr{X}$ has the *property* (r), r being a natural number, $1 \leqq r \leqq f - 1$, if there exists $J \subset F = \{1, 2, \ldots, f\}$ such that card $J = r$ and either conv $\mathscr{X}(J) \cap$ conv $\mathscr{X}(F - J) = \{o\}$ or cone $\mathscr{X}(J) \cap$ cone $\mathscr{X}(F - J)$ contains a ray. (By $\mathscr{X}(J)$ we denote the n-tuple $(x_{i_1}, \ldots, x_{i_n})$ with indices $J = \{i_1, \ldots, i_n\} \subset F$.)

Let $x_i = (x_{i1}, \ldots, x_{id})$ for $i = 1, \ldots, f$ in a basis $\mathscr{K}$. We shall consider the f by d matrix

$$X = \begin{pmatrix} x_{11} \ldots x_{1d} \\ \ldots\ldots\ldots \\ x_{f1} \ldots x_{fd} \end{pmatrix}$$

and we put $L(X) = \mathrm{lin}\,(x^{(1)}, \ldots, x^{(d)})$, where $x^{(i)} \in R^f$ is the ith column in X, $D(X)$ its orthogonal complement in R^f. It is $\dim L(X) = d$, $\dim D(X) = f - d$.

Forming the matrix

$$\bar{X} = \begin{pmatrix} \alpha_{11} \ldots \alpha_{1\,f-d} \\ \ldots\ldots\ldots\ldots \\ \alpha_{f1} \ldots \alpha_{f\,f-d} \end{pmatrix}$$

whose columns $\alpha^{(j)}$, $j = 1, \ldots, f - d$ form a basis of $D(X)$, we shall assign the ith row $\bar{x}_i \in R^{f-d}$ of $\bar{X}$ to each x_i, $i \in F$; the f-tuple $\bar{\bar{\mathscr{X}}} = (\bar{x}_1, \ldots, \bar{x}_f)$ of these points in R^{f-d} is called a *linear representation of* $\mathscr{X}$ (see [5]).

By an *affine representation* (or *Gale transform*) we understand an f-tuple $\tilde{\mathscr{X}} = (\tilde{x}_1, \ldots, \tilde{x}_f)$, $\tilde{x}_i = (\beta_{i1}, \ldots, \beta_{i\,f-d-1})$ $i = 1, \ldots, f$ of points in R^{f-d-1}, where the columns of the matrix

$$\begin{pmatrix} \beta_{11} \ldots \beta_{1\,f-d-1} \\ \ldots\ldots\ldots\ldots \\ \beta_{f1} \ldots \beta_{f\,f-d-1} \end{pmatrix}$$

form a basis of the orthogonal complement of the $(d + 1)$-space lin $(x^{(1)}, \ldots, x^{(d)}, \mathbf{1})$ in R^f (see [3], 5.4, $o \in \mathscr{X}$ possibly.)

(*) Under the assumption of $o \in \mathrm{conv}\,\mathscr{X}$ we denote by K the k-face of the polytope conv $\mathscr{X}$, $1 \leqq k \leqq d$, for which $o \in$ relint K and put $G = \{i \in F \mid x_i \in K\}$, $g =$ card G.

By $h(X)$ we shall denote the dimension of the projection of $D(X)$ on the coordinate $(f - g)$-space in R^f determined by the axes with indices $F - G$ in the direction of the complementary coordinate g-space; according to the definition we put $h(X) = -1$ if $G = F$.

1. *Under the situation* (*) *it is* $h(X) = h(X')$, *where* X *or* X' *are the matrices belonging to* $\mathscr{X}$ *in two arbitrary Cartesian systems* $\mathscr{K}$ *or* $\mathscr{K}'$ *in* R^d, *respectively, with the same origin* o.

Proof. We simplify the denotation as follows: We denote by G or $H = G^{\perp}$ the coordinate g-space or its orthogonal complement, resp. and let $1, 2, \ldots, g$ be the indices of the coordinate axes of G. We shall write briefly D, D', h, h' instead of $D(X)$, $D(X')$, $h(X)$, $h(X')$.

Putting $\beta = D \cap H$, $\beta' = D' \cap H$ we shall prove that $\dim \beta = \dim \beta'$. To this purpose we denote the g by $(f - d)$ matrix formed from the first g rows of $\bar{X}$ or $\bar{X}'$ by X^* or X'^*, resp. Then it is $\beta = \{x \in R^f | x = \bar{X}\bar{\lambda}$, where $\bar{\lambda} \in \Lambda = \{\lambda \in R^{f-d} \mid X^*\lambda = o\}\}$, $\beta' = \{x \in R^f \mid x = \bar{X}'\bar{\lambda}$, where $\bar{\lambda} \in \Lambda' = \{\lambda \in R^{f-d} \mid X'^*\lambda = o\}\}$. Since the columns of $\bar{X}$ and $\bar{X}'$ are linearly independent, it is $\dim \beta = \dim \Lambda$, $\dim \beta' = \dim \Lambda'$. Considering that $X'^* = X^*R$ for a suitable regular matrix R (As R we can take a regular matrix such that $\bar{X}' = \bar{X}R$ which exists because the column vectors of both matrices form the spaces of the same dimension $f - d$), it is $\dim \Lambda = \dim \Lambda'$ and hence $\dim \beta = \dim \beta'$.

Replacing H by G we shall prove that $\dim \gamma = \dim \gamma'$, where $\gamma = G \cap D$, $\gamma' = G \cap D'$. If we put $\delta = \beta^{\perp}_{D}$ (the orthogonal complement of β in D), $\varepsilon = \gamma^{\perp}_{\delta}$, $\delta' = \beta'^{\perp}_{D'}$, $\varepsilon' = \gamma'^{\perp}_{\delta'}$, we have $\dim \delta = \dim \delta'$, $\dim \varepsilon = \dim \varepsilon'$ and since $h = \dim \beta + \dim \varepsilon$, $h' = \dim \beta' + \dim \varepsilon'$, it follows $h = h'$.

Remark. Thus the number $h(\mathscr{X})$ can be defined by the relation $h(\mathscr{X}) = h(X)$, where X corresponds to arbitrary basis in R^d with the origin o (under conditions (*)).

2. $\mathscr{X}$ *has the property* (r) *if and only if there exists* $J \subset F$, caxd $J = r$ *and a hyperplane* H *in* R^{f-d}, $o \in H$ *such that* $\bar{\mathscr{X}}(J) \subset H_1$, $\bar{\mathscr{X}}(F - J) \subset H_2$, *where* H_1, H_2 *are closed halfspaces determined by* H *and* $\operatorname{int} H_1 \cap \bar{\mathscr{X}}(J) \neq \emptyset \neq \operatorname{int} H_2 \cap \bar{\mathscr{X}}(F - J)$.

Such a separation is called the *semiseparation of* points.

Proof. I. Let $\mathscr{X}$ have the property (r), i.e. there exists $J \subset F$, card $J = r$ and a point $b \in R^f$, $b = (\beta_1, \ldots, \beta_f)$ such that $\sum_{i \in F} \beta_i x_i = o$, $\beta_i \geqq 0$ for $i \in J$, $\beta_i \leqq 0$ for $i \in F - J$ and at least in one case there holds the inequality. Since $b \in D(X)$, it is $b = \sum_{j=1}^{f-d} \gamma_j \alpha^{(j)}$. Put $c = (\gamma_1, \ldots, \gamma_{f-d})$. It holds $(c, \bar{x}_i) = \beta_i$ for each $i \in F$. Thus the hyperplane in R^{f-d} whose normal is determined by c semiseparates the $\bar{\mathscr{X}}(J)$ and $\bar{\mathscr{X}}(F - J)$.

II. On the contrary, let $\bar{\mathscr{X}}(J)$, $\bar{\mathscr{X}}(F - J)$ be semiseparated by the hyperplane with $c = (\gamma_1, \ldots, \gamma_{f-d})$ as its normal. Put $\beta_i = (c, \bar{x}_i)$ for $i \in F$, $b = (\beta_1, \ldots, \beta_f)$. Then $\beta_i \geqq 0$ for $i \in J$, $\beta_i \leqq 0$ for $i \in F - J$ and in both cases at leest one inequality appears. It holds $b = \sum_{j=1}^{f-d} \gamma_i \alpha^{(j)}$ and hence $\sum_{i=1}^{f} \beta_i x_i = o$. From this it follows that $\sum_{i \in J} \beta_i x_i$ is the common point of cone $\mathscr{X}(J)$ and cone $\mathscr{X}(F - J)$. If $\sum_{i \in J} \beta_i x_i = o$, it is $o \in \operatorname{conv} \mathscr{X}(J) \cap \operatorname{conv} \mathscr{X}(F - J)$ and if $\sum_{i \in J} \beta_i x_i \neq o$, then cone $\mathscr{X}(J)$ and cone $\mathscr{X}(F - J)$ have the common ray.

3. (see [2], 358). *If the points* $x_1, \ldots, x_f \in R^d$ *satisfy the condition* $o \in \operatorname{int} \operatorname{conv} \{x_1, \ldots, x_f\}$, *then there exist positive numbers* λ_i, $i = 1, \ldots, f$ *such that* $o = \sum_{i=1}^{f} \lambda_i x_i$.

4. *Under the situation* (*) *a hyperplane H of R^{f-d}, $o \in H$ exists for which* $\bar{\mathscr{X}}(F - G) \subset \subset H$, $\bar{\mathscr{X}}(G)$ *lies in one of open halfspaces determined by* H, cone $\bar{\mathscr{X}}(F - G) =$ lin $\bar{\mathscr{X}}(F - G)$ *and* $h(\mathscr{X}) =$ dim cone $\bar{\mathscr{X}}(F - G)$.

Proof. Since $o \in$ relint K, it is (according to 3) $o = \sum_{i=1}^{f} \beta_i \bar{x}_i$ for a suitable $(\beta_1, \ldots, \beta_f)$ where $\beta_i > 0$ for $i \in G$, $\beta_i = 0$ for $i \in F - G$. From this it follows that $b = (\beta_1, \ldots, \ldots, \beta_f) \in D(X)$ and thus $b = \sum_{j=1}^{f-d} \gamma_j \alpha^{(j)}$ for some $c = (\gamma_1, \ldots, \gamma_{f-d})$; c is the normal vector of the required hyperplane H because of $(c, \bar{x}_i) = \beta_i$ for $i \in F$. Further on it holds that, for no supporting hyperplane of conv $\bar{\mathscr{X}}$ through o, more than g points from $\bar{\mathscr{X}}$ lie in the corresponding open halfspace. (In fact, if more than g points from $\bar{\mathscr{X}}$ lay in the open halfspace determined by such hyperplane, then there would exist more than g points of $\mathscr{X}$ lying in K.) From this it follows cone $\bar{\mathscr{X}}(F - G) =$ lin $\bar{\mathscr{X}}(F - G)$. We put $h^*(X) =$ dim lin $\bar{\mathscr{X}}(F - G)$. Then the equality $h^*(X) = h(\mathscr{X})$ holds. ($h^*(X)$ equals the rank of the $f - g$ by $f - d$ matrix formed from the rows of $\bar{X}$ with indices $F - G$, which is also equal to the rank of the f by $(f - d)$ matrix if we replace the rows with indices G by the zero rows and hence it equals the dimension of the projection of $D(X)$ on the coordinate $(f - g)$-space.); q.e.d. Note that evidently $f - d > h(\mathscr{X}) \geqq -1$.

5. (see [3], 5.4. iii)

If $Z = (z_1, \ldots, z_f)$ *is an f-tuple of points in* R^{f-d-1} *for which* $\sum_{i=1}^{f} z_i = o$ *and* dim lin $Z = f - d - 1$, then there exists an *f-tuple* $\mathscr{X} \subset R^d$ *such that* dim aff$\mathscr{X} = d$ *and Z is its affine representation.*

6. (see [3], 7.1.4)

If P is a k-neighbourly d-polytope (*i.e. each k-membered subset* $K \subset$ vert P *forms a face S of P for which* $K =$ vert S) *and* $k > \left[\frac{d}{2}\right]$, *then P is a d-simplex.*

Note. Corollary. If $\mathscr{X}$ is the set of all vertices of some d-polytope (with f vertices) and $f \geqq d + 2$, then there exists $k \leqq \left[\frac{d}{2}\right]$ such that conv $\mathscr{X}$ is an l-neighbourly polytope for each $1 \leqq l \leqq k$ and for $l > k$ it is not l-neighbourly.

7. (see [4], lemma 2)

For each affine representation $\bar{X} \subset R^{f-d-1}$, $f \geqq d + 2$ *of an* f-tuple $\mathscr{X} \subset R^d$, dim$\mathscr{X} = d$ *it holds:*

Every open halfspace of R^{f-d-1} *determined by a hyperplane* H, $o \in H$ *contains,*

(i) *at least one point of* $\bar{X}$; *and some of them contains exactly one point if* $\mathscr{X} \neq$ vert P *for every convex d-polytope P with f vertices,*

(ii) *at least $k + 1$ points of* $\bar{X}$ *if* $\mathscr{X} =$ vert P *for some k-neighbourly convex d-polytope P with f vertices; and some of such halfspaces contains exactly $k + 1$ points of* $\bar{X}$ *if P is k- but not $(k + 1)$-neighbourly convex d-polytope.*

8. *The range of the value r for which the given f-tuple* $\mathscr{X} \subset R^d$ *has the property* (r) *forms the integer interval.*

Proof. Let H, $o \in H$ be a hyperplane of R^{f-d} that semiseparates r points of $\bar{\mathscr{X}}$. There exists a point $x \neq o$ such that $x \in H \cap$ int cone $\bar{\mathscr{X}}$. Let λ a be any $(f-d-2)$-space going through o, x and lying in H. If H rotates around λ from 0° to 180°, then for every r', $r \leqq r' \leqq f - r$ there exists the position of H such that r' points from $\tilde{\mathscr{X}}$ are semiseparated.

9. *Let $\tilde{\mathscr{X}} = (\tilde{x}_1, \ldots, \tilde{x}_f)$ be an f-tuple of points in R^l, $l \geqq 1$, $f \geqq l + 1$,* $\dim \tilde{\mathscr{X}} = l$, $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$. *Then for every natural number r for which* $\dfrac{f - l - 1}{2} < r < \dfrac{f + l + 1}{2}$ *there exists a hyperplane containing o that semiseparates r points from $\tilde{\mathscr{X}}$; this interval cannot be enlarged.*

Proof. In 8 it is shown that the range of r is an interval. Since $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$, there exist (see 3) numbers $\lambda_1, \ldots, \lambda_f > 0$ such that $o = \sum_{i=1}^{f} \lambda_i \tilde{x}_i$. According to 5 there exists an f-tuple $\mathscr{X} \subset R^{f-l-1}$ such that the f-tuple $\lambda_1 \tilde{x}_1, \ldots, \lambda_f \tilde{x}_f$ is its affine representation and $\dim \operatorname{aff} \mathscr{X} = f - l - 1$. The semiseparation of $(\tilde{x}_1, \ldots, \tilde{x}_f)$ is equivalent to the semiseparation of $(\lambda_1 \tilde{x}_1, \ldots, \lambda_f \tilde{x}_f)$. If $f = l + 1$, r points can be semiseparated for arbitrary r, $1 \leqq r \leqq l$ because $\tilde{\mathscr{X}}$ is the set of vertices of an l-simplex and $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$. Thus the assertion holds.

Let $f \geqq l + 2$. A) If $\mathscr{X}$ is the set of vertices of some convex $(f - l - 1)$-polytope P (card vert $P = f$), then there exists exactly one k, $1 \leqq k \leqq \left[\dfrac{f - l - 1}{2}\right]$ such that P is a k-neighbourly polytope and not m-neighbourly for every $m > k$ (see 6). According to 7 (put $l = f - d - 1$) every open halfspace in R^l determined by a hyperplane going through o contains at least $k + 1$ points and some of them contains exactly $k + 1$ points from $\tilde{\mathscr{X}}$. In general, the semiseparation of $\left[\dfrac{f - l - 1}{2}\right] + 1$ points from $\tilde{\mathscr{X}}$ is guaranteed and no less. B) If A) does not work, then $\mathscr{X}$ is not the set of vertices of the convex $(f - l - 1)$-polytope with f vertices and by 7 one point of $\tilde{\mathscr{X}}$ can be semiseparated by a suitable hyperplane; q.e.d.

10. (see [4], theorem)

Let $\mathscr{X}$ be an f-tuple of points in R^d, card$\mathscr{X} \geqq d + 3$. *Then*

(i) *if $\mathscr{X}$ is not the set of vertices of a convex polytope with f vertices, $\mathscr{X}$ has a Radon partition of the type $\{r, f - r\}$ for arbitrary $r = 1, \ldots, f - 1$.*

(ii) *If $\mathscr{X}$ is the set of a k-neighbourly convex polytope P, then there is no partition of the type $\{r, f - r\}$ for $r \leqq k$, and if P is exactly k-neighbourly, then it admits Radon partitions for every r, $f - k - 1 \geqq r \geqq k + 1$.*

11. (see [1], 3.2.)

If $y \in \operatorname{int} \operatorname{conv} X$, $X \subset R^d$, then $y \in \operatorname{int} \operatorname{conv} Y$ where $Y \subset X$, card $Y \leqq 2d$.

Let $\mathscr{X}$ be an f-tuple of points in R^d, $f \geqq d + 2$, $o \notin \mathscr{X}$ $\dim \mathscr{X} = d$. Let us define for it the number $s(\mathscr{X})$ as follows:

1. In the case of $o \notin \operatorname{conv} \mathscr{X}$ put $s(\mathscr{X}) = \dfrac{d - 1}{2}$

2. In the case of $o \in \operatorname{conv} \mathscr{X}$, i.e. if (*) is fulfilled, we put

2.1. $s(\mathscr{X}) = 0$ for $g > 2k$

and for $g \leqq 2k$ we define

2.2.1. $s(\mathscr{X}) = \dfrac{g}{f - d} - 1$ if $h(\mathscr{X}) = 0$ or $= -1$

2.2.2. $s(\mathscr{X}) = \dfrac{d - g}{2}$ if $h(\mathscr{X}) = f - d - 1$

2.2.3. $s(\mathscr{X}) = \min\left\{\frac{g}{f-d-h(\mathscr{X})} - 1\ ,\ \frac{f-g-h(\mathscr{X})-1}{2}\right\}$
if $1 \leqq h(\mathscr{X}) \leqq f-d-2$.

Theorem. *Let $\mathscr{X}$ be an f-tuple of points in R^d, $f \geqq d+2$, $o \notin \mathscr{X}$, $\dim \mathscr{X} = d$. Then for every natural number r for which $s(\mathscr{X}) < r < f - s(\mathscr{X})$ $\mathscr{X}$ has the property (r) an this interval is the maximal one.*

Proof. According to 8 the range of admissible value of r is an interval.

Case 1. We choose a hyperplane H in R^d in order that it may strictly separate o and $\operatorname{conv} \mathscr{X}$ and project $\mathscr{X}$ from o on H; we denote the projection by $\mathscr{X}'$. It holds $\dim \operatorname{conv} \mathscr{X}' = d-1$ (because of $\dim \operatorname{conv} \mathscr{X} = d$) and $\mathscr{X}$ has the property (r) if and only if there exists $J \subset F$, card $J = r$ such that $\operatorname{conv} \mathscr{X}'(J) \cap \operatorname{conv} \mathscr{X}'(F-J) \neq \emptyset$. If for some $i, j \in F$, $i \neq j$, it is $x_i' = x_j'$, $\mathscr{X}$ has the property (r) for every $r = 1, 2, \ldots, f-1$. In other case the assumptions of 10 are satisfied because of card $\mathscr{X}' \geqq$ $\geqq (d-1)+3$. From 10 and 6 it follows that $\mathscr{X}$ has the property (r) for every r, $\left[\frac{d-1}{2}\right] + 1 \leqq r \leqq f - \left[\frac{d-1}{2}\right] - 1$ or, equivalently, for $\frac{d-1}{2} < r < f - \frac{d-1}{2}$.
The remaining part of the assertion follows from the existence of the $\left[\frac{d-1}{2}\right]$ — neighbourly polytope (see [3]).

Case 2.1. Under the situation (*) it is $o \in \operatorname{relint}_k \operatorname{conv} \mathscr{X}(G)$ and card $G = g > 2k$. Hence (by 11) there exists $j \in G$ such that $o \in \operatorname{relint}_k \operatorname{conv} \mathscr{X}(G - \{j\})$ and the property (r) for $r = 1$ can be achieved by the choice of $J = \{j\}$.

Case 2.2.1. If $h(\mathscr{X}) = -1$ (i.e. $G = F$), all the points of $\bar{\mathscr{X}}$ lie in the open half-space ϱ determined by the hyperplane H from 4. If $h(\mathscr{X}) = 0$, it is $\bar{x}_i = o$ for at least one $i \in F$ and for the remaining $j \in F$ it is $\bar{x}_j = o$ or $\bar{x}_j$ lies in ϱ. In both cases, consequently, $\operatorname{cone} \bar{\mathscr{X}}$ is the sharp cone of dimension $f-d$. Let E be the set of all $i \in F$ for which cone $\{\bar{x}_i\}$ is an extreme ray in cone $\bar{\mathscr{X}}$ and k_i be the multiplicity of that ray. Put $k = \min_{i \in E} k_i$. Then there exists a hyperplane in R^{f-d} going through o which semiseparates k points of $\bar{\mathscr{X}}$. Since $\max k = \left[\frac{g}{f-d}\right]$ when max operates on all the f-tuples $\mathscr{X}$ for which $h(\mathscr{X}) = 0$ or $= -1$ (the inequality $\leqq$ is evident, the equality is proved by the following example), it is (by 2) $s(\mathscr{X}) = \frac{g}{f-d} - 1$; q.e.d.

Example. Let $\mathscr{X}$ be an f-tuple of points in R^d its matrix X of which is the f by $f-d$ matrix whose rows $\bar{x}_i$, $i = 1, \ldots, f$ are the vectors

$$\begin{aligned}
\bar{x}_1 = \ldots = \bar{x}_k &= (1, 0, 0, \ldots, 0)\\
\bar{x}_{k+1} = \ldots = \bar{x}_{2k} &= (0, 1, 0, \ldots, 0)\\
&\ldots\ldots\ldots\ldots\\
\bar{x}_{(f-d-1)\,k+1} = \ldots = \bar{x}_g &= (0, 0, \ldots, 0, 1)\\
\bar{x}_{g+1} = \ldots = \bar{x}_f &= (1, 1, \ldots, 1)
\end{aligned}$$

where $k = \left[\frac{g}{f-d}\right]$, Such an f-tuple $\mathscr{X}$, $o \notin \mathscr{X}$ exists and has not the property (r) for every $r < \left[\frac{g}{f-d}\right]$ because $\left[\frac{g}{f-d}\right]$ point sin $\bar{\mathscr{X}}$ are the least number of points which may be semiseparated.

Case 2.2.2. Let H be the hyperplane in R^{f-d} from 4. Since for every hyperplane $H' \neq H$ in R^{f-d} going through o $H' \cap H$ semiseparates in H at least one point of $\bar{\mathscr{X}}(F - G)$ (because of cone $\bar{\mathscr{X}}(F - G) = H$), the least number of points in $\bar{\mathscr{X}}$ that can be semiseparated equals the minimal number of points from $\bar{\mathscr{X}}(F - G)$ which can be semiseparated by a hyperplane in H going through o. According to 9, for every r where $\frac{d - g}{2} < r < f - \frac{d - g}{2}$, r points of $\bar{\mathscr{X}}$ can be semiseparated and this interval cannot be enlarged. By 2 it is $s(\mathscr{X}) = \frac{d - g}{2}$.

Case 2.2.3. First of all it holds $f - d \geqq \dim \operatorname{cone} \bar{\mathscr{X}}(G) \geqq f - d - h(\mathscr{X})$ and cone $\bar{\mathscr{X}}(G)$ is a sharp cone. Denote by τ $(f - d - h)$-dimensional orthogonal complement to h-space cone $\bar{\mathscr{X}}(F - G)$ and project the g-tuple $\bar{\mathscr{X}}(G)$ on τ in the direction of this h-space; denote by $\bar{\mathscr{X}}_\tau(G)$ the projected g-tuple. The semiseparation of some points from $\bar{\mathscr{X}}(G)$ by a hyperplane in R^{f-d} going through the h-space cone $\bar{\mathscr{X}}(F - G)$ is equivalent to the semiseparation of points from $\bar{\mathscr{X}}_\tau(G)$ by a hyperplane in R^{f-d-h}. According to the case 2.2.1 $\left[\frac{g}{f - d - h}\right]$ points from $\bar{\mathscr{X}}_\tau(G)$ can be semiseparated and this number is generally the minimal one. At the same time it equals the least number of points which can be semiseparated in $\bar{\mathscr{X}}$ if the separating hyperplane contains cone $\bar{\mathscr{X}}(F - G)$. If the separating hyperplane (note it by H') is not of this kind, then $H \cap H'$ is such a hyperplane that in each of its open halfspaces there lies at least one point of $\bar{\mathscr{X}}(F-G)$. According to 9 the semiseparation of $\leqq \left[\frac{f - g - h - 1}{2}\right]$ points from $\bar{\mathscr{X}}(F - G)$ by $H \cap H'$ cannot be guaranteed and this estimation is the best one. This number is the same even for the semiseparation of points from $\bar{\mathscr{X}}$ Since every separating hyperplane in R^{f-d} is one of the above types and the estimations in 9 and 2.2.1 are the best ones, our assertion follows from 2.

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ON STRONG HOMOMORPHISMS OF FINITELY SEMIGENERATED LANGUAGES

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0. INTRODUCTION

M. Novotný [3] proved that finitely characterizable languages are preserved under strong homomorphisms. Finitely characterizable languages have been defined by the use of configurations. Taking semiconfigurations instead of configurations we can define similarly finitely semigenerated languages. The problem is, whether these languages are preserved under strong homomorphisms. In this paper the positive answer to the question mentioned above is given.

1. SOME DEFINITIONS

Let V be a set. We denote V^* the free monoid over V, i.e. the set of all finite sequences of elements of the set V including the empty sequence Λ. We identify one-member sequences with elements of the set V; it follows $V \subseteq V^*$. If $x \in V^*$, $x = x_1 \dots x_n$, where $x_i \in V (i = 1, \dots, n)$ and n is a natural number, we put $|x| = n$; further, we define $|\Lambda| = 0$.

Let V, U be sets, f a surjection of V onto U. Then there exists the only homomorphism of V^* onto U^*. This homomorphism (denoted as f_*) is defined as follows: for every $x \in V^*$, $x = x_1x_2 \dots x_n$, where n is a natural number and $x_i \in V (i = 1, 2, \dots, n)$, we put $f_*(x) = f(x_1) f(x_2) \dots f(x_n)$; further, we define $f_*(\Lambda) = \Lambda$.

Let us assume that $x \in V^*$, $f_*(x) = y_1y_2 \dots y_m$, where m is a natural number and $y_i \in U^* (i = 1, 2, \dots, m)$. It follows that there exist elements $x_1, x_2, \dots, x_m$ of the set V^* such that $x = x_1x_2 \dots x_m$ and $f_*(x_i) = y_i$ for $i = 1, 2, \dots, m$. It is obvious that $|x| = |f_*(x)|$ for every $x \in V^*$.

2. LANGUAGES AND GENERALIZED GRAMMARS

2.1. Definition. Let V be a set, L a subset of the set V^*. The ordered pair (V, L) is called a *language*. The elements of the set V are called *word-forms*, the elements of the set V^* are called *strings*, the elements of the set L are called *marked strings*. The set V is called a *vocabulary of the language* (V, L). The ordered pair (V, L) is called the *language over the vocabulary* V.

2.2. Definition. Let V, R be sets with the property $R \subseteq V^* \times V^*$. For $x, y \in V^*$ we put $x \to y(R)$ if $(x, y) \in R$. Further, for $x, y \in V^*$ we put $x \Rightarrow y(R)$ if there exist elements $u, v, t, z \in V^*$ such that $x = utv$, $y = uzv$, $t \to z(R)$. For $x, y \in V^*$ we put

$x \overset{*}{\Rightarrow} y(R)$ if there exists an integer number $p \geqq 0$ and elements $t_0, t_1, \dots, t_p \in V^*$ such that $x = t_0$, $t_p = y$ and $t_{i-1} \Rightarrow t_i(R)$ for $i = 1, 2, \dots, p$. The sequence of strings $(t_i)_{i=0}^{p}$ is called an *x-derivation of y of the length p in R. An x-derivation of length* 0 is called a *trivial derivation of x.*

2.3. Definition. Let V, V_T, S, R be sets such that $V_T \subseteq V$, $S \subseteq V^*$, $R \subseteq V^* \times V^*$. Then the quadruple $G = \langle V, V_T, S, R \rangle$ is called a *generalized grammar.*

2.4. Definition. Let $G = \langle V, V_T, S, R \rangle$ be a generalized grammar. We put $\mathscr{L}(G) = \{x;\ x \in V_T^*$ and there exists some $s \in S$ such that $s \overset{*}{\Rightarrow} x(R)\}$. The language $(V_T, \mathscr{L}(G))$ is called the *language generated by the generalized grammar G.*

2.5. Definition. A generalized grammar $G = \langle V, V_T, S, R \rangle$ is called a generalized special grammar if $V = V_T$. In this case we write $\langle V, S, R \rangle$ instead of $\langle V, V, S, R \rangle$.

2.6. Definition, A generalized grammar $G = \langle V, V_T, S, R \rangle$ is called a *grammar* if the sets V, S, R are finite.

2.7. Definition. A grammar $G = \langle V, V_T, S, R \rangle$ is called a *special grammar* if $V = V_T$, in this case we write $\langle V, S, R \rangle$ instead of $\langle V, V, S, R \rangle$.

3. SEMICONFIGURATIONS AND STRONG HOMOMORPHISMS

3.1. Definition. Let (V, L), (U, M) be languages, f be a surjection of V onto U. The surjection f is called a *weak homomorphism* of the language (V, L) onto (U, M) if $f_*(L) = M$. The surjection f is called a *strong homomorphism* of the language (V, L) onto (U, M) if $f_*^{-1}(M) = L$.

3.2. Remark. *It is obvious that each strong homomorphism is at the same time a weak homomorphism and a bijective weak homomorphism is strong.*

3.3. Definition. A bijective strong homomorphism of (V, L) onto (U, M) is called an *isomorphism.*

3.4. Definition. Let $G = \langle V, V_T, S, R \rangle$, $H = \langle U, U_T, P, Q \rangle$ be generalized grammars, f a surjection of V onto U. The surjection f is called a strong homomorphism of G onto H if following conditions are satisfied:

(A) For every $x \in V$, the condition $x \in V_T$ is equivalent to $f(x) \in U_T$.
(B) For every $x \in V^*$, the condition $x \in S$ is equivalent to $f_*(x) \in P$.
(C) For every $x, y \in V^*$, the condition $(x, y) \in R$ is equivalent to $(f_*(x), f_*(y)) \in Q$.

3.5. Definition. A bijective strong homomorphism of G onto H is called an *isomorphism.*

3.6. Theorem. *Let $G = \langle V, V_T, S, R \rangle$, $H = \langle U, U_T, P, Q \rangle$ be generalized grammars, f a strong homomorphism of G onto H. Then the following assertions hold true:*

(I) *For $t' \in U^*$, $s \in V^*$, $(x', y') \in Q$ the following conditions are equivalent:*
(A) *$t' \Rightarrow f_*(s)$ $(\{(x', y')\})$.*
(B) *there exist $t \in f_*^{-1}(t')$, $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ such that $t \Rightarrow s(\{(x, y)\})$.*

(II) *For $t' \in U^*$, $s \in V^*$ the condition $t' \Rightarrow f_*(s)$ (Q) is equivalent to the existence of $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R)$.*

(III) *For $t' \in U^*$, $s \in V^*$ the condition $t' \overset{*}{\Rightarrow} f_*(s)$ (Q) is equivalent to the existence of $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R)$.*

(IV) *For $x \in V^*$ the condition $x \in \mathscr{L}(G)$ is equivalent to $f_*(x) \in \mathscr{L}(H)$.*

(V) *f / V_T is the strong homomorphism $(V_T, \mathscr{L}(G))$ onto $(V_T, \mathscr{L}(H))$.*

Proof. $t' \Rightarrow f_*(s)$ $(\{(x', y')\})$ means, in other words, that there exist elements $u', v' \in U^*$ such that $t' = u'x'v'$, $f_*(s) = u'y'v'$. It means that there exist elements $u \in f_*^{-1}(u')$, $x \in f_*^{-1}(x')$, $v \in f_*^{-1}(v')$ such that $s = uyv$, $uxv \Rightarrow s(\{(x, y)\})$. Thus, assertion (I) is proved.

It is obvious that assertion (II) follows from (I). The assertion $t' \stackrel{*}{\Rightarrow} f_*(s)$ (Q) is equivalent with the existence of an integer $p \geqq 0$ and strings $t' = t_0', t_1', \ldots, t_p' = f_*(s)$ in U^* such that $t_{i-1}' \Rightarrow t_i'$ (Q) for $i = 1, 2, \ldots, p$. According to (II) there is possible to prove by induction that the existence of such a t_i' is equivalent with the existence of elemensts $t_i \in V^*(i = 0, 1, \ldots, p-1)$ such that $t_i \in f_*^{-1}(t_i')$, $t_{i-1} \Rightarrow t_i(R)$ for $i = 1, 2, \ldots, p-1$, $t_{p-1} \Rightarrow s(R)$. This is equivalent with the existence of $t \in f_*^{-1}(t')$ such that $t \stackrel{*}{\Rightarrow} s(R)$. So the assertion (III) is proved.

Now, the following conditions are equivalent for $x \in V^*$: (1) $f_*(x) \in U_T^*$ and there exists $s' \in P$ such that $s' \stackrel{*}{\Rightarrow} f_*(x)$ (Q); (2) $x \in V_T^*$ and there exists $s \in f_*^{-1}(s') \subseteq S$ such that $s \stackrel{*}{\Rightarrow} x(R)$. From it follows the proof of the assertion (IV).

Assertion (V) follows obviously from (IV) and from 3.1.

3.7. Definition. Let (V, L) be a language. The element $x \in V^*$ is called *necessary* in the language (V, L) (which we symbolize by $x\, \nu\, (V, L)$) if there exist elements $u, v \in V^*$ such that $uxv \in L$.

3.8. Definition. Let (V, L) be a language. For elements $x, y \in V^*$ we put $x > y(V, L)$ (x can be substituted by y in the language (V, L)), if $u, v \in V^*$, $uxv \in L$ imply $uyv \in L$.

3.9. Lemma. *Let (V, L), (U, M) be languages, f a strong homomorphism of the language (V, L) onto (U, M). Then following assertions hold:*

(A) *For every $x \in V^*$ the condition $x\, \nu\, (V, L)$ is equivalent to $f_*(x)\, \nu\, (U, M)$.*

(B) *For every $x, y \in V^*$ the condition $x > y(V, L)$ is equivalent to $f_*(x) > f_*(y)$ (U, M).*

This lemma can be found in [3] as the lemma 1.

3.10. Definition. Let (V, L) be a language, $x, y \in V^*$. We say that x is a *semiconfiguration* in the language (V, L) with the resultant y if following conditions are satisfied:

(1) $y\, \nu\, (V, L)$

(2) $y > x(V, L)$, $y \neq x$, $|y| \leqq |x|$.

We denote by $E(V, L)$ the set of all pairs (y, x), where x is a semiconfiguration of the language (V, L) with the resultant y.

3.11. Definition. We put $R(V, L) = \{(y, x);\, y\, \nu\, (V, L),\; y > x(V, L), |y| \leqq |x|\}$.

3.12. Remark. *From the definitions of the sets $E(V, L)$ and $R(V, L)$ it follows that $E(V, L) \subseteq R(V, L)$ and also, for every $s, t \in V^*$, the condition $s \Rightarrow t(E(V, L))$ implies $s \Rightarrow t(R(V, L))$ and for every $s, t \in V^*$, $s \neq t$, the condition $s \Rightarrow t(R(V, L))$ implies $s \Rightarrow t(E(V, L))$.*

3.13. Lemma. *Let be $s, t \in V^*$. Then $s \stackrel{*}{\Rightarrow} t(E(V, L))$ iff $s \stackrel{*}{\Rightarrow} t(R(V, L))$.*

Proof. I. Let us have $s, t \in V^*$, $s \stackrel{*}{\Rightarrow} t(E(V, L))$. Then, by **3.12**, $s \stackrel{*}{\Rightarrow} t(R(V, L))$ holds.

II. If $s, t \in V^*$, $s \stackrel{*}{\Rightarrow} t(R(V, L))$, then there exist elements $t_0, t_1, \ldots, t_p \in V^*$ such that $s = t_0$, $t_p = t$ and $t_{i-1} \Rightarrow t_i(R(V, L))$ for $i = 1, 2, \ldots, p$. If $t_{i-1} \neq t_i$ is valid for $i = 1, 2, \ldots, p$, then a by **3.12**, the condition $t_{i-1} \Rightarrow t_i(E(V, L))$ is satisfied for $i = 1, 2, \ldots, p$ and so $s \stackrel{*}{\Rightarrow} t(E(V, L))$.

III. Suppose $s, t \in V^*$. Let $(t_i)_{i=0}^p$ be an s-derivation of t of length p in $R(V, L)$ and k an integer $k \in \{1, 2, \ldots, p\}$ such that $t_{k-1} = t_k$. If we cancel the string t_k in $(t_i)_{i=0}^p$ we obtain an s-derivation of t of the length $p-1$ in $R(V, L)$. Repeating this procedure we obtain an s-derivation of t of length $l \leq p$ in $R(V, L)$ such that $t_{i-1} \neq t_i$ for $i = 1, 2, \ldots, l$. By II, this s-derivation is an s-derivation of t in $E(V, L)$, and so $s \overset{\cdot}{\Rightarrow} t(E(V, L))$.

3.14. Definition. Let (V, L) be a language. For $x \in L$ we put $x \in B_E(V, L)$ iff, for every $t \in L$, the condition $t \overset{\cdot}{\Rightarrow} x(E(V, L))$ implies $|t| = |x|$.

For $x \in L$ we put $x \in B_R(V, L)$ iff for every $t \in L$, the condition $t \overset{\cdot}{\Rightarrow} x(R(V, L))$ implies $|t| = |x|$.

3.15. Theorem. Let (V, L) be a language. Then $B_E(V, L) = B_R(V, L)$.

Proof. If $x \in L$, $x \in B_E(V, L)$, $t \in L$, $t \overset{\cdot}{\Rightarrow} x(R(V, L))$, then, by 3.13, it holds true that $t \overset{\cdot}{\Rightarrow} x(E(V, L))$ and according to the definition of $B_E(V, L)$, we have $|t| = |x|$. Thus, $x \in B_R(V, L)$.

If $x \in L$, $x \in B_R(V, L)$, $t \in L$, $t \overset{\cdot}{\Rightarrow} x(E(V, L))$, then, by 3.13, it holds true that $t \overset{\cdot}{\Rightarrow} x(R(V, L))$ and so $|t| = |x|$. It follows that $x \in B_E(V, L)$.

3.16. Lemma. *Let (V, L) be a language. Then the following assertions hold:*
(A) *For every $x \in L$ there exists $s \in B_E(V, L)$ such that $s \overset{\cdot}{\Rightarrow} x(E(V, L))$.*
(B) *For every $x \in L$ there exists $s \in B_R(V, L)$ such that $s \overset{\cdot}{\Rightarrow} x(R(V, L))$.*

Proof. There exists at least one element $s \in L$ such that $s \overset{\cdot}{\Rightarrow} x(E(V, L))$. One can consider for example the trivial s-derivation in $E(V, L)$. If the element of minimum length from those mentioned above is chosen, there is evident that this element belongs to $B_E(V, L)$.

That is the proof of assertion (A).

Assertion (B) follows from 3.15, 3.13 and (A).

3.17. Definition. Two definitions are condensed in 3.17; the first is obtained when reading the conditions denoted by 1° the second is obtained when reading the conditions denoted by 2°. 3.23 must be interpreted similarly.
Let (V, L) be a language. If $s, t \in V^*$ are the strings such that 1° $s \Rightarrow t(E(V, L))$, 2° $s \Rightarrow t(R(V, L))$, we put 1° $|(s, t)|_E = \min\{|q|;\ (p, q) \in E(V, L),\ s \Rightarrow t(\{(p, q)\})\}$, 2° $|(s, t)|_R = \min\{|q|;\ (p, q) \in R(V, L),\ s \Rightarrow t(\{(p, q)\})\}$.

If $s, t \in V^*$ are strings and $(t_i)_{i=0}^p$ is an s-derivation of t in 1° $E(V, L)$, 2° $R(V, L)$, $p > 0$, then we put 1° $\|(t_i)_{i=0}^p\|_E = \max\{|(t_{i-1}, t_i)|_E;\ i = 1, 2, \ldots, p\}$, 2° $\|(t_i)_{i=0}^p\|_R = \max\{|(t_{i-1}, t_i)|_R;\ i = 1, 2, \ldots, p\}$. The number 1° $\|(t_i)_{i=0}^p\|_E$, 2° $\|(t_i)_{i=0}^p\|_R$ is called the norm of the s-derivation $(t_i)_{i=0}^p$ of t in 1° $E(V, L)$, 2° $R(V, L)$. The norm of a trivial s-derivation in 1° $E(V, L)$, 2° $R(V, L)$ is defined to be zero.

If $s, t \in V^*$ are the strings such that 1° $s \overset{\cdot}{\Rightarrow} t(E(V, L))$, 2° $s \overset{\cdot}{\Rightarrow} t(R(V, L))$, we define the norm 1° $\|(s, t)\|_E$, 2° $\|(s, t)\|_R$ of the ordered pair (s, t) to be the minimum of norms of all s-derivations of t in 1° $E(V, L)$, 2° $R(V, L)$. If $t \in L$, we put 1° $\|t\|_E = \min\{\|(s, t)\|_E;\ s \in B_E(V, L),\ s \overset{\cdot}{\Rightarrow} t(E(V, L))\}$, 2° $\|t\|_R = \min\{\|(s, t)\|_R;\ s \in B_R(V, L),\ s \overset{\cdot}{\Rightarrow} t(R(V, L))\}$. The number 1° $\|t\|_E$, 2° $\|t\|_R$ is called the norm of t in 1° $E(V, L)$, 2° $R(V, L)$.

3.18. Lemma. *Suppose $s, t \in V^*$. If $|(s, t)|_E$, exists, then $|(s, t)|_R$ exists and it holds that $|(s, t)|_E = |(s, t)|_R$. Further, for $t \in V^*$, $t \Rightarrow t(R(V, L))$ it holds evidently that $|(t, t)|_R = 0$.*

Proof. The proof follows from 3.12, and 3.17.

3.19. Lemma. *Suppose $s, t \in V^*$, let $(t_i)_{i=0}^p$ be an s-derivation of t in $E(V, L)$. Then $(t_i)_{i=0}^p$ is also an s-derivation of t in $R(V, L)$ and the equation $\|(t_i)_{i=0}^p\|_E = \|(t_i)_{i=0}^p\|_R$ holds. On the contrary, if $s, t \in V^*$, and if $(t_i)_{i=0}^p$ is an s-derivation of t in $R(V, L)$ such that $t_{i-1} \neq t_i$ for $i = 1, 2, \ldots, p$, then $(t_i)_{i=0}^p$ is an s-derivation in $E(V, L)$ and the equation $\|(t_i)_{i=0}^p\|_E = \|(t_i)_{i=0}^p\|_R$ holds.*

Proof. The proof follows from 3.18 and 3.17.

3.20. Remark. *Suppose $s, t \in V^*$. If $(t_i)_{i=0}^p$ is an s-derivation of t in $R(V, L)$, then it is obvious from 3.18 and 3.17 that the elements t_i of the s-derivation of t, such that $t_{i-1} = t_i$ have no influence on the value of $\| (t_i)_{i=0}^p \|_R$.*

3.21. Lemma. *If $s, t \in V^*$, $s \overset{\cdot}{\Rightarrow} t(E(V, L))$, then $\| (s, t) \|_E = \| (s, t) \|_R$.*

Proof. The proof follows from 3.17, 3.19 and 3.20.

3.22. Theorem. *If $t \in L$, then $\| t \|_E = \| t \|_R$.*

Proof. Assume $t \in L$. Then, by 3.15 and 3.13, for every $s \in B_E(V, L)$, the condition $s \overset{\cdot}{\Rightarrow} t(E(V, L))$ implies $s \in B_R(V, L)$, $s \overset{\cdot}{\Rightarrow} t(R(V, L))$. Further, by 3.21 it holds that $\| (s, t) \|_E = \| (s, t) \|_R$, thus, according to the definition of $\| t \|_E$ and $\| t \|_R$ it holds that $\| t \|_E \geqq \| t \|_R$. Similarly, it is possible to prove that $\| t \|_E \leqq \| t \|_R$; thus $\| t \|_E = \| t \|_R$.

3.23. Lemma. *Let (V, L) be a language. Then, for every $t \in L$, there exists a string 1° $s \in B_E(V, L)$, 2° $s \in B_R(V, L)$ and an s-derivation of t in 1° $E(V, L)$, 2° $R(V, L)$ such that the norm of this s-derivation is equal to 1° $\| t \|_E$, 2° $\| t \|_R$.*

Proof. According to 3.17, there exists an element $s \in B_E(V, L)$ such that $\| (s, t) \|_E = \| t \|_E$. It means that there exists such an s-derivation of t in $E(V, L)$ that its norm is equal to $\| t \|_E$.

Similar proof takes place in the case of $R(V, L)$.

3.24. Definition. Let (V, L) be a language. Then we put $X_E(V, L) = \{(y, x);$ $(y, x) \in E(V, L), | x | > \| t \|_E$ for every $t \in L\}$, $X_R(V, L) = \{(y, x); (y, x) \in R(V, L),$ $| x | > \| t \|_R$ for every $t \in L\}$, $Z_E(V, L) = E(V, L) - X_E(V, L)$, $Z_R(V, L) = R(V, L) - X_R(V, L)$:

3.25. Lemma. *It holds that $X_E \subseteq X_R$, $Z_E \subseteq Z_R$.*

Proof. The proof follows from 3.11 and 3.24.

3.26. Lemma. *Let be $s, t \in V^*$. Then $s \overset{\cdot}{\Rightarrow} t(Z_E(V, L))$ iff $s \overset{\cdot}{\Rightarrow} t(Z_R(V, L))$.*

Proof. I. Assume $s, t \in V^*$, $s \overset{\cdot}{\Rightarrow} t(Z_E(V, L))$. Then it follows, by 3. 25, that $s \overset{\cdot}{\Rightarrow} t(Z_R(V, L))$.

II. Suppose $s, t \in V^*$, $s \overset{\cdot}{\Rightarrow} t(Z_R(V, L))$. Then there exists an s-derivation $(t_i)_{i=0}^p$ of t in $Z_R(V, L)$. If $t_{i-1} \neq t_i (i = 1, 2, ..., p)$, then this is an s-derivation of t in $Z_E(V, L)$. If $t_{i-1} = t_i$ for some $i \in \{1, 2, ..., p\}$, then it is possible to omit these t_i and to obtain again an s-derivation of t in $Z_R(V, L)$ which is at the same time the s-derivation of t in $Z_E(V, L)$.

3.27. Theorem. *Let (V, L) be a language. Then the following assertions hold:*

(A) *For every $t \in L$ there exists at least one element $s \in B_E(V, L)$ such that $s \overset{\cdot}{\Rightarrow} t$ $(Z_E(V, L))$.*

(B) *For every $t \in L$ there exists at least one element $s \in B_R(V, L)$ such that $s \overset{\cdot}{\Rightarrow} t$ $(Z_R(V, L))$.*

Proof. I. According to 3.23 for every $t \in L$ there exists a string $s \in B_E(V, L)$ and an s-derivation $(t_i)_{i=0}^p$ of t in $E(V, L)$ such that $\| (t_i)_{i=0}^p \|_E = \| t \|_E$. It follows that $| (t_{i-1}, t_i) |_E \leqq \| t \|_E$ for $i = 1, 2, ..., p$ by 3.17. Further, for every $i = 1, 2, ..., p$ there exists an element $(p_i, q_i) \in E(V, L)$ such that $t_{i-1} \Rightarrow t_i$ $(\{(p_i, q_i)\})$ and $| q_i | = | (t_{i-1}, t_i) |_E \leqq \| t \|_E$. Thus, $(p_i, q_i) \in Z_E(V, L)$ for $i = 1, 2, ..., p$ and consequently $s \overset{\cdot}{\Rightarrow} t(Z_E(V, L))$.

That is the proof of assertion (A).

The proof of assertion (B) follows from (A), 3.15 and 3.26.

3.28. Definition. Let (V, L) be a language. We put $K_E(V, L) = \langle V, B_E(V, L), Z_E(V, L)\rangle$, $K_R(V, L) = \langle V, B_R(V, L), Z_R(V, L)\rangle$. $K_E(V, L)$, respectively $K_R(V, L)$ is a generalized special grammar called further a *generalized semiconfigurational* respectively *R-semiconfigurational* grammar.

3.29. Theorem. *Let (V, L) be a language. Then $\mathscr{L}(K_E(V, L)) = \mathscr{L}(K_R(V, L)) = L$.*

Proof. I. According to 3.27 we have $L \subseteq \mathscr{L}(K_E(V, L))$.

II. Let $V(n)$ be the following assertion: if $t \in \mathscr{L}(K_E(V, L))$ and if there exists an element $s \in B_E(V, L)$ and an s-derivation of t of length n in $Z_E(V, L)$, then $t \in L$.

If $t \in \mathscr{L}(K_E(V, L))$ and if there exists an element $s \in B_E(V, L)$ and a trivial s-derivation of t in $Z_E(V, L)$, then $t = s \in B_E(V, L) \subseteq L$. Thus, $V(0)$ is valid.

Let be now $m \geqq 0$ and assume that $V(m)$ holds. Suppose further that $t \in \mathscr{L}(K_E(V, L))$, $s \in B_E(V, L)$ and that $(t_i)_{i=0}^{m+1}$ is an s-derivation of length $m + 1$ in $Z_E(V, L)$. Then, according to $V(m)$, it holds that $t_m \in L$. Further, $t_m \Rightarrow t\,(Z_E(V, L))$. It means that there exist elements $x, y, u, v \in V^*$ such that $t_m = uxv$, $t = uyv$, $(x, y) \in Z_E(V, L) \subseteq E(V, L)$. It follows that $x > y(V, L)$ which implies $t \in L$. Thus $V(m + 1)$ holds. Hence $V(m)$ holds for $m = 0, 1, \ldots$. It means that $\mathscr{L}(K_E(V, L)) \subseteq L$.

The assertion $\mathscr{L}(K_E(V, L)) = L$ has been proved.

III. Suppose $t \in \mathscr{L}(K_E(V, L))$ and $s \in B_E(V, L)$, $s \stackrel{*}{\Rightarrow} t(Z_E(V, L))$. Then, by 3.15, it holds that $s \in B_R(V, L)$ and, by 3.26, it holds that $s \stackrel{*}{\Rightarrow} t(Z_R(V, L))$. Thus, $t \in \mathscr{L}(K_R(V, L))$ and we have $\mathscr{L}(K_E(V, L)) \subseteq \mathscr{L}(K_R(V, L))$.

Assume $t \in \mathscr{L}(K_R(V, L))$ and $s \in B_R(V, L)$, $s \stackrel{*}{\Rightarrow} t(Z_R(V, L))$. Then, by 3.15, it holds that $s \in B_E(V, L)$ and by 3.26, it holds that $s \stackrel{*}{\Rightarrow} t(Z_E(V, L))$. It means that $t \in \mathscr{L}(K_E(V, L))$ and we have $\mathscr{L}(K_R(V, L)) \subseteq \mathscr{L}(K_E(V, L))$.

Thus, the assertion $\mathscr{L}(K_E(V, L)) = \mathscr{L}(K_R(V, L))$ has been proved.

3.30 Lemma. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M). Let us have $t' \in U^*$, $s \in V^*$. Then the following assertions hold:*

(A) *If $t' \Rightarrow f_*(s)\ (R(U, M))$, then there exists $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R(V, L))$ and $|\,(t, s)\,|_R \leqq |\,(t', f_*(s))\,|_R$.*

(A') *If there exists $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R(V, L))$, then $t' \Rightarrow f_*(s)\ (R(U, M))$ and $|\,(t', f_*(s))\,|_R \leqq |\,(t, s)\,|_R$.*

(B) *If $(t'_i)_{i=0}^{p}$ is a t'-derivation of the string $f_*(s)$ in $R(U, M)$, then there exist $t_i \in f_*^{-1}(t'_i)$ for $i = 0, 1, \ldots, p$, $t_p = s$ such that $(t_i)_{i=0}^{p}$ is t_0-derivation of the string s in $R(V, L)$ such that $\|\,(t_i)_{i=0}^{p}\,\|_R \leqq \|\,(t'_i)_{i=0}^{p}\,\|_R$.*

(B') *If $t \in f_*^{-1}(t')$ and if $(t_i)_{i=0}^{p}$ is a t-derivation of the string s in $R(V, L)$, then $(f_*(t_i))_{i=0}^{p}$ is a t'-derivation of the string $f_*(s)$ in $R(U, M)$ such that $\|\,(f_*(t_i))_{i=0}^{p}\,\|_R \leqq \|\,(t_i)_{i=0}^{p}\,\|_R$.*

(C) *If $t' \Rightarrow f_*(s)\ (R(U, M))$, then there exists $t \in f_*^{-1}(t')$ such that $t \stackrel{*}{\Rightarrow} s(R(V, L))$ and $\|\,(t, s)\,\|_R \leqq \|\,(t', f_*(s))\,\|_R$.*

(C') *If $t \in f_*^{-1}(t')$ and $t \stackrel{*}{\Rightarrow} s(R(V, L))$, then $t' \stackrel{*}{\Rightarrow} f_*(s)\ (R(U, M))$ and $\|\,(t', f_*(s))\,\|_R \leqq \|\,(t, s)\,\|_R$.*

Proof. O. If f is a strong homomorphism of the language (V, L) onto (U, M), then it is also a strong homomorphism of the generalized grammar $\langle V, V, L, R(V, L)\rangle$ onto $\langle U, U, M, R(U, M)\rangle$. It follows from 3.30.

1. Assume $(x', y') \in R(U, M)$, $t' \Rightarrow f_*(s)\ (\{(x', y')\})$ and $|\,(t', f_*(s))\,|_R = |\,y'\,|$. According to 3.6 there exist $t \in f_*^{-1}(t')$, $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ that $t \Rightarrow s(\{(x, y)\})$. It follows $|\,(t, s)\,|_R \leqq |\,y\,| = |\,y'\,| = |\,(t', f_*(s))\,|_R$ and (A) holds.

1'. Assume $(x, y) \in R(V, L)$, $t \Rightarrow s(\{(x, y)\})$ and $|\,(t, s)\,|_R = |\,y\,|$. According to 3.6 there is $t' \Rightarrow f_*(s)\ (\{(f_*(x), f_*(y))\})$ and, further, $|\,(t', f_*(s))\,|_R \leqq |\,f_*(y)\,| = |\,y\,| = |\,(t, s)\,|_R$ and (A') holds.

2. We put $t_p = s$. Then $\|\,(t_i)_{i=p}^{p}\,\|_R = 0 = \|\,(t'_i)_{i=p}^{p}\,\|_R$. Suppose $0 < k \leqq p$ and assume that we have such $t_i \in f_*^{-1}(t'_i)$ for $i = k, k + 1, \ldots, p$ that $(t_i)_{i=k}^{p}$ is a t_k-derivation of the string s in $R(V, L)$ with the property $\|\,(t_i)_{i=k}^{p}\,\|_R \leqq \|\,(t'_i)_{i=k}^{p}\,\|_R$. Then $t'_{k-1} \Rightarrow f_*(t_k)\ (R(U, M))$. According to (A) there exists $t_{k-1} \in f_*^{-1}(t'_{k-1})$ such that

$t_{k-1} \Rightarrow t_k(R(V, L))$ and $|(t_{k-1}, t_k)|_R \leqq |(t'_{k-1}, t'_k)|_R$. It follows $\|(t_i)_{i=k-1}^p\|_R = \max\{|(t_{k-1}, t_k)|_R, \|(t_i)_{i=k}^p\|_R\} \leqq \max\{|(t'_{k-1}, t'_k)|_R, \|(t'_i)_{i=k}^p\|_R\} = \|(t'_i)_{i=k-1}^p\|_R$. Assertion (B) could be proved by induction.

2'. There is $\|(f_*(t_i)_{i=0}^0\|_R = 0 = \|(t_i)_{i=0}^0\|_R$. Suppose $0 \leqq k < p$ and assume $\|(f_*(t_i))_{i=0}^k\|_R \leqq \|(t_i)_{i=0}^k\|_R$. There exists the string $t_k \in f_*^{-1}(t'_k)$ such that $t_k \Rightarrow$ $\Rightarrow t_{k+1}(R(V, L))$. Then $f_*(t_k) \Rightarrow f_*(t_{k+1})$ $(R(U, M))$ and $|(f_*(t_k), f_*(t_{k+1}))|_R \leqq$ $\leqq |(t_k, t_{k+1})|_R$ according to (A'). It follows $\|(f_*(t_i))_{i=0}^{k+1}\|_R = \max\{\|(f_*(t_i)_{i=0}^k\|_R,$ $|(f_*(t_k), f_*(t_{k+1}))|_R \leqq \max\{\|(t_i)_{i=0}^k\|_R, |(t_k, t_{k+1})|_R\} = \|(t_i)_{i=0}^{k+1}\|_R$. Assertion (B') could be proved by induction.

3. Let be $(t'_i)_{i=0}^p$ a t'-derivation of the string $f_*(s)$ in $R(U, M)$ such that $\|(t', f_*(s))\|_R = \|(t'_i)_{i=0}^p\|_R$. According to (B) there exist $t_i \in f_*^{-1}(t'_i)$ for $i = 0, 1, \ldots, p$, $t_p = s$ such that $(t_i)_{i=0}^p$ is a t_0-derivation of the string s in $R(V, L)$ with the property $\|(t_i)_{i=0}^p\|_R \leqq \|(t'_i)_{i=0}^p\|_R$. For $t = t_0$, it follows that $\|(t, s)\|_R \leqq \|(t_i)_{i=0}^p\|_R$ and this is the proof of the assertion (C).

3'. Let $(t_i)_{i=0}^p$ be a t-derivation of the string s in $R(V, L)$ such that $\|(t, s)\|_R =$ $= \|(t_i)_{i=0}^p\|_R$. According to (B'), $(f_*(t_i))_{i=0}^p$ is a t'-derivation of the string $f_*(s)$ in $R(U, M)$ such that $\|(f_*(t_i))_{i=0}^p\|_R \leqq \|(t_i)_{i=0}^p\|_R$. It follows that $\|(t', f_*(s))\|_R \leqq$ $\leqq \|(f_*(t_i))_{i=0}^p\|_R \leqq \|(t_i)_{i=0}^p\|_R = \|(t, s)\|_R$ and this is the proof of the assertion (C').

3.32 Lemma. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M). Then $B_R(V, L) = f_*^{-1}(B_R(U, M))$ and $\|z\|_{R(V,L)} = \|f_*(z)\|_{R(U,M)}$ for every $z \in L$.*

Proof. 1. It holds that $B_R(V, L) \subseteq f_*^{-1}(B_R(U, M))$. Indeed, suppose $z \in B_R(V, L)$. Then we have $z \in L$ and consequently $f_*(z) \in M$. Suppose $s' \in M$ and $s' \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$. According to 3.31 (C) there exists $s \in f_*^{-1}(s')$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$. We have $s \in L$ and it follows $|s| = |z|$. Further, it implies $|s'| = |f_*(s)| =$ $= |s| = |z| = |f_*(z)|$. It follows $f_*(z) \in B_R(U, M)$ and, thus, $z \in f_*^{-1}(B_R(U, M))$. This proves immediatly the assertion.

2. It holds that $f_*^{-1}(B_R(U, M)) \subseteq B_R(V, L)$. Indeed, suppose $z \in f_*^{-1}(B_R(U, M))$. Then we have $z \in f_*^{-1}(M) = L$. Suppose $s \in L$, $s \overset{*}{\Rightarrow} z(R(V, L))$. According to (C'), it holds that $f_*(s) \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $f_*(s) \in M, f_*(z) \in B_R(U, M)$. Thus, $|f_*(s)| =$ $= |f_*(z)|$ and it follows $|s| = |f_*(s)| = |f_*(z)| = |z|$. Therefore $z \in B_R(V, L)$.

3. Assertions 1 and 2 imply that $B_R(V, L) = f_*^{-1}(B_R(U, M))$.

4. For every $z \in L$, the condition $\|z\|_{R(V,L)} \leqq \|f_*(z)\|_{R(U,M)}$ holds. Indeed if $z \in L$ then $f_*(z) \in M$ and there exists $s' \in B_R(U, M)$ such that $s' \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $\|(s', f_*(z))\|_{R(U,M)} = \|f_*(z)\|_{R(U,M)}$. According to **3.31** (C) there exists $s \in$ $\in f_*^{-1}(s')$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$ and $\|(s, z)\|_{R(V,L)} \leqq \|(s', f_*(z))\|_{R(U,M)} =$ $= \|f_*(z)\|_{R(U,M)}$. Now, we have $s \in f_*^{-1}(B_R(U, M)) = B_R(V, L)$ according to 3 and therefore $\|z\|_{R(V,L)} \leqq \|(s, z)\|_{R(V,L)} \leqq \|f_*(z)\|_{R(U,M)}$.

5. For every $z \in L$ the condition $\|f_*(z)\|_{R(U,M)} \leqq \|z\|_{R(V,L)}$ holds. Indeed there exists $s \in B_R(V, L)$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$ and $\|(s, z)\|_{R(V,L)} = \|z\|_{R(V,L)}$. According to 3.31 (C'), we have $f_*(s) \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $\|(f_*(s), f_*(z))\|_{R(U,M)} \leqq$ $\leqq \|(s, z)\|_{R(V,L)} = \|z\|_{R(V,L)}$. Now, we have $f_*(s) \in f_*(B_R(V, L)) =$ $= f_*(f_*^{-1}(B_R(U, M))) = B_R(U, M)$ according to 3 and therefore $\|f_*(z)\|_{R(U,M)} \leqq$ $\leqq \|(f_*(s), f_*(z))\|_{R(U,M)} \leqq \|z\|_{R(V,L)}$.

6. It follows from 4 and 5 that $\|f_*(z)\|_{R(U,M)} = \|z\|_{R(V,L)}$ for every $z \in L$.

3.33. Lemma. *Let (V, L), (U, M) be languages, f a strong homomorphism (V, L) onto (U, M). Then, for every $x, y \in V^*$, the following assertions hold:*

(A) $(y, x) \in X_R(V, L)$ *iff* $(f_*(y), f_*(x)) \in X_R(U, M)$.
(B) $(y, x) \in Z_R(V, L)$ *iff* $(f_*(y), f_*(x)) \in Z_R(U, M)$.

Proof. Suppose $x, y \in V^*$, $(y, x) \in X_R(V, L)$. Then, by 3.24, we have $(y, x) \in \in R(V, L)$ and $t \in L$ implies $|x| > ||t||_{R(V,L)}$. By 3.30, the condition $(f_*(y), f_*(x)) \in \in R(U, M)$ holds. Now suppose $z \in M$. It follows from the definition of a strong homomorphism that there exists a string $z' \in L$ such that $f_*(z') = z$. For this string, the condition $|x| > ||z'||_{R(V,L)}$ holds. By 3.32, we have $||z'||_{R(V,L)} = ||f_*(z')||_{R(U,M)} = ||z||_{R(U,M)}$. Thus, it holds that $|f_*(x)| > ||z||_{R(U,M)}$ for every $z \in M$. It means that $(f_*(y), f_*(x)) \in X_R(U, M)$. Now, suppose $x, y \in V^*$, $(f_*(y), f_*(x)) \in X_R(U, M)$. Then $(f_*(y), f_*(x)) \in R(U, M)$ and, by 3.30, it follows $(y, x) \in R(V, L)$. If $t \in L$ then $f_*(t) \in M$ and $|f_*(x)| > ||f_*(t)||_{R(U,M)}$. By 3.32, $||f_*(t)||_{R(U,M)} = ||t||_{R(V,L)}$ and therefore $|x| > ||t||_{R(V,L)}$ for every $t \in L$. It means that $(y, x) \in X_R(V, L)$.

This is the proof of the assertion (A).

The assertion (B) follows from (A) and 3.30.

3.34. Theorem. *Let (V, L), (U, M) be languages, f a strong homomorphism (V, L) onto (U, M). Then f is a strong homomorphism $K_R(V, L)$ onto $K_R(U, M)$.*

Proof. The proof follows from 3.32 and 3.33.

3.35. Theorem. *Let (V, L), (U, M) be languages.*

(A) *If f is a strong homomorphism of $K_E(V, L)$ onto $K_E(U, M)$, then f is also the strong homomorphism of the language (V, L) onto (U, M).*
(B) *If f is a strong homomorphism of $K_R(V, L)$ onto $K_R(U, M)$, then f is also the strong homomorphism of the language (V, L) onto (U, M).*

Proof. The proof follows from 3.6 and 3.29.

3.36. Theorem. *Let (V, L), (U, M) be languages, f a surjection V onto U. Then f is a strong homomorphism of the language (V, L) onto (U, M) iff f is the strong homomorphism of $K_R(V, L)$ onto $K_R(U, M)$.*

Proof. The proof follows immediately by 3.34 and 3.35.

4. FINITELY SEMIGENERATED LANGUAGES

4.1. Definition. A language (V, L) is called finitely semigenerated if the sets V, $B_E(V, L)$, $Z_E(V, L)$ are finite.

A language (V, L) is called *finitely R-semigenerated* if the sets V, $B_R(V, L)$, $Z_R(V, L)$ are finite.

4.2. Theorem. *Let (V, L) be a language. Then*
(A) *(V, L) is finitely semigenerated iff the following two conditions are satisfied:*
  (a) *The sets V, $B_E(V, L)$ are finite.*
  (b) *There exists a number N such that $||z||_E \leqq N$ for every $z \in L$.*
(B) *(V, L) is finitely R-semigenerated iff the following two conditions are satisfied:*
  (a) *The sets V, $B_R(V, L)$ are finite.*
  (b) *There exists a number N such that $||z||_R \leqq N$ for every $z \in L$.*

Proof. 1. If the language (V, L) is finitely semigenerated, the sets V, $B_E(V, L)$, $Z_E(V, L)$ are finite. We put $N = max\{|q|; (p, q) \in Z_E(V, L)\}$. Let us have an arbitrary $z \in L$. By 3.27, there exists $s \in B_E(V, L)$ such that $s \overset{*}{\Rightarrow} z(Z_E(V, L))$. Let $(s_i)_{i=0}^n$ be an s-derivation of z in $Z_E(V, L)$.

Then $|(s_{i-1}, s_i)|_E \leqq N$ for $i = 1, 2, \ldots, n$. If follows $\|(s_i)_{i=0}^n\|_E \leqq N$ and that implies $\|(s, z)\|_E \leqq N$ and finally $\|z\|_E \leqq N$.

2. Let V, $B_E(V, L)$ be finite and suppose the existence of a number N such that $\|z\|_E \leqq N$ for every $z \in L$. Let us have an arbitrary $(p, q) \in Z_E(V, L)$. Then there exists $z \in L$ such that $|p| \leqq |q| \leqq \|z\|_E \leqq N$. It follows that the set $Z_E(V, L)$ is finite.

That is the proof of the assertion (A).

The assertion (B) could be proved in a similar way.

4.3. Theorem. *Let (V, L) be a language. Then (V, L) is finitely semigenerated iff it is finitely R-semigenerated.*

Proof. The proof follows from 3.15, 3.22 and 4.2.

4.4. Theorem. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M).*

(A) *If (V, L) is finitely R-semigenerated, then (U, M) is also finitely R-semigenerated.*

(B) *If V is a finite set and (U, M) a finitely R-semigenerated language, then (V, L) is also finitely R-semigenerated.*

Proof. The proof follows from 3.32 and 4.2.

4.5. Corollary. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M).*

(A) *If (V, L) is finitely semigenerated, then (U, M) is also finitely semigenerated.*

(B) *If V is a finite set and (U, M) a finitely semigenerated language, then (V, L) is also finitely semigenerated.*

It follows by 4.3 and 4.4.

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NOTE ON THE THEORY OF DISPERSIONS OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$

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1.1. Consider a differential equation

$$y'' = q(t)y,\ q \in C^\circ[a, b),\ q(t) < 0,\ t \in [a, b),\ b \leqq \infty, \tag{q}$$

where $C^n[a, b)$ (n being a nonnegative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$. In all the work we suppose that (q) is an oscillatoric ($t \to b_-$) differential equation, i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$.

Let y_1 (y_2) be a non-trivial solution of (q) such that $y_1(t) = 0$ ($y_2'(t) = 0$), $t \in [a, b)$. If $\varphi(t)$ ($\psi(t)$) is the first zero of y_1 (y_2') lying on the right of t, then $\varphi(\psi)$ is called the basic central dispersion of the 1-st (2-nd) kind (briefly, dispersion of the 1-st (2-nd) kind).

The properties of dispersions can be found in [3]. If δ is the dispersion of the k-th kind, $k = 1, 2$, then

$$\begin{array}{lll} & 1.\ \delta \in C^3[a, b) & \text{if } k = 1 \\ & \quad\ \delta \in C^1(a, b) & \text{if } k = 2 \\ (1) & 2.\ \delta'(t) > 0 & \text{on } [a, b) \\ & 3.\ \delta(t) > t & \text{on } [a, \mathrm{b}) \\ & 4.\ \lim\limits_{t \to b_-} \delta(t) = b & \end{array}$$

hold (see [3] § 13). Let y be an arbitrary non-trivial solution of (q). Then (see [3] § 13.3)

$$\begin{aligned} \psi'(t) &= \frac{q(t)}{q(\psi(t))} \frac{y'^2(\psi(t))}{y'^2(t)} \qquad \text{if } y'(t) \neq 0, \\ &= \frac{q(t)}{q(\psi(t))} \frac{y^2(t)}{y^2(\psi(t))} \qquad \text{if } y'(t) = 0. \end{aligned} \tag{2}$$

The dispersion φ of the first kind of (q) fulfils the following non-linear differential equation

$$-\frac{1}{2} \frac{\varphi'''}{\varphi'} + \frac{3}{4} \frac{\varphi''^2}{\varphi'^2} + q(\varphi)\, \varphi'^2 = q(t), \qquad t \in (a, b). \tag{3}$$

1.2. In our later considerations we shall need some results being derived in [1], [4].

(i) Let $\varphi(\psi)$ be the dispersion of the 1-st (2-nd) kind of (q), $\mathrm{q} \in C^\circ[a, b]$, $q(t) < 0$

on $[a, b)$, $b \leqq \infty$, (q) oscillatoric on $[a, b)$. Let $t_0 \in (a, b)$. Then

1) $\varphi(t_0) < \psi(t_0)$ iff $\varphi''(t_0) > 0$

2) $\varphi(t_0) = \psi(t_0)$ iff $\varphi''(t_0) = 0$

3) $\varphi(t_0) > \psi(t_0)$ iff $\varphi''(t_0) < 0$

4) $\varphi(t_0) = \psi(t_0)$ iff $\varphi'(t_0)\ \psi'(t_0) = \dfrac{q(t_0)}{q(\psi(t_0))}$

5) $\varphi(t_0) \neq \psi(t_0)$ iff $\varphi'(t_0)\ \psi'(t_0) < \dfrac{q(t_0)}{q(\psi(t_0))}$

(ii) Let (q), $q \in C^\circ\ [a, b)$, $b \leqq \infty$ be oscillatoric on $[a, b)$ and let φ be its dispersion of the 1-st kind.

a) If $\varphi'(t) \leqq 1$ on $[a, b)$, then every solution of (q) is bounded on $[a, b)$.

b) If $\varphi'(t) \leqq$ const < 1 on $[a, b)$, then $b < \infty$ and every solution of (q) tends to zero for $t \to b_-$.

2. In [1] relations between the dispersions of the 1-st and 2-nd kind were examined. The following theorem completes the results derived there.

Theorem 1. Let (q), $q \in C^\circ\ [a, b)$, $q(t) < 0$, $t \in [a, b)$ be an oscillatoric $(t \to b_-)$ differential equation and $\varphi(\psi)$ its dispersion of the 1-st (2-nd) kind. Let $t_0 \in [a, b)$ and

$$f(t) = \frac{q(\psi(t))}{q(t)}\ \psi'(t), \qquad t \in [a, b).$$

Then

a) $\varphi(t_0) < \psi(t_0)$ if, and only if $f'(t_0) < 0$

b) $\varphi[t_0) = \psi(t_0)$ if, and only if $f'(t_0) = 0$

c) $\varphi(t_0) > \psi(t_0)$ if, and only if $f'(t_0) > 0$.

Proof. a) Let y be a solution of (q) such that $y'(t_0) > 0$, $y\ (t_0) = 0$. It follows from (2) that the function f has the derivative and

$$f'(t_0) = \left(\frac{y'^2(\psi(t))}{y'^2(t)}\right)'\Bigg|_{t = t_0} = 2\ \psi'^2_0\ \frac{q^2(\psi_0)}{q(t_0)}\ \frac{y(\psi_0)}{y'(\psi_0)} \tag{4}$$

holds where $\psi_0 = \psi(t_0)$, $\psi'_0 = \psi'(t_0)$.

Let $\varphi(t_0) < \psi(t_0)$. Then $y(\psi_0) < 0$, $y'(\psi_0) < 0$ and according (4) we have

$$f'(t_0) < 0. \tag{5}$$

Let (5) ve valid. As $y'(\psi_0) < 0$, it follows from (4) that $y(\psi_0) < 0$ and thus $\varphi(t_0) < \psi(t_0)$ b) c) These cases can be proved in the same way.

The following theorem sums up the results of 1.2. and Theorem 1 concerning the important case $\varphi(t_0) = \psi(t_0)$, $t_0 \in [a, b)$.

Theorem 2. Let $\varphi(\psi)$ be the dispersion of the 1-st (2-nd) kind of an oscillatoric $(t \to b_-)$ differential equation (q), $q \in C^\circ\ [a, b)$, $q(t) < 0$ on $[a, b)$. Then the following assertions are equivalent:

a) $\varphi(t_0) = \psi(t_0)$

b) $\varphi''(t_0) = 0$

c) $\left(\frac{q(\psi(t))}{q(t)}\psi'(t)\right)'\Big|_{t=t_0} = 0$

d) $\varphi'(t_0)\cdot\frac{q(\psi(t_0))}{q(t_0)}\psi'(t_0) = 1.$

Remark 1. Theorem 2 indicates that there exists a more profound dependence between the functions φ' and $\frac{q(\psi)}{q(t)}\cdot\psi'$. The following theorem expresses this dependence more in detail.

Theorem 3. Let (q), $q \in C^\circ [a, b)$, $q(t) < 0$ on $[a, b)$ be oscillatoric on $[a, b)$ and φ, ψ be its dispersions of the 1-st and 2-nd kind. Let us put:

$$f(t) = \frac{q(\psi(t))}{q(t)}\psi'(t), \qquad t \in [a, b).$$

Then

a) The function φ' has a local maximum (minimum) at $t = t_0$ if, and only if f has a local minimum (maximum) at the point t_0. Moreover,

$$\varphi'(t_0) = \frac{1}{f(t_0)} \tag{6}$$

holds if the point t_0 is an extremant of φ' or f.

b) The function φ' is increasing (decreasing) at $t = t_0$ if, and only if f is decreasing (increasing) at $t = t_0$.

c) If $\varphi'(t) \geqq 1$ ($f(t) \geqq 1$) holds on $[a, b)$, then $f(t) \leqq 1$ ($\varphi'(t) \leqq 1$) on $[a, b)$. If $\varphi'(t) \leqq 1$ ($f(t) \leqq 1$) holds on (a, b), then there exists a number $\bar{t}$, $\bar{t} \in [a, b)$ such that $f(t) \geqq 1$ ($\varphi'(t) \geqq 1$) on $[\bar{t}, b)$.

Proof. a) b) The relation (6) from the case a) follows from Theorem 2 because if the function $\varphi'(f)$ has a local extreme at the point t_0, then $\varphi''(t_0) = 0$ ($f'(t_0) = 0$). Further, it follows from Theorem 1 and 1. 2. that $\varphi''(t_0) < 0$, resp. $= 0$, resp. > 0 if, and only if $f'(t_0) > 0$, resp. $= 0$, resp. < 0. Thus if $\varphi''(t_0) \neq 0$ ($f'(t_0) \neq 0$) holds, then the statement b) is valid. If $\varphi''(t_0) = 0$ ($f'(t_0) = 0$), then the statements a) b) follows from the following assertions.

1) If $\varphi'(t) \geqq \varphi'(t_0)$ ($f(t) \geqq f(t_0)$), $t \in J$, then $f(t) \leqq f(t_0)$ ($\varphi'(t) \leqq \varphi'(t_0)$), $t \in J$ holds.

2) If $\varphi'(t) \leqq \varphi'(t_0)$ ($f(t) \leqq f(t_0)$, $t \in J$, then $f(t) \geqq f(t_0)$ ($\varphi'(t) \geqq \varphi'(t_0)$), $t \in J_1$ holds, where $J = [t_0, t_0 + \varepsilon)$, resp. $(t_0 - \varepsilon, t_0]$, $\varepsilon > 0$ is an arbitrary number, $\varepsilon \leqq t_0 - a$ and $J_1 = [t_0, t_0 + \varepsilon_1)$, resp. $(t_0 - \varepsilon_1, t_0]$, $\varepsilon_1 \leqq \varepsilon$ is a suitable number and $\varphi''(t_0) = 0$ ($f'(t_0) = 0$).

The assertion 1) follows directly from 1.2. and Theorem 1. The assertion 2): Let $\varphi'(t) \leqq \varphi'(t_0)$, $t \in J$ and $\bar{\bar{t}} \in J$, $\varphi''(\bar{\bar{t}}) = 0$. Then according to Theorem 2 we have:

$$f(\bar{\bar{t}}) = \frac{1}{\varphi'(\bar{\bar{t}})} \geqq \frac{1}{\varphi'(t_0)} = f(t_0),$$

Let a number t_1, $t_1 \in J$ exist such that $\varphi''(t_1) = 0$, $t_1 \neq t_0$. If $\bar{t} \in J$, $\varphi''(\bar{t}) \neq 0$, $|\bar{t} - t_0| < |t_1 - t_0|$, then φ' is monotone in some neighourhood of the point $\bar{t}$ and there exist numbers t_2, $t_3 \in J$ such that $\varphi''(t_2) = \varphi''(t_3) = 0$, $\varphi''(t) \neq 0$, $t \in (t_2, t_3)$, $\bar{t} \in (t_2, t_3)$. We have: $f(t_2) \geqq f(t_0)$, $f(t_3) \geqq f(t_0)$. As the function f is monotone on (t_2, t_3), we have

$f(t) \geqq f(t_0)$ and the statement is valid in this case. If the above mentioned number t_1 does not exist, then $\varphi''(t) > 0$, resp. < 0 for $t \in J$, $t \neq t_0$ where $J = (t_0 - \varepsilon, t_0]$, resp. $J = [t_0, t_0 + \varepsilon)$. From here it follows (by use of 1.2.) that the function f is increasing, resp. decreasing and in both cases $f(t) \geqq f(t_0)$, $t \in J$ holds. The rest of the statement

$$f(t) \leqq f(t_0), t \in J \Rightarrow \varphi'(t) \geqq \varphi'(t_0), t \in J_1$$

we can proved in the same way.

c) I. Let $\varphi'(t) \geqq 1$ $(f(t) \geqq 1)$, $t \in [a, b)$. Then according to 1.2. we have: $f(t) \leqq 1$ $(\varphi'(t) \leqq 1)$, $t \in [a, b)$ and so the statement is valid in this case.

II. Let $\varphi'(t) \leqq 1$ $(f(t) \leqq 1)$, $t \in [a, b)$ and let M be the set of all numbers $t \in [a, b)$ such that the function $\varphi'(f)$ has a local maximum at $t \in M$. If the infinity is the accumulation point of M, then it follows from a) that $f(\varphi')$ has all local minima at the points $t \in M$ and we have

$$\varphi'(t) \, . \, f(t) = 1, \, t \in M.$$

From this $f(t) \geqq 1$ $(\varphi'(t) \geqq 1)$, $t \in [t_0, b)$ and the statement is valid in this case.

If the infinity is not the accumulation point of M, then there exists a number $\bar{t} \in [a, b)$ such that the function $\varphi'(f)$ is monotone on $[\bar{t}, b)$.

A. Let $\varphi'(t) \leqq 1$, $t \in [\bar{t}, b)$. Suppose that $\lim\limits_{t \to b} f(t) = c < 1$. Let y be an arbitrary non-trivial solution of (q) and $\{x_k\}_0^\infty$ the sequence of all zeros of y', $x_k \in [\bar{t}, b)$. So $x_k = \psi(x_{k-1})$, $k \geqq 1$ and $|y(x_k)|$ are local maxima of $|y|$. It follows from (2) that

$$0 < \frac{y^2(x_0)}{y^2(x_k)} = \prod_{n=1}^{k} \frac{y^2(x_{n-1})}{y^2(x_n)} = \prod_{n=1}^{k} f(x_{n-1}) \xrightarrow[k \to \infty]{} 0$$

Thus y is unbounded and it is in contradiction with 1.2. (ii). Thus $\lim\limits_{t \to b} f(t) \geqq 1$.

Let φ' be non-decreasing. Then f is non-increasing (see b)) and the statement is valid.

Let φ' be non-increasing. Then $\lim\limits_{t \to b} \varphi'(t) < 1$ and according to 1.2. (ii) we have that an arbitrary solution of (q) converges to zero for $t \to b_-$. Suppose that $\lim\limits_{t \to b} f(t) = 1$. As f is non-decreasing we have $f(t) \leqq 1$, $t \in [\bar{t}, b)$. Let y be an arbitrary non-trivial solution of (q) and $\{x_n\}_0^\infty$ a sequence of the zeros of y', $x_n \in [t_0, b)$. Then y has a local extreme at x_n and by use of (2) we have:

$$\infty \xleftarrow[n \to \infty]{} \frac{y^2(x_0)}{y^2(x_n)} = \prod_{k=1}^{n} \frac{y^2(x_{k-1})}{y^2(x_k)} = \prod_{k=1}^{n} f(x_{k-1}) \leqq 1. \tag{7}$$

But this is the contradiction. So $\lim\limits_{t \to b} f(t) > 1$ and the statement is valid.

B. Let $f(t) \leqq 1$, $t \in [\bar{t}, b)$ and let $\lim\limits_{t \to b} \varphi'(t) = c < 1$. Then (7) is valid and it is the contradiction. Thus $\lim\limits_{t \to b} \varphi'(t) \geqq 1$. If f is non-decreasing, then φ' is non-increasing and the statement is valid. Let f be non-increasing. Then $\lim\limits_{t \to b} f(t) = c < 1$ and φ' is non-decreasing on $(\bar{t}, b)$. In the first part of c) II. A) we proved that the conditions $\varphi'(t) \leqq 1$, $t \in [\bar{t}, b)$, $\lim\limits_{t \to b_-} f(t) < 1$ can not be valid at the same time. From this $\lim\limits_{t \to b}$ $\varphi'(t) > 1$ and the statement of the theorem is proved.

Remark 2. The case c) of Theorem 3 is valid, too if we replace the inequalities $\leqq$, $\geqq$ by $<$, $>$, resp. because if $\varphi'(t_0) = 1$ $(f(t_0) = 1)$, $t_0 \in [a, b)$, then according to Theorem 2 we have $f(t_0) = 1$ $(\varphi'(t_0) = 1)$.

The results of Theorem 3 gives us a possibility to generalize a theorem from [2] (Theorem 10) concerning the behaviour of solutions of (q).

Theorem 4. *Let* (q), $q \in C^\circ[a, b)$, $q(t) < 0$, $t \in [a, b)$, *be oscillatoric on* $[a, b)$ *and let* φ, ψ *be its dispersions of the 1-st, 2-nd kind, resp. Consider the following assertions on* $[a, b)$:

A) *The sequence of the absolute values of all local extremes (of the derivative) of an arbitraty non-trivial solution of (q) is non-increasing.*

B) *The sequence of the absolute values of all local extremes of the derivative of an arbitrary non-trivial solution (of an arbitrary non-trivial solution) of (q) is non-decreasing.*

C) $\dfrac{q(\psi)}{q(t)}\psi' \geqq 1$ $\quad(\varphi(t) - t$ *is non-decreasing*)

D) $\varphi(t) - t$ $\quad$ *is non-increasing* $\quad\left(\dfrac{q(\psi)}{q(t)}\psi' \leqq 1\right)$.

Then $A \Leftrightarrow C \Rightarrow D \Leftrightarrow B$ *and there exists a number* t_0, $t_0 \in [a, b)$ *such that we have* $D \Rightarrow C$ *on* $[t_0, b)$.

Proof. According to Theorem 10 from [2] we must only prove that there exists a number t_0, $t_0 \in [a, b)$ such that $D \Rightarrow C$ on $[t_0, b)$ holds. But this fact follows directly from Theorem 3c).

Remark 3. If we replace „non-increasing", „non-decreasing", $\leqq$, $\geqq$ by „decreasing", „increasing", $<$, $>$, respectively, then Theorem 4 is valid, too.

Theorem 5. *Let* (q), $q \in C^\circ[a, b)$, $q(t) < 0$, $t \in [a, b)$ *be oscillatoric on* $[a, b)$ *and let* φ, ψ *be its dispersions of the 1-st and 2-nd kind. Let* $t_0 \in [a, b)$. *Then the following assertions are equivalent.*

A. $\varphi(t_0) = \psi(t_0)$, $\varphi'(t_0) = \psi'(t_0)$

B. $\varphi''(t_0) = \varphi'''(t_0) = 0$.

Moreover, if there exists $q'(t_0)$, *then the assertion*

C) $f'(t_0) = f''(t_0) = 0$ *where* $f(t) = \dfrac{q(\psi(t))}{q(t)}\psi'(t)$ *is equivalent with A) and B).*

Proof. $A \Rightarrow B$: According to Theorem 2 we have:

$$\varphi''(t_0) = 0, \qquad \varphi'^2(t_0) = \varphi'(t_0)\,\psi'(t_0) = \frac{q(t_0)}{q(\psi(t_0))} = \frac{q(t_0)}{q(\varphi(t_0))}.$$

From this and from (3)

$$-\frac{1}{2}\frac{\varphi'''(t_0)}{\varphi'(t_0)} + \frac{3}{4}\left(\frac{\varphi''(t_0)}{\varphi'(t_0)}\right)^2 = 0$$

holds and thus $\varphi'''(t_0) = 0$.

$B \Rightarrow A$. It follows from Theorem 2 that $\varphi(t_0) = \psi(t_0)$ holds and from (3) we have:

$$q(t_0) = q(\varphi(t_0))\,\varphi'^2(t_0) = q(\psi(t_0))\,.\,\varphi'^2(t_0).$$

From this and from theorem 2 we get: $\varphi'^2(t_0) = \varphi'(t_0)\,.\,\psi'(t_0)$ and thus $\varphi'(t_0) = \psi'(t_0)$

$A \Leftrightarrow C$. Let y be a non-trivial solution of (q) such that $y(t_0) = 0$. Then it follows from (2) that f has the derivative and

$$f'(t) = 2 \, . f^2(t)q(t) \frac{y(\psi(t))}{y'(\psi(t))} - 2 f(t) \, q(t) \, . \frac{y(t)}{y'(t)}$$

holds in some neighbourhood of the point t_0. Thus we can see that the function $\frac{f'}{q}$ has the derivative and if $q'(t_0)$ exists, then we have at $t = t_0$:

$$\left(\frac{f'}{q}\right)' = \frac{f''}{q} - \frac{f'q'}{q^2} = \frac{3}{2} \frac{f'^2}{f \, . \, q} + 2f(f \, . \, \psi' - 1). \tag{8}$$

$C \Rightarrow A$: According to (8) we have $f(f\psi' - 1) = 0$ for $t = t_0$ and because $f \neq 0$ we get.

$$f(t_0) \, \psi'(t_0) = 1. \tag{9}$$

Theorem 2 gives us: $\varphi(t_0) = \psi(t_0)$,

$$f(t_0) \, \varphi'(t_0) = 1. \tag{10}$$

Thus $\varphi'(t_0) = \psi'(t_0)$ and the statement is proved.

$A \Rightarrow C$. It follows from the assumptions and Theorem 2 that $f'(t_0) = 0$ and (10) and (9) hold. Then the statement follows from (8).

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ASYMPTOTISCHE EIGENSCHAFTEN DER LÖSUNGEN DES SYSTEMS $\{A_{n-1}^{-1}(x)\ldots[A_1^{-1}(x)y']'\ldots\}' = A_n(x)y$

Ivo Res, Brno

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1. EINLEITUNG

Es seien $A_i(x) = (a^i_{j,k}(x))$, $i = 1, 2, \ldots, n$ Quadratmatrizen der Ordnung m, $a^i_{j,k}(x) \in C_{n-1}(I)$, $I = \langle x_0, \infty)$. Wir werden voraussetzen, daß die Matrizen $A_i(x)$, $i = 1, 2, \ldots, n - 1$ in I regulär sind und bezeichnen $A_i^{-1}(x)$ inverse Matrizen, y einen Spaltenvektor der Dimension m.

Wir werden uns mit den asymptotischen Eigenschaften der Lösungen des Systems

$$\{A_{n-1}^{-1}(x)\ldots[A_1^{-1}(x)\, y']'\ldots\} = A_n(x)\, y \tag{1,1}$$

beschäftigen.

Um die Lösung y des Systems (1,1) und seine Ableitungen zu finden, werden wir Reihen konstruieren, die in I oder in $I_1 \subset I$ zu y gleichmäßig konvergieren. Diese Methode wurde von U. Richard für die Gleichung zweiter Ordnung in der Arbeit [1] veröffentlicht. Sie ermöglicht approximative Lösungen von (1,1) zu suchen, welche besonders für große Werte von x passend sind.

Mit der Untersuchung der asymptotischen Eigenschaften des Systems (1,1) für $n = 2$ beschäftigten sich viele Autoren — A. Wintner [3], M. Ráb [4], [5], R. Bellman [2]. Dabei zeigte es sich vorteilhaft, das sogennante Peano-Baker Verfahren zu benutzen. Dieses Verfahren ermöglicht, die Lösung des Systems in der Form von unendlichen Reihen auszudrücken, die im ganzen Intervall I gleichmäßig konvergieren. Es ist besonders vorteilhaft darum, daß es eine sehr einfache und dabei genügend genaue Abschätzung des Fehlers ermöglicht, s. M. Ráb [6].

Gleichzeitig mit dem System (1,1) betrachten wir das Matrizensystem

$$\{A_{n-1}^{-1}(x)\ldots[A_1^{-1}(x)Y']'\ldots\}' = A_n(x)Y, \tag{1,2}$$

wobei Y eine $m \times m$ Matrix bezeichnet.

Wir machen jetzt folgende Verabredungen:

1. Wenn A eine Quadratmetrix der Ordnung m mit stetigen Elementen $a_{j,k}$ im Intervall I ist, dann verstehen wir unter der Norm $||A||$ der Matrix A den Ausdruck

$$||A|| = \frac{1}{m}\sum_{j,k}^{m} |a_{j,k}|.$$

2. Wenn a eine reelle Zahl ist, dann bedeutet das Symbol $\int_{a_j}^{x} A_j(t)\, dt$:

a) das Riemansche Integral, für $a_j = a$,

b) $\lim_{\tau\to\infty} \int_{\tau}^{x} A_j(t)\, dt$, für $a_j = \infty$.

3. Ist kein Mißverständnis zu befürchten, so wollen wir das Argument x weglassen.

2. DEFINITIONEN UND BEZEICHNUNGEN

Definition 2,1. *Wir bezeichnen* $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}x}$ *und definieren die Differentialoperatoren*

$$L_s = \mathbf{A}_s^{-1}\,\mathrm{D}\ldots\mathrm{D}\mathbf{A}_1^{-1}\,\mathrm{D},\ \text{für } s = 1, 2, \ldots, n-1 \tag{2,1}$$

$$L_n = \mathrm{D}L_{n-1}. \tag{2,2}$$

Definition 2,2. *Es sei* $\mathbf{F}$ *die Menge aller Quadratmatrizen* $\mathbf{F} = (f_{j,k}(x))$ *der Ordnung* m, $f_{j,k} \in C_0(I)$. *Wir bezeichnen mit* $Q_j, j = 1, 2, \ldots, n$ *den Integraloperator, der die Menge* $\mathbf{F}$ *in sich selbst abbildet nach der Regel*

$$Q_j(\mathbf{F}) = \int_{a_j}^{x} \mathbf{A}_j(t)\,\mathbf{F}(t)\,\mathrm{d}t. \tag{2,3}$$

Unter einem Produkt Q_jQ_k *der Operatoren* Q_j, Q_k *für* j, $k = 1, 2, \ldots, n$ *verstehen wir den durch die Beziehung*

$$Q_jQ_k(\mathbf{F}) = \int_{a_j}^{x} \mathbf{A}_j(t) \int_{a_k}^{t} \mathbf{A}_k(s)\,\mathbf{F}(s)\,\mathrm{d}s\,\mathrm{d}t \tag{2,4}$$

definierten Operator.

Das durch die Formel (2,4) definierte Produkt der Operatoren ist eine Superposition dieser Operatoren. Auch die Superposition anderer Operatoren, die in der Arbeit vorkommen, werden wir in der Gestalt eines Produktes schreiben, zum Beispiel

$$\mathrm{D}Q_j(\mathbf{F}) = \mathbf{A}_j\mathbf{F}.$$

Definition 2,3. *Es sei* $a \in I$ *eine reelle Zahl und es bezeichne* a_i, $i = 1, 2, \ldots, n$ *entweder* a *oder* ∞. *Wir definieren die Operatoren* U_j, $j = 1, 2, \ldots, n$ *durch die Beziehung*

$$U_j(\mathbf{F}) = Q_jQ_{j+1}\ldots Q_nQ_{n+1}\ldots Q_{n+j-1}(\mathbf{F}), \tag{2,5}$$

wobei wir $Q_{n+i} = Q_i$ *setzen. Weiter setzen wir*

$$U_j^0(\mathbf{F}) = \mathbf{F},\ U_j^i(\mathbf{F}) = U_jU_j^{i-1}(\mathbf{F}),\ i = 1, 2, \ldots \tag{2,6}$$

Definition 2,4. *Es seien* j, k *natürliche Zahlen* $1 \leqq j, k \leq n$. *Wir definieren* $\mathbf{H}_{j,j}(x) = \mathbf{E}$, $\mathbf{E} =$ *Einheitsmatrix,*

$$\mathbf{H}_{j,k}(x) = Q_jQ_{j+1}\ldots Q_{k-1}(\mathbf{E}),\ \text{für } j < k, \tag{2,7}$$

$$\mathbf{H}_{j,k}(x) = Q_jQ_{j+1}\ldots Q_nQ_{n+1}\ldots Q_{n+k-1}(\mathbf{E}),\ \text{für } j > k,$$

wo bei $Q_{n+i} = Q_i$ *gesetzt ist.*

Definition 2,5. *Es sei* $f(x)$ *eine stetige positive Funktion. Wir bezeichnen mit* q_j, $j = 1, 2, \ldots, n$ *den Integraloperator, der durch die Beziehung*

$$q_j(f) = \int_{a_j}^{x} \|\mathbf{A}_j(t)\|\,f(t)\,\mathrm{d}t \tag{2,8}$$

definiert ist.

Unter einem Produkt der Operatoren q_jq_k, j, $k = 1, 2, \ldots, n$ *verstehen wir den Operator*

$$q_jq_k(f) = \int_{a_j}^{x} \|\mathbf{A}_j(t)\| \int_{a_k}^{t} \|\mathbf{A}_k(s)\|\,f(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{2,9}$$

Definition 2,6. *Es sei j eine natürliche Zahl, $1 \leqq j \leqq n$. Definieren wir die Funktionen*

(2,10) $$\gamma_j(x) = | q_j q_{j+1} \dots q_n q_{n+1} \dots q_{n+j-1}(1) |,$$

mit $q_{n+i} = q_i$.

Definition 2.7. *Es seien j, k natürliche Zahlen, $1 \leqq j, k \leqq n$. Wir definieren Funktionen $\varkappa_{j,k}(x)$ durch die Beziehungen*

$$\varkappa_{j,j}(x) = 1$$

(2,11) $$\varkappa_{j,k}(x) = | q_j q_{j+1} \dots q_{k-1}(1) |, \text{ für } j < k,$$

$$\varkappa_{j,k}(x) = | q_j q_{j+1} \dots q_n q_{n+1} \dots q_{n+k-1}(1) |, \text{ für } j > k.$$

Bemerkung 2,8. Wenn wir

$$\begin{aligned} \boldsymbol{u}_1' &= \boldsymbol{A}_1 \boldsymbol{u}_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ \boldsymbol{u}_{n-1}' &= \boldsymbol{A}_{n-1} u_n \\ \boldsymbol{u}_n' &= \boldsymbol{A}_n u_1 \end{aligned}$$

setzen, können wir das System (1,1) in der Gestalt

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u}$$

schreiben, wobei die Matrix $\boldsymbol{A}$ von Blöcken $\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n$ und von Nullmatrizen der Ordnung m gebildet wird,

$$\boldsymbol{A} = \begin{pmatrix} 0, \boldsymbol{A}_1, 0, \dots, 0 \\ 0, 0, \boldsymbol{A}_2, \dots, 0 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{A}_n, 0, 0, \dots, 0 \end{pmatrix}$$

Offenbar sind die ersten m Komponenten des Vektors $\boldsymbol{u}$ die Komponenten der Lösung $\boldsymbol{y}$. Daraus sieht man, daß folgender Existenzsatz für das System (1,1) gilt.

Satz 2,9, *Es seien $\boldsymbol{A}_i$, $i = 1, 2, \dots, n$ eine Matrix mit stetigen Elementen im Intervall I, $\boldsymbol{A}_i$ regulär für $i = 1, 2, \dots, n - 1$. Es seien weiter $\boldsymbol{y}_0, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{n-1}$, beliebige konstante Vektoren. Dann existiert genau eine im ganzen Intervall I definierte Lösung $\boldsymbol{y}$ des Systems* (1, 1) *und es gilt*

$$\boldsymbol{y}(x_0) = \boldsymbol{y}_0^0, \; L_s\boldsymbol{y}(x) \,|_{\,x = x_0} = \boldsymbol{y}_s^0, \; s = 1, 2, \dots, n - 1$$

wobei der Differentialoperator L_s durch die Formel (2,1) *definiert ist.*

Bemerkung 2,10. Mit den Bezeichnungen (2,1), (2,2) können wir das System (1,2) in der Form

(2,12) $$L_n\boldsymbol{Y} = \boldsymbol{A}_n\boldsymbol{Y}$$

schreiben.

3. BEZIEHUNGEN ZWISCHEN OPERATOREN

Hilfssatz 3,1. *Es seien U_j^i und $\boldsymbol{H}_{j,k}$, j, $k = 1, 2, \dots, n$ durch die Formeln* (2,5), (2,6), (2,7) *definiert. Dann gilt*

(3,1) $$U_j^i(\mathbf{H}_{j,k}) = Q_j U_{j+1}^i(\mathbf{H}_{j+1,k}), \text{ für } j \neq k,\ i = 0, 1, \ldots$$

(3,2) $$U_j^i(\mathbf{H}_{j,j}) = Q_j U_{j+1}^{i-1}(\mathbf{H}_{j+1,j}), \text{ für } i = 1, 2, \ldots$$

Diese Beziehungen werden durch die vollständige Induktion bewiesen. Da die Beweise analog sind, werden wir nur die Beziehung (3,1) beweisen.

Es gilt

$$U_j^1(\mathbf{H}_{j,k}) = U_j(\mathbf{H}_{j,k}) = Q_j Q_{j+1} \ldots Q_n \ldots Q_{n+j-1} Q_{n+j}(\mathbf{H}_{j+1,k}) = Q_j U_{j+1}(\mathbf{H}_{j+1,k}),$$

so daß Beziehung (3,1) für $i = 1$ erfüllt ist. Es sei jetzt (3,1) gültig. Dann ist nach (2,6)

$$U_j^{i+1}(\mathbf{H}_{j,k}) = U_j U_j^i(\mathbf{H}_{j,k}) = Q_j Q_{j+1} \ldots Q_n \ldots Q_{n+j} U_{j+1}^i(\mathbf{H}_{j+1,k}) =$$
$$= Q_j U_{j+1} U_{j+1}^i(\mathbf{H}_{j+1,k}) = Q_j U_{j+1}^{i+1}(\mathbf{H}_{j+1,k}),$$

und die Behauptung ist bewiesen.

4. FORMALLÖSUNG DES SYSTEMS (1,2)

In diesem Absatz werden wir oft folgende Voraussetzungen machen: (*) *Es seien* $\mathbf{A}_i = (a_{j,k}^i)$, $a_{j,k}^i \in C_{n-i}(I)$, $i = 1, 2, \ldots, n$ *Quadratmatrizen der Ordnung* m *und* $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_{n-1}$ *seien regulär.*

Hilfssatz 4,1. *Es gelte* (*). *Werden* $\mathbf{H}_{1,k}$ *durch* (2,7) *und* L_n *durch Definition* 2,1 *gegeben, so ist*

(4,1) $$L_n \mathbf{H}_{1,k} = 0,\ 0 = \textit{Nullmatrix}.$$

Beweis. Für $k = 1$ ist die Behauptung offenbar. Durch die Induktion beweist man leicht, daß für $s = 1, 2, \ldots, k-1$, $k = 2, 3, \ldots, n$

$$\mathrm{L}_s \mathbf{H}_{1,k}(x) = \mathbf{H}_{s+1,k}(x),$$

gilt. Es ist bessonders für $s = k - 1$

$$L_{k-1} \mathbf{H}_{1,k}(x) = \mathbf{E}.$$

Hieraus erhalten wir die Behauptung auf Grund der Relation

$$L_n = \mathrm{D}\mathbf{A}_{n-1}^{-1} D \ldots \mathbf{A}_k^{-1} \mathrm{D} \mathrm{L}_{k-1}.$$

Sazt 4,2. *Es gelte* (*). *Dann ist die Formallösung des Systems* (1,2) *durch die Reihe*

(4,2) $$\mathbf{Y}_k(x) = \sum_{i=0}^{\infty} U_1^i(\mathbf{H}_{1,k}),\ k = 1, 2, \ldots, n$$

gegeben, wobei U_1^i *durch* (2,5), (2,6) *und* $\mathbf{H}_{1,k}$ *durch* (2,7) *gegeben sind.*

Beweis. Es gilt offenbar

$$L_n U_1(\mathbf{F}) = \mathbf{A}_n \mathbf{F}$$

und

$$L_n \mathbf{Y}_k = L_n \mathbf{H}_{1,k} + L_n \sum_{i=1}^{\infty} U_1^i(\mathbf{H}_{1,k}) = L_n \mathbf{H}_{1,k} + \sum_{i=1}^{\infty} L_n U_1 U_1^{i-1}(\mathbf{H}_{1,k}) =$$
$$= L_n \mathbf{H}_{1,k} + \mathbf{A}_n \sum_{i=1}^{\infty} U_1^{i-1}(\mathbf{H}_{1,k}) = L_n \mathbf{H}_{1,k} + \mathbf{A}_n \sum_{i=0}^{\infty} U_1^i(\mathbf{H}_{1,k}) =$$
$$= L_n \mathbf{H}_{1,k} + \mathbf{A}_n \mathbf{Y}_k.$$

Nach Hilfssatz 4,1 gilt aber (4,1). Es ist also

$$L_n \mathbf{Y}_k = \mathbf{A}_n \mathbf{Y}_k,$$

was ein mit (1,2) äquivalentes System ist. Damit ist der Beweis durchgeführt.

Satz 4,3. *Es gelte (*) und es sei die Lösung des Systems (1, 2) durch die Reihen (4, 2) gegeben. Dann kann man formal*

$$(4,3) \qquad L_s \mathbf{Y}_k(x) = \sum_{i=0}^{\infty} U_{s+1}^i(\mathbf{H}_{s+1,k}),\ s = 0, 1, \ldots, n-1 \qquad k = 1, 2, \ldots, n$$

schreiben, wobei $L_0\, \mathbf{Y}_k(x) = \mathbf{Y}_k(x)$ ist und die Formeln (2,1), (2,5), (2,6) (2,7) *gelten.*

Bemerkung 4,4. Im folgenden Absatz wird gleichmäßige Konvergenz der Reihen (4,3) untersucht. Wir machen aufmerksam darauf, daß $\mathbf{Y}_k$ Lösung des Systems (1,2) im Intervall I oder $I_1 \subset I$ ist, wenn die Reihen (4,3) in diesem Intervall für $s = 0, 1, \ldots, n-1$ gleichmäßig konvergieren.

Satz 4,3 wird durch vollständige Induktion bewiesen. Für $s = 1$ gilt nach (2,1)

$$L_1 \mathbf{Y}_k = \mathbf{A}_1^{-1} \mathrm{D} \mathbf{Y}_k.$$

Es sei $k \neq 1$. Dann ist nach (3,1)

$$\mathbf{A}_1^{-1} \mathrm{D} \mathrm{Y}_k = \mathbf{A}_1^{-1} \mathrm{D} \sum_{i=0}^{\infty} U_1^i(\mathbf{H}_{1,k}) = \mathbf{A}_1^{-1} \mathrm{D} \sum_{i=0}^{\infty} Q_1 U_2^i(\mathbf{H}_{2,k}) = \sum_{i=0}^{\infty} U_2^i(\mathbf{H}_{2,k}).$$

Wenn $k = 1$, dann ist wegen (3,2)

$$\mathbf{A}_1^{-1} \mathrm{D} \mathbf{Y}_1 = \mathbf{A}_1^{-1} \mathrm{D} \left[\mathbf{E} + \sum_{i=1}^{\infty} U_1^i(\mathbf{H}_{1,1})\right] = \mathbf{A}_1^{-1} \mathrm{D} \left[\mathbf{E} + \sum_{i=1}^{\infty} Q_1 U_2^{i-1}(\mathbf{H}_{2,1})\right] =$$

$$= \sum_{i=1}^{\infty} U_2^{i-1}(\mathbf{H}_{2,1}) = \sum_{i=0}^{\infty} U_2^i(\mathbf{H}_{2,1}).$$

Für $s = 1$ gilt also (4,3). Es gelte jetzt (4,3) für $s = j - 1$. Nach (2,1) ist

$$L_j \mathbf{Y}_k = \mathbf{A}_j^{-1} \mathrm{D} L_{j-1} \mathbf{Y}_k = \mathbf{A}_j^{-1} \mathrm{D} \sum_{i=0}^{\infty} U_j^i(\mathbf{H}_{j,k}).$$

Wenn $j \neq k$ ist, dann gilt nach (3,1)

$$\mathbf{A}_j^{-1} \mathrm{D} \sum_{i=0}^{\infty} U_j^i(\mathbf{H}_{j,k}) = \mathbf{A}_j^{-1} \mathrm{D} \sum_{i=0}^{\infty} Q_j U_{j+1}^i(\mathbf{H}_{j+1,k}) = \sum_{i=0}^{\infty} U_{j+1}^i(\mathbf{H}_{j+1,k}).$$

Für $j = k$ bekommen wir mit Hilfe (3,2)

$$\mathbf{A}_j^{-1} \mathrm{D} \sum_{i=0}^{\infty} U_j^i(\mathbf{H}_{j,j}) = \mathbf{A}_j^{-1} \mathrm{D} \left[\mathbf{E} + \sum_{i=1}^{\infty} U_j^i(\mathbf{H}_{j,j})\right] = \mathbf{A}_j^{-1} \mathrm{D} \left[\mathbf{E} + \sum_{i=1}^{\infty} Q_j U_{j+1}^{i-1}(\mathbf{H}_{j+1,j})\right] =$$

$$= \sum_{i=1}^{\infty} U_{j+1}^{i-1}(\mathbf{H}_{j+1,j}) = \sum_{i=0}^{\infty} U_{j+1}^i(\mathbf{H}_{j+1,j}).$$

Damit sind die Beziehungen (4,3) bewiesen.

Bemerkung 4,5. Wenn $\mathbf{Y}_k$ eine Lösung des Systems (1,2) ist, dann kann die Lösung $\mathbf{y}_k$ des Systems (1,1) in der Gestalt

$$(4,4) \qquad \mathbf{y}_{k,i}(x) = \mathbf{Y}_k(x)\, \mathbf{y}_{k,i}^0,\ i = 1, 2, \ldots, m$$

geschrieben werden, wobei $\mathbf{y}_{k,i}^0$ ein beliebiger konstanter Vektor ist.

5. GLEICHMÄSSIGE KONVERGENZ DER REIHEN (4,3).

Hilfssatz 5,1. *Die Matrizenfunktionen* $\mathbf{H}_{j,k}$ *seien durch Definition 2,4 gegeben. Dann gilt*

$$\| \mathbf{H}_{j,k}(x) \| \leqq \varkappa_{j,k}(x), \text{ für } j \neq k,$$

$$\| \mathbf{H}_{j,j}(x) \| = \| \mathbf{E} \| = 1,$$

wobei die Funktionen $\varkappa_{j,k}$ *durch* (2,11) *festgelegt sind.*

Beweis. Es ist für $j < k$

$$\| \mathbf{H}_{j,k}(x) \| = \| Q_j Q_{j+1} \dots Q_{k-1}(\mathbf{E}) \| \leqq | q_j q_{j+1} \dots q_{k-1}(1) | = \varkappa_{j,k}(x)$$

und für j > k

$$\| \mathbf{H}_{j,k}(x) \| = \| Q_j Q_{j+1} \dots Q_n Q_{n+1} \dots Q_{n+k-1}(\mathbf{E}) \| \leqq$$

$$\leqq | q_j q_{j+1} \dots q_n q_{n+1} \dots q_{n+k-1}(1) | = \varkappa_{j,k}(x).$$

Nach der Verabredung 1., Absatz 1. ist $\| \mathbf{E} \| = 1$.

Satz 5,2. *Es gelte* (*) *und es sei* $a_i = \infty$ *für* $i = 1, 2, \dots, n$. *Ist für irgendein Paar* s, k, $0 \leqq s \leqq n - 1$, $1 \leqq k \leqq n$

$$\gamma_{s+1}(x) < \infty, \tag{5,1}$$

$$\varkappa_{s+1}(x) < \infty, \tag{5,2}$$

so konvergiert die Reihe (4,3), *wobei* $\mathbf{H}_{j,k}$ *und* U_j^i *durch* (2,5), (2,6), (2,7) *gebegen sind, im Intervall* I *gleichmäßig.*

Beweis. Die gleichmäßige Konvergenz der Reihe (4,3) in I wird so bewiesen, daß man zu ihr eine konvergente Majorante konstruiert. Mit der vollständigen Induktion beweisen wir, daß

$$\| U_{s+1}^i(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(x) \frac{\gamma_{s+1}^i(x)}{i!}, \; x \in I \tag{5,3}$$

gilt.

Tatsächlich erhalten wir für $i = 1$

$$\| U_{s+1}^1(\mathbf{H}_{s+1,k}) \| = \| U_{s+1}(\mathbf{H}_{s+1,k}) \| = \| Q_{s+1} Q_{s+2} \dots Q_{n+s}(\mathbf{H}_{s+1,k}) \| \leqq$$

$$\leqq | q_{s+1} q_{s+2} \dots q_{n+s}(\| \mathbf{H}_{s+1,k} \|) |.$$

Nach Hilfssatz 5,1 und (2,10) kann man

$$\| U_{s+1}^1(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(x) \, \gamma_{s+1}(x),$$

schreiben und das ist die Ungleichung (5,3) für $i = 1$.

Es gelte jetzt (5,3). Dann ist wegen (2,5)

$$\| U_{s+1}^{i+1}(\mathbf{H}_{s+1,k}) \| = \| U_{s+1} U_{s+1}^i(\mathbf{H}_{s+1,k}) \| = \| Q_{s+1} Q_{s+2} \dots Q_{n+s} U_{s+1}^i(\mathbf{H}_{s+1,k}) \| \leqq$$

$$\leqq | q_{s+1} q_{s+2} \dots q_{n+s}[\, \| U_{s+1}^i(\mathbf{H}_{s+1,k}) \| \,] | \leqq \varkappa_{s+1,k}(x) \left| q_{s+1} q_{s+2} \dots q_{n+s} \frac{\gamma_{s+1}^i}{i!} \right| \leqq$$

$$\leqq \varkappa_{s+1,k}(x) \left| \int_{\infty}^{x} \gamma_{s+1}'(t) \frac{\gamma_{s+1}^i(t)}{i!} \, dt \right| = \varkappa_{s+1,k}(x) \frac{\gamma_{s+1}^{i+1}(x)}{(i+1)!}$$

Damit ist die Ungleichung (5,3) bewiesen. Für $x \in I$ gilt offenbar

$$\sum_{i=0}^{\infty} \| U_{s+1}^{i}(\boldsymbol{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(x) \sum_{i=0}^{\infty} \frac{\gamma_{s+1}^{i}(x)}{i!} \leqq \varkappa_{s+1}(x_0) \sum_{i=0}^{\infty} \frac{\gamma_{s+1}^{i}(x_0)}{i!}$$

und das bedeutet, daß die Reihe $\varkappa_{s+1,k}(x_0) \sum_{i=0}^{\infty} \frac{\gamma_{s+1}^{i}(x_0)}{i!}$ konvergente Majorante der Reihe (4,3) ist. Die Behauptung des Satzes ist bewiesen.

Bemerkung 5,3. *Gilt für alle $i = 1, 2, \ldots, n$*

$$\int_{x_0}^{\infty} \| \boldsymbol{A}_i(t) \| \, dt < \infty, \tag{5,4}$$

dann konvergiert die Reihe (4,3) *gleichmäßig im Intervall I.*

Satz 5,4. *Es gelten die Voraussetzungen des Satzes 5,2.* Dann gelten *die Abschätzungen*

$$\| L_s \boldsymbol{Y}_k(x) - \sum_{i=0}^{n} U_{s+1}^{i}(\boldsymbol{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(x) \frac{\gamma_{s+1}^{n+1}(x)}{(n+1)!} \exp\{\gamma_{s+1}(x)\} \tag{5,5}$$

für $x \in I$.

Beweis. Bezeichnen wir

$$\mathrm{R}_{n+1}(x) = \sum_{i=n+1}^{\infty} U_{s+1}^{i}(\boldsymbol{H}_{s+1,k})$$

Nach (5,3) ist

$$\| \mathrm{R}_{n+1}(x) \| \leqq \varkappa_{s+1,k}(x) \sum_{i=n+1}^{\infty} \frac{\gamma_{s+1}^{i}(x)}{i!} = \varkappa_{s+1,k}(x) \left[\frac{\gamma_{s+1}^{n+1}(x)}{(n+1)!} + \frac{\gamma_{s+1}^{n+2}(x)}{(n+2)!} + \ldots \right] =$$

$$= \varkappa_{s+1,k}^{(x)} \frac{\gamma_{s+1}^{n+1}(x)}{(n+1)!} \left[1 + \frac{\gamma_{s+1}(x)}{n+2} + \frac{\gamma_{s+1}^{2}(x)}{(n+2)(n+3)} + \ldots \right] <$$

$$< \varkappa_{s+1,k}(x) \frac{\gamma_{s+1}^{n+1}(x)}{(n+1)!} \exp\{\gamma_{s+1}(x)\},$$

womit (5,5) bewiesen ist.

Satz 5,5. *Es gelte* (*). *Dann konvergiert dieReihe* (4,3) *gleichmäßig im Intervall $I_1 = \langle x_0, b \rangle$, wobei U_j^i und $\boldsymbol{H}_{j,k}$ durch* (2,5,) (2,6), (2,7) *gegeben sind mit $a_i = a$, für $i = 1, 2, \ldots, n$ und $b > x_0$.*

Der Beweis dieses Satzes wird ganz analog wie bei Satz 5,2 durchgeführt. Es gilt nämlich

$$\| U_{s+1}^{i}(\boldsymbol{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(b) \frac{\gamma_{s+1}^{i}(b)}{i!}, \ x \in \langle x_0, b \rangle \tag{5,6}$$

und so ist die Reihe $\sum_{i=0}^{\infty} \varkappa_{s+1,k}(b) \frac{\gamma_{s+1}^{i}(b)}{i!}$ eine konvergente Majorante der Reihe (4,3). Daraus folgt die gleichmäßige Konvergenz dieser Reihe in I_1.

Satz 5,6, *Es seien die Voraussetzungen des Satzes 5,5 erfüllt. Dann gilt für $x \in I_1$*

$$\| L_s \boldsymbol{Y}_k(x) - \sum_{i=0}^{n} U_{s+1}^{i}(\boldsymbol{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(x) \frac{\gamma_{s+1}^{n+1}(x)}{(n+1)!} \exp\{\gamma_{s+1}(x)\}. \tag{5,7}$$

Der Beweis wird analog wie bei Satz 5,4 durchgeführt und darum lassen wir ihn fort.

Hilfssatz 5,7. *Es sei* $\varkappa_{j,k}(x) < \infty$, $\gamma_j(x) < \infty$, $1 \leqq j, k \leqq n$. *Dann gilt*

$$\| U_j^i(\mathbf{H}_{j,k}) \| \leqq (\sup_{t\in I} \varkappa_{j,k}(t)) (\sup_{t\in I} \gamma_j(t))^i \tag{5,8}$$

Beweis. Wir werden die Behauptung durch die vollständige Induktion beweisen. Für $i = 1$ haben wir wegen (2,5), (2,6), (2,7) (2,10), (2,11)

$$\| U_j^1(\mathbf{H}_{j,k}) \| \leqq | q_j q_{j+1} \dots q_{n+j-1} (\| \mathbf{H}_{j,k} \|) | \leqq \sup_{t\in I} \varkappa_{j,k}(t) \, | q_j q_{j+1} \dots q_{n+j-1}(1) | \leqq$$
$$\leqq \sup_{t\in I} \varkappa_{j,k}(t) \sup_{t\in I} \gamma_j(t).$$

Nehmen wir jetzt an, daß (5,8) gilt. Dann ist mit Berücksichtigung auf (2,6)

$$\| U_j^{i+1}(\mathbf{H}_{j,k}) \| = \| U_j U_j^i(\mathbf{H}_{j,k}) \| \leqq | q_j q_{j+1} \dots q_{n+j-1} [\| U_j^i(\mathbf{H}_{j,k}) \|] | \leqq$$
$$\leqq \sup_{t\in I} \varkappa_{j,k}(t) (\sup_{t\in I} \gamma_j(t))^i \, | q_j q_{j+1} \dots q_{n+j-1}(1) | =$$
$$= \sup_{t\in I} \varkappa_{j,k}(t) (\sup_{t\in I} \gamma_j(t))^i \gamma_j(x) \leqq \sup_{t\in I} \varkappa_{j,k}(t) (\sup_{t\in I} \gamma_j(t))^{i+1}$$

Damit ist der Beweis durchgeführt.

Satz 5,8. *Es gelte* $(*)$. *Es sei* $a_{s+1} = \infty$ *für irgedein Paar* s, k, $0 \leqq s \leqq n - 1$ $1 \leqq k \leqq n$ *und es existiere wenigstens ein* i, $1 \leqq i \leqq n$ *so, daß* $a_i = a$. *Nehmen wir an, daß*

$$\gamma_{s+1}(x) < \infty, \tag{5,9}$$

$$\varkappa_{s+1,k}(x) < \infty \tag{5,10}$$

gilt. Dann konvergiert di Reihe (4,3) gleichmäßig auf jedem Intervall $\langle a, \infty)$, *dessen Endpunkt* a *die Bedingung* $\gamma_{s+1}(a) < 1$ *erfüllt.*

Beweis. Unter den Voraussetzungen (5,9), (5,10) sind die Funktionen $\gamma_{s+1}(x)$ und $\varkappa_{s+1,k}(x)$ endlich, stetig und abnehmend. Es gibt also eine Zahl $a > x_0$ so, daß $\gamma_{s+1}(x) \leqq \gamma_{s+1}(a) < 1$ für $x \in \langle a, \infty)$ ist. Nach Hilfssatz 5,7 gilt für $x \geqq a$

$$\| U_{s+1}^i(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,k}(a) \, \gamma_{s+1}^i(a). \tag{5,11}$$

Wegen $\gamma_{s+1}(a) < 1$, ist die geometrische Reihe $\sum\limits_{i=0}^{\infty} \varkappa_{s+1,k}(a) \, \gamma_{s+1}^i(a)$ eine konvergente Majorante der Reihe (4,3). Die Reihe (4,3) konvergiert also gleichmäßig im Intervall $\langle a, \infty)$.

Satz 5,9. *Es seien die Voraussetzungen des Satzes 5,8 erfüllt. Dann gilt für* $x \geqq a$

$$\| L_s \mathbf{Y}_k(x) - \sum_{i=0}^{n} U_{s+1}^i(H_{s+1,k}) \| \leqq \gamma_{s+1}^{n+1}(a) \frac{\varkappa_{s+1,k}(a)}{1 - \gamma_{s+1}(a)}. \tag{5,12}$$

Beweis. Wir bezeichnen

$$\mathrm{R}_{n+1}(x) = \sum_{i=n+1}^{\infty} U_{s+1}^i(\mathbf{H}_{s+1,k}).$$

Nach (5,11) gilt

$$\| R_{n+1}(x) \| \leqq \varkappa_{s+1,k}(a) \sum_{i=n+1}^{\infty} \gamma_{s+1}^{i}(a) = \gamma_{s+1}^{n+1}(a) \frac{\varkappa_{s+1,k}(a)}{1 - \gamma_{s+1}(a)},$$

womit die Ungleichung (5,12) bewiesen ist.

Satz 5,10. *Es gelte* (*). *Es sei* $a_{s+1} = a$ *und es existiere ein* i, $1 \leqq i \leqq n$ *so, daß* $a_i = \infty$. *Es sei* a_l *die erste Zahl* $a_{s+1}, \ldots, a_{n+s}$ *für welche* $a_l = \infty$. *Nehmen wir an, daß*

$$\gamma_l(x) < \infty, \tag{5,13}$$

$$\varkappa_{l,k}(x) < \infty, \tag{5,14}$$

gilt. Dann konvergiert die Reihe (4,3) *gleichmäßig in jedem Intervall* $\langle a, b \rangle$, *dessen Endpunkt* a *die Bedingung* $\gamma_l(a) < 1$ *erfüllt.*

Beweis. Durch die Verwendung des Hilfssatzes 5,7 kann man leicht folgende Behauptungen ableiten.

1° Wenn $s + 1 < l \leqq k$ oder $l \leqq k < s + 1$ oder $k < s + 1 < l$ ist, dann gilt für $x \in \langle a, b \rangle$

$$\| U_{s+1}^{i}(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{i}(a)\, \varkappa_{l,k}(a), \quad i = 0, 1, \ldots \tag{5,15}$$

2° Wenn $l < s + 1 \leqq k$ oder $s + 1 \leqq k < l$ oder $k < l < s + 1$ ist, dann gilt für $x \in \langle a, b \rangle$

$$\| U_{s+1}^{i}(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{i-1}(a)\, \varkappa_{l,k}(a), \quad i = 1, 2, \ldots \tag{5,16}$$

In den beiden Fällen existiert unter der Voraussetzung $\gamma_l(a) < 1$ konvergente geometrische Reihe, die Majorante der Reihe (4,3) ist. Die Reihen (4,3) konvergieren also gleichmäßig im Intervall $\langle a, b \rangle$.

Satz 5,11. *Es seien die Voraussetzungen des Satzes 5,10 erfüllt. Dann gilt für* $x \in \langle a, b \rangle$

$$\| L_s \mathbf{Y}_k(x) - \sum_{i=0}^{n} U_{s+1}^{i}(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{n+1}(a) \frac{\varkappa_{l,k}(a)}{1 - \gamma_l(a)}, \tag{5,17}$$

$$\| L_s \mathbf{Y}_k(x) - \mathbf{H}_{s+1,k}(x) - \sum_{i=1}^{n} U_{s+1}^{i}(\mathbf{H}_{s+1,k}) \| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{n}(a) \frac{\varkappa_{l,k}(a)}{1 - \gamma_l(a)} \tag{5,18}$$

Der Beweis kann analog wie bei Satz 5,9 durchgeführt werden.

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A REMARK ON THE OSCILLATORY BEHAVIOUR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF ORDER 3 AND 4

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Consider a differential equation of the form

$$y''' + q(x)y' + r(x)y = f(x) \tag{1}$$

with $q(x) \in C_1 \langle x_0, \infty)$, $r(x) \in C_0 \langle x_0, \infty)$, $f(x) \in C_0 \langle x_0, \infty)$ and $\int\limits_{x_0}^{\infty} |f(t)| \, dt < \infty$ where $x_0 \in (-\infty, \infty)$. Suppose further that

$$F(y(x)) = y(x)y''(x) + \frac{1}{2} q(x) \, y^2(x) - \frac{1}{2} y'^2(x),$$

where $y(x)$ is a solution of (1). Then we have

Theorem 1. (*Theorem* 4 *in* [3]). *Let for any* $x \in \langle x_0, \infty)$ *the following condition hold:*

$$q(x) \geqq 0, \; r(x) \geqq k_1 > 0, \; 2r(x) - q'(x) - 1 \geqq k_2 > 0, \; \left| \int\limits_{x_0}^{x} f(t) dt \right| \leqq K < \infty.$$

If $y(x)$ *is a solution of* (1) *such that*

$$F(y(x_0)) + \frac{1}{2} \int_{x^0}^{\infty} f^2(t) \, dt \leqq 0,$$

then $y(x)$ *is oscillatory or* $\lim\limits_{x \to \infty} y(x) = 0$.

The following result is of the similar character

Theorem 2. *For any* $x \in \langle x_0, \infty)$, *let the following conditions hold:*

$$q(x) \geqq 0 \text{ and } 2r(x) - q'(x) - |f(x)| \geqq 0.$$

If

$$\int\limits_{x_0}^{\infty} q(t) \, dt = +\infty, \tag{2}$$

then a solution $y(x)$ *of* (1) *which satisfies*

$$F(y(x_0)) + \frac{1}{2} \int_{x_0}^{\infty} |f(t)| \, dt \leqq 0$$

is oscillatory or $\lim\limits_{x \to \infty} y(x) = 0$.

Proof. Suppose that the hypotheses hold and that $y(x)$ is not oscillatory. Thus there exists a number $x_1 \geqq x_0$ such that $y(x) \neq 0$ for $x \in \langle x_1, \infty)$. Them from (1) we have

$$F(y(x)) + \int_{x_0}^{x} \left[r(t) - \frac{1}{2} q'(t) - \frac{1}{2} \mid f(t) \mid\right] y^2(t)\, dt \leqq F(y(x_0)) + \frac{1}{2} \int_{x_0}^{x} \mid f(t) \mid dt$$

and therefore

$$y(x) y''(x) - \frac{1}{2} y'^2(x) \leqq -\frac{1}{2} q(x)\, y^2(x). \tag{3}$$

From (3) we get

$$y(x)\, y''(x) - y'^2(x) \leqq y(x)\, y''(x) - \frac{1}{2} y'^2(x) \leqq -\frac{1}{2} q(x)\, y^2(x)$$

thus for $x \geqq x_1$

$$\frac{d}{dx}\left(\frac{y'(x)}{y(x)}\right) \leqq -\frac{1}{2} q(x)$$

and therefore

$$\frac{y'(x)}{y(x)} \leqq \frac{y'(x_1)}{y(x_1)} - \frac{1}{2} \int_{x_1}^{x} q(t)\, dt. \tag{4}$$

Since (2) holds, there exists $x_2 \geqq x_1$ such that for $x \geqq x_2$ from (4) we have

$$\frac{y'(x)}{y(x)} \leqq -k, \text{ where } k > 0. \tag{5}$$

Suppose that $y(x) > 0$ for $x \geqq x_1$. From (5) we have $y'(x) < 0$ for $x \geqq x_2$. Therefore it is necessary that $y(x) \geqq C = \lim_{x \to \infty} y(x) \geqq 0$ for any $x \geqq x_2$. Let $C > 0$. Then for $x \geqq x_2$:

$$\frac{y'(x)}{C} \leqq \frac{y'(x)}{y(x)} \leqq -k,$$

so that $y(x) \to -\infty$ for $x \to \infty$ which is a contradiction. Hence $C = 0$ and $\lim_{x \to \infty} y(x) = 0$.

The following part of this paper is concerned with the oscillatory behaviour of solutions of the differential equation

$$y^{(4)} + p(x) y'' + q(x)\, y' + r(x)\, y = f(x), \tag{6}$$

with $p(x) \in C_0 \langle x_0, \infty)$, $q(x) \in C_1 \langle x_0, \infty)$, $r(x) \in C_0 \langle x_0, \infty)$, $f(x) \in C_0 \langle x_0, \infty)$ and $\int_{x_0}^{\infty} \mid f(t) \mid dt < \infty$, where $x_0 \in (-\infty, \infty)$. Suppose further that

$$F_1(y(x)) = y(x)\, y'''(x) - y'(x)\, y''(x) + \frac{1}{2} q(x)\, y^2(x),$$

where $y(x)$ is a solution of (6). Then we have

Theorem 3. *Suppose that for all* $x \in \langle x_0, \infty)$

$$q(x) \geqq 0, \mid p(x) \mid \leqq 2, 2r(x) - \mid p(x) \mid - q'(x) - \mid f(x) \mid \geqq 0.$$

If (2) holds, then a solution y(x) of (6) which satisfies

$$F_1(y(x_0)) + \frac{1}{2} \int_{x_0}^{\infty} \mid f(t) \mid \mathrm{d}t \leqq 0 \tag{7}$$

is oscillatory on $\langle x_0, \infty)$.

Proof. Suppose that a solution $y(x)$ of (6) satisfies (7) and that $y(x) \neq 0$ for $x \in \langle x_1, \infty)$, $x_1 \geqq x_0$. From (6) we get

$$F_1(y(x)) + \int_{x_0}^{x} \left[1 - \frac{1}{2} \mid p(t) \mid\right] y''^2(t) \, \mathrm{d}t + \int_{x_0}^{x} \left[r(t) - \frac{1}{2} \mid p(t) \mid - \frac{1}{2} q'(t) - \right.$$

$$\left. - \frac{1}{2} \mid f(t) \mid \right] y^2(t) \, \mathrm{d}t \leqq F_1(y(x_0)) + \frac{1}{2} \int_{x_0}^{x} \mid f(t) \mid \mathrm{d}t$$

and therefore

$$y(x)\, y'''(x) - y'(x)\, y''(x) \leqq -\frac{1}{2} q(x)\, y^2(x)$$

thus for $x \geqq x_1$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y''(x)}{y(x)} \right) \leqq -\frac{1}{2} q(x)$$

and hence

$$\frac{y''(x)}{y(x)} \to -\infty \text{ for } x \to \infty. \tag{8}$$

Suppose that $y(x) > 0$ for $x \geqq x_1$. From (8) we can see that $y''(x) < 0$ for $x \in \langle x_2, \infty)$ with suitable $x_2 \geqq x_1$. Since $y'(x)$ is monotonous, only the following two cases are possible:

1) $y'(x) > 0$ for all $x > x_2$

2) there exists $x_3 \geqq x_2$ such that $y'(x_3) < 0$.

Evidently in the second case there exists $\xi \geqq x_2$ such that $y(\xi) = 0$ which contradicts the hypothesis. Therefore let $y'(x) > 0$ for all $x \geqq x_2$; thus $y(x)$ is an increasing function on $\langle x_2, \infty)$ and therefore

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \text{ for } x \in \langle x_2, \infty),$$

so that, owing to (8), $\lim_{x \to \infty} y''(x) = -\infty$, which is agai contradictory to the assumption that $y(x) > 0$ for $x \geqq x_1$.

Analogously we prove that (6) has no solution $y(x)$ satisfying (7) such that $y(x) < 0$ for all $x \geqq x_1 \geqq x_0$.

This completes the proof.

Remark. *Theorem* 3 *is a generalization of certain theorems in* [1] *and* [2] *and of Theorem* 6 *in* [4] *which deal with equations*

$$y^{(4)} + 2A(x)\,y' + [b(x) + A'(x)]\,y = 0,$$

or

$$y^{(4)} + q(x)\,y' + r(x)\,y = f(x).$$

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КАНОНИЧЕСКИЕ ПРЕДСТАВЛЕНИЯ И НОМОГРАММЫ ЭЛЕМЕНТАРНЫХ (ПРЕИМУЩЕСТВЕННО) НОМОГРАФИРУЕМЫХ ФУНКЦИЙ КЛИФФОРДОВА КОМПЛЕКСНОГО АРГУМЕНТА[1]

Павел Галайда, Кошице

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§ 0. Сделаем несколько предварительных замечаний.

В силу суждения 1 стр. 200 работы [4] каноническими представлениями аналитической зависимости

$$F(\omega; z) = 0, \tag{0.0}$$

где $z = a + bi$,[2] $\omega = p + qi$, $i = \sqrt{-1}$ (гауссов случай) первого номографического класса, называется пара совместных вещественных уравнений Коши

$$\begin{cases} S(p)\,X(a) + Y(b) + H(p) = O, \\ T(q)\,X(a) + Y(b) + R(q) = O, \end{cases} \tag{0.1}$$

где $X(a)$ a $Y(b)$ — прямолинейные, а $S(p)$, $H(p)$, $T(q)$, $R(q)$ — криволинейные, вообще говоря, характеристики (см. там же определение 2).

Если

$$F(X, Y) = O, \tag{0.2}$$

уравнение общего канонического носителя p и q, а $Y = O$ и $X = O$ — уравнения шкал a и b, то, поскольку из предыдущих параграфов видно, что в результате преобразования принципа перенесения

$$\begin{cases} a \to \tilde{a}, \\ b \to \varepsilon i\tilde{b}, \\ p \to \tilde{p}, \\ q \to -\varepsilon i\tilde{q} \\ X(a) \to \tilde{X}(\tilde{a}) = X(\tilde{a}), \\ Y(a) = O \to \tilde{Y}(\tilde{a}) = O, \end{cases} \tag{0.3}$$

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[1]) Эта статья является второй частью работы. Первая часть вместе с литературой была опубликована в Arch. Math. 4, VII: 167 — 188, 1971.

[2]) Отметим, что вместо $z = a + bi$ автор в следующих параграфах настоящей работы пишет $z = x + yi$.

$$(0.3)\qquad \begin{cases} Y(b) & \to \quad \tilde{Y}(\tilde{b}) = Y(-\varepsilon i\tilde{b}), \\ X(b) = O & \to \quad \tilde{X}(\tilde{b}) = O, \\ S(p) & \to \quad \tilde{S}(\tilde{p}) = S(\tilde{p}), \\ H(p) & \to \quad \tilde{H}(\tilde{p}) = H(\tilde{p}), \\ T(q) & \to \quad \tilde{T}(\tilde{q}) = T(-\varepsilon i\tilde{q}), \\ R(q) & \to \quad \tilde{R}(\tilde{q}) = R(-\varepsilon i\tilde{q}), \end{cases}$$

все функции $\tilde{X}(\tilde{a})$, $\tilde{Y}(\tilde{b})$, $\tilde{S}(\tilde{p})$, $\tilde{H}(\tilde{p})$, $\tilde{T}(\tilde{q})$, $\tilde{R}(\tilde{q})$ для всевозможных вещественных $\tilde{a}$, $\tilde{b}$, $\tilde{p}$, $\tilde{q}$ имеют вещественные значения, то и номограмма для уравнения

$$(0.4)\qquad \tilde{F}(w;\, \tilde{z}) = O,$$

соответствующего по принципу перенесения уравнению (0.0) (а мы только такие уравнения (0.0) и (0.4) рассматривали) определяется каноническими представлениями

$$(0.5)\qquad \begin{aligned} \tilde{S}(\tilde{p})\, \tilde{X}(\tilde{a}) + \tilde{Y}(\tilde{b}) + \tilde{H}(\tilde{p}) = O, \\ \tilde{T}(\tilde{q})\, \tilde{X}(\tilde{a}) + \tilde{Y}(\tilde{b}) + \tilde{R}(\tilde{q}) = O, \end{aligned}$$

получаемыми из (0.1) нормальным применением преобразования (0.3) и, следовательно, эта номограмма имеет своим носителем *то же самое каноническое сечение* (0.2), несущее шкалы $\tilde{p}$ и $\tilde{q}$, *и те же самые прямые* (оси координат в данном проективном преобразовании)

$$(0.6)\qquad Y = O,\ X = O,$$

на которых расположены соответственно $\tilde{a}$ и $\tilde{b}$.

Тем не менее, как это не трудно усмотреть из сопоставления формул всех параграфов предыдущей работы с формулами соответствующих, т. е. одинаково занумерованных параграфов настоящей работы совпадения градуировок номограмм соответствующих по принципу перенесения уравнений

$$(0.7)\qquad F(w;\, z) = O,\ \tilde{F}(\tilde{w};\, \tilde{z}) = O,$$

для соответствующих аргументов b и $\tilde{b}$, q и $\tilde{q}$ не происходит, а эти градуировки дополняют одна другую до полного покрытия несущей их линии. Градуировки же a и $\tilde{a}$, p и $\tilde{p}$ — совпадают. Возникает естественный вопрос возникает ли то, что носитель (скелет, т. е. носители всех шкал без градуировок) номограмм, соответствующих между собой по принципу перенесения уравнений (0.7) — общий. Или же это обстоятельство должно в каждом отдельном случае отдельно доказываться фактической проверкой того, что получается в каждой паре соответствующих между собой по принципу перенесения случаев (0.7) одно и то же вещественное уравнение (0.2) для общего конического носителя шкал p, q, $\tilde{p} = p$, $\tilde{q}$ и пара прямолинейных шкал, из которых одна шкала несет инцидентные градуировки для $a = \tilde{a}$, а другая — неинцидентные градуировки b и $\tilde{b}$.

Общее доказательство кажется на первый взгляд излишним, поскольку в силу (0.1) за координаты точек шкал ($\tilde{X}$ и $\tilde{Y}$) номограммы уравнения (0.0) принимаются (см. § 1, начало стр. 201 работы [4] надо исправить ошибку, поменяв местами X_b и Y_b)

$$(0.8)\quad \begin{aligned} &X_a = \frac{1}{X(a)}, \qquad Y_a = O; \qquad X_b = O, \qquad Y_b = \frac{1}{Y(b)};\\ &X_p = -\frac{S(p)}{H(p)}, \qquad Y_p = -\frac{1}{H(p)}; \qquad X_q = -\frac{T(q)}{R(q)}, \qquad Y_q = -\frac{1}{R(q)} \end{aligned}$$

а за координаты точек шкал номограмм соответствующего по принципу перенесения уравнения (0.4) принимаются аналогичные выражения, составленные при помощи канонических представлений (0.5). Но тогда в силу (0.3) инцидентивность носителей шкал a и $\tilde{a}$, p и $\tilde{p}$, b и $\tilde{b}$, q и $\tilde{q}$ представляется, на первый взгляд, очевидной, т. к. вместо того, чтобы трактовать переход от (0.0) и (0.1) к (0.4) и (0.5), как преобразование (0.3), т. е. (выписываем главную часть преобразования (0.3)):

$$(0.9)\quad \begin{cases} a \to \tilde{a},\\ b \to -\varepsilon i\tilde{b},\\ p \to \tilde{p},\\ q \to -\varepsilon i\tilde{q}, \end{cases}$$

мы можем, казалось бы, говорить просто о замене

$$(0.10)\quad \begin{cases} a = \tilde{a},\\ b = -\varepsilon i\tilde{b},\\ p = \tilde{p},\\ q = -\varepsilon i\tilde{q}, \end{cases}$$

а тогда вместо (0.3) имеем ли мы просто равенства

$$(0.11)\quad \begin{cases} X(a) = \tilde{X}(\tilde{a}),\\ Y(a) = O = Y(\tilde{a}),\\ X(b) = O = \tilde{X}(\tilde{b}),\\ Y(b) = \tilde{Y}(\tilde{b}),\\ S(p) = \tilde{S}(\tilde{p}),\\ H(p) = \tilde{H}(\tilde{p}),\\ T(q) = \tilde{T}(\tilde{q}),\\ R(q) = \tilde{R}(\tilde{q}), \end{cases}$$

в силу которых уравнение (0.2), связывающее X и Y, немедленно превращается в точно такое же уравнение

$$(0.12)\quad F(\tilde{X};\ \tilde{Y}) = O,$$

связывающее X и Y.

Тем не менее это не верно, т. к. мы имеем не замены (0.10) и их следствия (0.11), а именно преобразования (0.9) и их следствия (0.3), т. к. a, b, p, q —

с одной стороны и $\tilde{a}$, $\tilde{b}$, $\tilde{p}$, $\tilde{q}$ — с другой стороны, имеют строго вещественные значения и поэтому трактование преобразования (0.9) как замену переменных по формулам (0.10), ввиду явного *несовпадения областей значений правых и левых частей последних равенств, является незаконным*, во всяком случае, не является законным без привлечения дополнительных обоснований, связанных с принципом аналитического продолжения для аналитических функций гиперкомплексного переменного $a + bi + c\varepsilon + d\varepsilon i$. Ведь вопрос приводится к следующему.

Имеется соотношение (0.2), в котором X и Y являются в силу (0.8) вещественными функциями вещественных параметров либо p либо q, причем как в первом, так и во втором случае вещественная функция двух вещественных переменных $F(u; v)$ одна и та же. Спрашивается можно ли считать соотношение (0.2) сохранившимся, если в X и Y в (0.2) вещественные аргументы p и q заменяются аргументами $\tilde{p}$ и $-\varepsilon i\tilde{q}$, где $\tilde{p}$ вещественно и $\tilde{q}$ вещественно и X, $X(p)$ и Y и $Y(p)$, $X(q)$, $Y(q)$ остаются вещественными? Иначе говоря, речь идет о сохранении соотношения (0.2) при аналитическом продолжении X и Y на всю плоскость гиперкомплексного аргумента $a + bi + c\varepsilon + d\varepsilon i$?

Это действительно имеет место, поскольку функции $X(p)$, $Y(p)$, $X(q)$, $Y(q)$ — аналитические функции гиперкомплексного аргумента $a + bi + c\varepsilon + d\varepsilon i$, что и имеет место, т. к. эллиптические функции, как степенные ряды, являются аналитическими функциями.

Другой подход заключается в том, что выражают $\tilde{X}(\tilde{q})$ и $\tilde{Y}(\tilde{q})$ через $X(p)$ и $Y(p)$ или $X(q)$ и $Y(q)$, удовлетворяющие соотношению (0.2), установив соответствующее вещественное соотношение между вещественными аргументами $\tilde{q}$ и p или $\tilde{q}$ и q. Заменив затем в (0.2) $X(p)$ и $Y(p)$ или $X(q)$ и $Y(q)$ через установленные их выражения через $\tilde{X}(\tilde{q})$ и $\tilde{Y}(\tilde{q})$ получают уравнение, связывающее $X(q)$ и $Y(q)$, которое оказывается таким же уравнением (0.2), иначе говоря, неградуированные скелеты номограмм соответствующих в силу принципа перенесения функции гауссова и клиффордова аргументов совпадают.

Это заключение позволяет утверждать, что и соответствующие пучки конических номограмм соответствующих гауссовых и клиффордовых зависимостей (0.7) также совпадают между собой.

Поэтому характеристику пучков конических номограмм пар зависимостей (0.7) мы можем дать на основе пучков номограмм гауссовых уравнений (0.7_1), описанных И. А. Вильнером в соответствующих параграфах главы IV основной работы [4].

§ 1. Мы будем считать ниже γ и N вещественными или чисто мнимыми числами. Тогда, выполняя прямое преобразование перенесения над формулами (1.1), (1.2), (1.4), (1.5) предыдущей работы, получим

$$\tilde{A} + \tilde{B}\varepsilon = \ln \sin (\tilde{P} + \tilde{Q}\varepsilon), \tag{1.1}$$

где

$$\begin{aligned} &\tilde{A} = \operatorname{Re} [\tilde{\gamma}(\tilde{z} - \tilde{z}_0)],\ \tilde{B} = \operatorname{Jm} [\tilde{\gamma}(\tilde{z} - \tilde{z}_0)],\ \tilde{P} = \operatorname{Re} [\tilde{N}(\tilde{w} - \tilde{w}_0)], \\ &\tilde{Q} = \operatorname{Jm} (\tilde{N}(\tilde{w} - \tilde{w}_0)], \end{aligned} \tag{1.2}$$

Пусть $\tilde{U}$ — прямоугольная матрица 3×4 из векторов-столбцов, определяющих 4 шкалы номограммы. Найдем $\tilde{U}$, мы, прежде всего, отметим, что принимая принцип перенесения (см. [1]), мы, наряду с прямым преобразованием перенесения

$$(1.2') \qquad \left\{\begin{array}{ccl} \gamma & \to & \tilde{\gamma}, \\ z & \to & \tilde{z}, \\ w & \to & \tilde{w}, \\ \gamma^2 & \to & \tilde{\gamma}^2, \\ \gamma_z & \to & \tilde{\gamma}_{\tilde{z}}, \\ A & \to & \tilde{A}, \\ B & \to & \tilde{B}, \end{array}\right.$$

получаем таблицу, полезную для практических преобразований при помощи принципа перенесения

$$(1.2'') \qquad \left\{\begin{array}{rcl} \cos 2P & \to & \cos 2\tilde{P}, \\ \sin 2P & \to & \sin 2\tilde{P}, \\ \sin^2 2P & \to & \sin^2 2\tilde{P}, \\ \cos^2 2P & \to & \cos^2 2\tilde{P}, \\ \operatorname{sh} 2P & \to & \operatorname{sh} 2\tilde{P}, \\ \operatorname{ch} 2P & \to & \operatorname{ch} 2\tilde{P}, \\ \sin 2Q & \to & -\varepsilon i \operatorname{sh} 2\tilde{Q}, \\ \sin^2 2Q & \to & -\operatorname{sh}^2 2\tilde{Q}, \\ \cos 2Q & \to & \operatorname{ch} 2\tilde{Q}, \\ \cos^2 2Q & \to & \operatorname{ch}^2 2\tilde{Q}, \\ \operatorname{tg}^2 2Q & \to & -\operatorname{th}^2 2\tilde{Q}, \\ \operatorname{ctg}^2 2Q & \to & -\operatorname{cth}^2 2\tilde{Q}, \\ \operatorname{sh} 2Q & \to & -\varepsilon i \sin 2\tilde{Q}, \\ \operatorname{sh}^2 2Q & \to & -\sin^2 2\tilde{Q}, \\ \operatorname{ch} 2Q & \to & \cos 2\tilde{Q}, \\ \operatorname{ch}^2 2Q & \to & \cos^2 2\tilde{Q}, \\ \operatorname{th}^2 2Q & \to & -\operatorname{tg}^2 2\tilde{Q}. \end{array}\right.$$

Само собой разумеется, что в таблице (1.2″) можно коэффициент „2“ заменить 1 или любым другим числом. Мы взяли коэффицент, равный 2 только потому, что этот случай нужен для наших формул.

Соотношение (1.4) предыдущей работы примет вид матрицы, определяя номограмму уравнения (1.1).

Подвергая прямому преобразованию перенесения

$$(1.3)\qquad \begin{cases} A \to \tilde{A}, \\ B \to -\varepsilon i\tilde{B}, \\ P \to \tilde{P}, \\ Q \to -\varepsilon i\tilde{Q}, \end{cases}$$

получим на основании (1.16) предыдущей работы

$$(1.4)\qquad \begin{Vmatrix} e^{2A} & O & \dfrac{\sin^2 2\tilde{P}}{2\cos 2\tilde{P}} & -\dfrac{\sin^2 2\tilde{Q}}{2\cos 2\tilde{Q}} \\ O & \dfrac{1}{\operatorname{ch} 2\tilde{B}} & -\dfrac{1}{\cos 2\tilde{P}} & \dfrac{1}{\cos 2\tilde{Q}} \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

Если заменить в (1.4) согласно (1.2)

$$(1.5)\qquad \tilde{A} = \mathrm{Re}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)] = \mathrm{Re}[\tilde{\gamma}(\tilde{z})] - \mathrm{Re}[\tilde{\gamma}\tilde{z}_0],$$

и затем умножить первую строку матрицы (1.4) на $e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]}$ то и придем к матрице U, фигурирующей вторым множителем в левой части матричного равенства

$$(1.6)\qquad A\tilde{U} = \tilde{U}',$$

или более подробно

После упрощений, при которых номограмма (1.6) претерпевает коллиниарные преобразования (умножение и деление рядов матрицы на отличные от нуля множители), получим, опуская эти множители, следующее выражение матрицы $\tilde{U}'$ следующую номограмму уравнения (1.1) настоящей работы:

$$(1.7)\qquad \tilde{U}' = \begin{Vmatrix} \tilde{\gamma}^2 & -1 + \operatorname{ch} 2\tilde{B} & \dfrac{\tilde{\gamma}^2}{\sin^2 \tilde{P}} & \dfrac{\tilde{\gamma}^2}{\cos^2 \tilde{Q}} \\ \tilde{\gamma}^2 & 1 + \operatorname{ch} 2\tilde{B} & -\dfrac{\tilde{\gamma}^2}{\cos^2 2\tilde{P}} & -\dfrac{\tilde{\gamma}^2}{\sin^2 \tilde{Q}} \\ e^{2\mathrm{Re}[\widetilde{\gamma}(\tilde{z} - \widetilde{z}_0)]} & 0 & 1 & 1 \end{Vmatrix}.$$

Заметим, что непосредственно отсюда или применяя преобразование принципа перенесения к равенствам (1.14) (1.15) предыдущей работы, получим следующие канонические представления и уравнения шкал и носителя номограммы зависимости (1.1) настоящей работы

$$(1.8)\qquad \begin{cases} \left(-\dfrac{\sin^2 2\tilde{P}}{2}\, e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]}\right)\left(e^{-2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}]}\right) + (\operatorname{ch} 2\tilde{B}) + (\cos 2\tilde{P}) = 0, \\ \left(-\dfrac{\sin^2 2\tilde{Q}}{2}\, e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]}\right)\left(e^{-2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}]}\right) + (\operatorname{ch} 2\tilde{B}) + (-\cos 2\tilde{Q}) = 0, \end{cases}$$

что соответствует (1.14) предыдущей работы

$$(1.9)\quad \begin{cases} X_{\widetilde{A}} = e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}]}; & Y_{\widetilde{A}} = O; \quad X_{\widetilde{B}} = O, \quad Y_{\widetilde{B}} = -\dfrac{1}{\mathrm{ch}\, 2\widetilde{B}}, \\ X_{\widetilde{P}} = \dfrac{\sin^2 2\widetilde{P}}{2\cos 2\widetilde{P}} e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]}, & Y_{\widetilde{P}} = -\dfrac{1}{\mathrm{ch}\, 2\widetilde{P}}, \\ X_{\widetilde{Q}} = -\dfrac{\sin^2 2\widetilde{Q}}{2\cos 2\widetilde{Q}} e^{2\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]}, & Y_{\widetilde{Q}} = \dfrac{1}{\cos 2\widetilde{Q}}; \end{cases}$$

$$Y^2_{(\widetilde{P};\widetilde{Q})} + 2e^{-\mathrm{Re}[\widetilde{\gamma}\widetilde{z}_0]} X_{(\widetilde{P};\widetilde{Q})} Y_{(\widetilde{P};\widetilde{Q})} - 1 = 0,$$

что соответствует (1.15) предыдущей работы.

§ 2. Применяя прямой принцип перенесения (см. работу [1]) к формулам § 2 предыдущей работы, получим зависимость

$$(2.1)\qquad \widetilde{z} - \widetilde{z}_0 = \frac{1}{\widetilde{\gamma}} \sin(\widetilde{P} + \widetilde{Q}\varepsilon),$$

и ее канонические представления

$$(2.2)\quad \begin{cases} \left(\dfrac{\widetilde{\gamma}^2}{\sin^2 \widetilde{P}}\right)\left[\dfrac{\mathrm{Re}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)]}{\widetilde{\gamma}}\right]^2 + \left(\dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{P}}\right)\left[\dfrac{\mathrm{Jm}\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)}{\widetilde{\gamma}}\right]^2 = 1, \\ \left(\dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{Q}}\right)\left[\dfrac{\mathrm{Re}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0]}{\widetilde{\gamma}}\right]^2 + \left(\dfrac{\widetilde{\gamma}}{\sin^2 \widetilde{Q}}\right)\left[\dfrac{\mathrm{Jm}\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)}{\widetilde{\gamma}}\right]^2 = 1. \end{cases}$$

Уравнения шкал номограммы зависимости (2.1) будут, очевидно,

$$(2.3)\quad \begin{cases} X_{\widetilde{a}} = \left\{\dfrac{\widetilde{\gamma}}{\mathrm{Re}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)]}\right\}^2, \quad Y_{\widetilde{a}} = 0; \quad X_{\widetilde{b}} = 0; \quad Y_{\widetilde{b}} = \left\{\dfrac{\widetilde{\gamma}}{\mathrm{Jm}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)]}\right\}^2, \\ X_{\widetilde{P}} = \dfrac{\widetilde{\gamma}^2}{\sin^2 \widetilde{P}}, \quad Y_{\widetilde{P}} = \dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{P}}; \quad X_{\widetilde{Q}} = \dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{Q}}, \quad Y_{\widetilde{Q}} = \dfrac{\widetilde{\gamma}^2}{\sin^2 \widetilde{Q}}, \end{cases}$$

$$\frac{1}{X_{(\widetilde{P};\widetilde{Q})}} + \frac{1}{Y_{(\widetilde{P};\widetilde{Q})}} = \frac{1}{\widetilde{\gamma}^2}.$$

Обозначая через $\widetilde{V}$ матрицу номограммы (2.3), получим

$$(2.4)\qquad \widetilde{V} \equiv \begin{Vmatrix} \left\{\dfrac{\widetilde{\gamma}}{\mathrm{Re}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)]}\right\}^2 & 0 & \dfrac{\widetilde{\gamma}^2}{\sin^2 \widetilde{P}} & \dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{Q}} \\ 0 & \left\{\dfrac{\widetilde{\gamma}}{\mathrm{Jm}[\widetilde{\gamma}(\widetilde{z}-\widetilde{z}_0)]}\right\}^2 & \dfrac{\widetilde{\gamma}^2}{\cos^2 \widetilde{P}} & \dfrac{\widetilde{\gamma}^2}{\sin^2 \widetilde{Q}} \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

Теперь мы поставили перед собой задачу *найти* очевидного совмещения номограмм (1.7) и (2.4) зависимостей (1.1) и (2.1) настоящей работы, номографически совместных по $\widetilde{W}$, поскольку [см. (1.19) и (2.4) предудыщей работы] номографически совместных по W соответствующие им по принципу перенесения зависимостей (1.1) и (2.1) предыдущей работы.

Мы можем матрицы (1.7) и (2.4) настоящей работы переписать в результате допустимых преобразований (коллиниаций) следующим образом: В матрице (1.7) делим элементы 3 и 4 столбцов на $\tilde{\gamma}^2$, затем умножаем 3-ю строку на $\tilde{\gamma}^2$, потом делим элементы первого столбца на $\tilde{\gamma}^2 e^{2\mathrm{Re}[\tilde{\gamma}(\tilde{z}-\tilde{z}_0)]}$ потом вычитаем от элементов первой строки и прибавляем к элементам второй строки соответствующие элементы, т. е. 1 0 1 1 третьей строки. Затем умножим 3 и 4 столбцы соответственно на $\mathrm{tg}^2\,\tilde{P}$ и $\mathrm{ctg}^2\,\tilde{Q}$.
Тогда получим вместо (1.7)

$$(1.7') \qquad \tilde{U}' = \begin{Vmatrix} e^{-2\mathrm{Re}[\tilde{\gamma}(\tilde{z}-\tilde{z}_0)]}-1 & -1+\mathrm{ch}2\tilde{B} & 1 & 1 \\ e^{-2\mathrm{Re}[\tilde{\gamma}(\tilde{z}-\tilde{z}_0)]}+1 & 1+\mathrm{ch}2\tilde{B} & -\mathrm{tg}^4\,\tilde{P} & -\mathrm{ctg}^4\,Q \\ 1 & 0 & \mathrm{tg}^2\,\tilde{P} & \mathrm{ctg}^2\,\tilde{Q} \end{Vmatrix}.$$

Но легко видеть, что мы получим в точности этот результат, если подвергнем матрицу (1.19') предыдущей работы преобразованию принципа перенесения, заменим B на $-\varepsilon i\tilde{B}$, P на $\tilde{P}$, γ на $\tilde{\gamma}$, z и z_0 на $\tilde{z}$ и $\tilde{z}_0$ и, наконец, Q на $-\varepsilon i\tilde{Q}$.
Тогда, как легко видеть

$$(\mathrm{A}) \qquad \left\{ \begin{array}{rcl} e^{-2\mathrm{Re}[\gamma(z-z_0)]} & \to & e^{-2\mathrm{Re}[\tilde{\gamma}(\tilde{z}-\tilde{z}_0)]} \\ \cos 2B & \to & \mathrm{ch}\,2\tilde{B} \\ \mathrm{tg}\,P & \to & \mathrm{tg}\,\tilde{P}, \\ \mathrm{sh}\,Q & \to & -\varepsilon i \sin\tilde{Q}, \\ \mathrm{ch}\,Q & \to & \cos\tilde{Q}, \\ \mathrm{cth}\,Q & \to & \varepsilon i\,\mathrm{ctg}\,\tilde{Q}, \\ \mathrm{cth}^2 Q & \to & -\mathrm{ctg}^2\,\tilde{Q}, \\ \mathrm{ctg}^4\,Q & \to & \mathrm{ctg}^4\,\tilde{Q}, \end{array} \right.$$

и матрица U (1.19') предыдущей работы превращается в матрицу $\tilde{U}'$ (1.7') настоящей работы.

Пользуясь обозначениями (1.2) настоящей работы, мы перепишем (1.7') настоящей работы в виде

$$(1.7'') \qquad \tilde{U}' = \begin{Vmatrix} e^{-2\tilde{A}}-1 & -1+\mathrm{ch}\,2\tilde{B} & 1 & 1 \\ e^{-2\tilde{A}}+1 & 1+\mathrm{ch}\,2\tilde{B} & -\mathrm{tg}^4\,\tilde{P} & -\mathrm{ctg}^4\,Q \\ 1 & 0 & \mathrm{tg}^2\,\tilde{P} & \mathrm{ctg}^2\,\tilde{Q} \end{Vmatrix}.$$

Подвергая теперь матрицу $\tilde{V}$ (2.4) аналогично приведенному сейчас преобразованию матрицы $\tilde{U}'$ (1.7) к виду (1.7''), а именно — деля все столбцы на $\tilde{\gamma}^2$, затем умножая последнюю строку на $\tilde{\gamma}^2$, затем вычитая получившуюся последнюю строку единиц из первой и второй строк, умножая затем 3-ий и 4-ый столбцы на соответственно $\mathrm{tg}^2\,\tilde{P}$ и $\mathrm{ctg}^2\,\tilde{Q}$, и, наконец, умножая потом вторую строку на (—1), получим

$$(2.4'') \qquad \tilde{V} = \left\| \begin{matrix} \frac{1}{\tilde{A}^2} - 1 & -1 & 1 & 1 \\ 1 & -\frac{1}{\tilde{B}^2} + 1 & -\operatorname{tg}^4 \tilde{P} & -\operatorname{ctg}^4 \tilde{Q} \\ 1 & 1 & \operatorname{tg}^2 \tilde{P} & \operatorname{ctg}^2 \tilde{Q} \end{matrix} \right\|.$$

Легко видеть, что матрица V (2.4″) предыдущей работы превращается в точности в эту матрицу (2.4″) для $\tilde{V}$, если в матрице V предыдущей работы сделать преобразование принципа перенесения

$$(\text{Б}) \qquad \left\{ \begin{array}{rcl} A & \to & \tilde{A}, \\ B & \to & -\varepsilon i \tilde{B}, \\ P & \to & \tilde{P}, \\ Q & \to & -\varepsilon i \tilde{Q}, \\ \operatorname{tg} P & \to & \operatorname{tg} \tilde{P}, \\ \operatorname{ch} Q & \to & \cos \tilde{Q}, \\ \operatorname{sh} Q & \to & -\varepsilon i \sin \tilde{Q}, \\ \operatorname{cth} Q & \to & \varepsilon i \operatorname{ctg} \tilde{Q}, \\ \operatorname{cth}^2 Q & \to & -\operatorname{ctg}^2 \tilde{Q}, \\ \operatorname{cth}^4 Q & \to & \operatorname{ctg}^4 \tilde{Q}. \end{array} \right.$$

Матрица (2.4″) предыдущей работы в результате преобразования (Б) примет вид

$$V \to \left\| \begin{matrix} \frac{1}{\tilde{A}^2} - 1 & -1 & 1 & 1 \\ 1 & -\frac{1}{\tilde{B}^2} + 1 & -\operatorname{tg}^4 \tilde{P} & -\operatorname{ctg}^4 \tilde{Q} \\ 1 & 1 & \operatorname{tg}^2 \tilde{P} & \operatorname{ctg}^2 \tilde{Q} \end{matrix} \right\|,$$

что в точности совпадает с матрицей (2.4″). Вернемся к формулам (2.3). Сопоставление их с формулами (2.3) предыдущей работы обнаруживаем несовпадение уравнений носителей шкал P и Q, $\tilde{P}$ и $\tilde{Q}$. Это кажущееся противоречие нашим общим выводам в вводном параграфе этой работы получилось только потому, что при преобразовании перенесения канонических представлений (2.2) зависимости (2.1) предыдущей работ в канонические представления (2.2) зависимости (2.1) настоящей работы мы переносили множитель минус единицу из множителя $\left[\frac{\operatorname{Jm}\tilde{\gamma}(\tilde{z} - \tilde{z}_0)}{\tilde{\gamma}}\right]^2$ в множитель $\frac{\tilde{\gamma}^2}{\cos^2 \tilde{P}}$ и в $\frac{\tilde{\gamma}^2}{\sin^2 \tilde{Q}}$.

Итак, вместо (2.2) должны из (2.2) предыдущей работы получить

$$(2.5)\quad \begin{cases} \left(\dfrac{\tilde\gamma^2}{\sin^2\tilde P}\right)\left[\dfrac{\mathrm{Re}[\tilde\gamma(\tilde z-\tilde z_0)]}{\tilde\gamma}\right]^2+\left(-\dfrac{\tilde\gamma^2}{\cos^2\tilde P}\right)\left\{-\left[\dfrac{\mathrm{Jm}[\tilde\gamma(\tilde z-\tilde z_0)]}{\tilde\gamma}\right]^2\right\}=1,\\ \left(\dfrac{\tilde\gamma^2}{\cos^2\tilde Q}\right)\left[\dfrac{\mathrm{Re}[\tilde\gamma(\tilde z-\tilde z_0)]}{\tilde\gamma}\right]^2+\left(-\dfrac{\tilde\gamma^2}{\sin^2\tilde Q}\right)\left\{-\left[\dfrac{\mathrm{Jm}\tilde\gamma(\tilde z-\tilde z_0)}{\tilde\gamma}\right]^2\right\}=1,\end{cases}$$

откуда

$$(2.6)\quad \begin{cases} X_{\tilde a}=\left\{\dfrac{\tilde\gamma}{\mathrm{Re}[\tilde\gamma(\tilde z-\tilde z_0)]}\right\}^2, \qquad Y_{\tilde a}=0; \qquad X_{\tilde b}=0;\\ Y_{\tilde b}=-\left\{\dfrac{\tilde\gamma}{\mathrm{Jm}[\tilde\gamma(\tilde z-\tilde z_0)]}\right\}^2\\ X_{\tilde P}=\dfrac{\tilde\gamma^2}{\sin^2\tilde P}; \qquad Y_{\tilde P}=-\dfrac{\tilde\gamma^2}{\cos^2\tilde P};\\ X_{\tilde Q}=\dfrac{\tilde\gamma^2}{\cos^2\tilde Q}; \qquad Y_{\tilde Q}=-\dfrac{\tilde\gamma^2}{\sin^2\tilde Q};\\ \dfrac{1}{X_{(\tilde P;\,\tilde Q)}}-\dfrac{1}{Y_{(\tilde P;\,\tilde Q)}}=\dfrac{1}{\tilde\gamma^2}.\end{cases}$$

Матрица номограммы (2.6) есть

$$(2.7)\quad \begin{Vmatrix} \left\{\dfrac{\tilde\gamma}{\mathrm{Re}[\gamma(z-z_0)]}\right\}^2 & 0 & \dfrac{\tilde\gamma^2}{\sin^2\tilde P} & \dfrac{\tilde\gamma^2}{\cos^2\tilde Q}\\ 0 & -\left\{\dfrac{\tilde\gamma}{\mathrm{Jm}[\tilde\gamma(\tilde z-\tilde z_0)]}\right\}^2 & -\dfrac{\tilde\gamma^2}{\cos^2\tilde P} & -\dfrac{\tilde\gamma^2}{\sin^2\tilde Q}\\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

Совместность номограмм (1.7) и (2.7) по переменным $\tilde P$ и $\tilde Q$ теперь усматривается непосредственно, не требуя даже проективного преобразования.

Однако полученная при этом номограмма — матрица 3×6 порядка имеет бесконечно удаленную шкалу

$$(2.8)\quad \begin{Vmatrix} \tilde\gamma^2 & -1+\mathrm{ch}\,2\tilde B & \dfrac{\tilde\gamma^2}{\sin^2\tilde P} & \dfrac{\tilde\gamma^2}{\cos^2\tilde Q} & \left\{\dfrac{\tilde\gamma}{\mathrm{Re}[\tilde\gamma(\tilde z-\tilde z_0)]}\right\}^2 & 0\\ \tilde\gamma^2 & 1+\mathrm{ch}\,2\tilde B & -\dfrac{\tilde\gamma^2}{\cos^2\tilde P} & -\dfrac{\tilde\gamma^2}{\sin^2\tilde Q} & 0 & -\left\{\dfrac{\tilde\gamma}{\mathrm{Jm}[\tilde\gamma(\tilde z-\tilde z_0)]}\right\}^2\\ e^{2\tilde A} & 0 & 1 & 1 & 1 & 1 \end{Vmatrix}.$$

С большой степенью произвола можно это устранить, помножив матрицу (2.8) слева на невырождающуюся числовую, вообще говоря, матрицу 3-го порядка.

Произвол выбора числовой матрицы ограничен априори лишь требованиями получить в матрице произведение, хотя бы одну из трех строк, не содержит ни одного равного нулю элемента. Разумеется, практически важное требование получить в какомто смысле хорошую номограмму внесет дополнительные ограничения на произвол выбора матричного множителя третьего порядка, определяющего проективное преобразование шестишкальной конической номограммы (2.8), второго жанра (коническое сечение, несущее шкалы $\tilde{P}$ и $\tilde{Q}$, 2 прямолинейные шкалы $\tilde{A}$ и $\tilde{B}$ и две прямолинейные шкалы cos $\mathrm{Re}[\tilde{\gamma}\tilde{z}]$ и для $\mathrm{Jm}[\tilde{\gamma}\tilde{z}]$, т. е. шкалы номограммы системы

$$\tilde{A} + \tilde{B}\varepsilon = e^{\tilde{\gamma}\tilde{z}} = \sin(\tilde{P} + \tilde{Q}\varepsilon). \tag{2.9}$$

По поводу ошибочной записи этого равенства в виде аналогичном (2.8) предыдущей работы см. замечание по поводу равенств (2.7) и (2.8).

Номограмма (2.8), это пучок ∞^1 конических номограмм (параметр $\tilde{\gamma}$), т. к. с изменением $\tilde{\gamma}$ конические сечения будут изменяться.

Если же разделить элементы 1,3 и 4 столбцов на $\tilde{\gamma}^2$, умножив затем элементы 3-ей строки на $\tilde{\gamma}^2$, то пучка номограмм не будет: все ∞^1 номограмм расположатся на четырех прямых и коническом сечении. Практически это вряд ли целесообразно. Эта шестишкальная номограмма но в гауссовом случае, была получена И. А. Вильнером на стр. 218в § 18, п. 1 работы [4]. Там же (на стр. 218), § 18, п. 2 И. А. Вильнером были указаны номограммы с бесконечно удаленными шкалами.

Сравнивая матрицу (2.8) с соответствующей матрицей (2.6) предыдущей работы, а также сравнивая соответствующие этим матрицам системы уравнений (2.9) этой работы и (2.7) предыдущей работы, видим, что если положить

$$\begin{cases} A \equiv \tilde{A}, \quad P \equiv \tilde{P}, \\ \mathrm{Re}[\gamma(z - z_0)] \equiv \mathrm{Re}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)], \end{cases} \tag{2.10}$$

то шкалы A и $\tilde{A}$, $\mathrm{Re}[\gamma(z-z_0)]$ и $\mathrm{Re}[\tilde{\gamma}(\tilde{z}-\tilde{z}_0)]$, P и $\tilde{P}$ совмещаются равнозначными значениями этих аргументов, тогда как остальные пары шкал не совмещаются. Шкалы B и $\tilde{B}$ лежат на одной прямой, но их градуировки не инциденты. Шкалы $\mathrm{Jm}[\gamma(z - z_0)]$ и $\mathrm{Jm}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)]$ лежат на одной прямой, но их равнозначные пометки не инцидентны. Шкалы Q и $\tilde{Q}$ лежат, как P и $\tilde{P}$, на том же коническом сечении, на котором расположены и инцидентные шкалы P и $\tilde{P}$, но шкалы Q и $\tilde{Q}$ не перекрывают одна другую (как и шкалы $\mathrm{Jm}[\gamma(z - z_0)]$ и $\mathrm{Jm}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)]$ и шкалы B и $\tilde{B}$.

Это означает, что если все же соединить номограмму (2.8) этой работы с номограммой (2.6) предыдущей работы так, чтобы совпали шкалы $A \equiv \tilde{A}$, $\mathrm{Re}[\gamma[z - z_0)] \equiv \mathrm{Re}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)]$, $P \equiv \tilde{P}$, то получим опять-таки коническую номограмму на двух прямых и коническом сечении.

На одной прямой будет общая шкала $A \equiv \tilde{A}$, на другой прямой — общая шкала $\mathrm{Re}[\gamma(z - z_0)] \equiv \mathrm{Re}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)]$; на коническом сечении общая шкала $P \equiv \tilde{P}$.

Кроме того, в этой номограмме будет еще одна прямая с непрекрывающимися шкалами B и $\tilde{B}$ и еще одна прямая с неперекрывающимися шкалами $\mathrm{Jm}[\gamma(z - z_0)]$ и $\mathrm{Jm}[\tilde{\gamma}(\tilde{z} - \tilde{z}_0)]$ и на коническом сечении на участке его, дополняющем шкалу P, будут расположены шкалы Q и $\tilde{Q}$. Эта номограмма, состоящая, таким образом, из одного конического сечения и четырех прямых будет 4-го жанра (4 шкалы на коническом сечении).

Эту номограмму можно также представить одной матрицей 3×12 порядка, на 3 пары столбцов в ней отождествляются поскольку имеют место соотношения (2.10).

§ 3. Соответственно зависимостям нулевого жанра (3.1), (3.2), (3.3) предыдущей работы, делая в них и в соответствующих им канонических представлениях (3.1_1), (3.1_2), (3.2_1), (3.2_2), (3.3_1), (3.3_2) прямое преобразование перенесения (2.5) главы 1 работы [1], получим (см. также (.1.2′), (.1.2″)) следующие результаты. Для зависимостей

$$
(3.1)\quad
\begin{gathered}
\underline{\tilde z = \ln \operatorname{th} \omega}, \qquad e^{\tilde z} = \operatorname{th} \tilde\omega, \qquad \operatorname{sh} 2\tilde w = -\frac{1}{\operatorname{sh}\tilde z},\\
\operatorname{th}\tilde z = -\frac{1}{\operatorname{ch} 2\tilde w},\\
\operatorname{th} 2\tilde w = \frac{1}{\operatorname{ch}\tilde z}, \qquad \underline{-2\tilde w = \ln \operatorname{th}\left(-\frac{\tilde z}{2}\right)},
\end{gathered}
$$

получим канонические представления

$$
(3.1_1)\quad
\begin{cases}
(\operatorname{ch}^2 2\tilde p)(\operatorname{ch} 2\tilde y) + (-\operatorname{sh}^2 2\tilde p)(\operatorname{ch} 2\tilde x) + (+1) = 0,\\
(\operatorname{ch}^2 2\tilde q)(\operatorname{ch} 2\tilde y) + (-\operatorname{sh}^2 2\tilde q)(\operatorname{ch} 2\tilde x) + (-1) = 0,
\end{cases}
$$

или

$$
(3.1_2)\quad
\begin{cases}
(-\operatorname{sh}^2 \tilde x)(\operatorname{ch} 4\tilde p) + (\operatorname{ch}^2 \tilde x)(\operatorname{ch} 4\tilde q) + (+1) = 0,\\
(-\operatorname{sh}^2 \tilde y)(\operatorname{ch} 4\tilde p) + (\operatorname{ch}^2 \tilde y)(\operatorname{ch} 4\tilde q) + (-1) = 0,
\end{cases}
$$

что равносильно (3.1).

Для равносильных между собой зависимостей нулевого жанра

$$
(3.2)\quad
\begin{gathered}
\tilde z = 2\operatorname{arctg} e^{\tilde w} - \frac{\pi}{2}, \qquad \frac{\tilde z}{2} + \frac{\pi}{4} = \operatorname{arctg} e^{\tilde w},\\
\tilde w = \ln \operatorname{tg}\left(\frac{\tilde z}{2} + \frac{\pi}{4}\right), \qquad \operatorname{tg}\left(\frac{\tilde z}{2} + \frac{\pi}{4}\right) = e^{\tilde w},\\
\underline{\operatorname{tg}\frac{\tilde z}{2} = \operatorname{th}\frac{\tilde w}{2}}, \qquad \sin\tilde z = \operatorname{th}\tilde w, \qquad \cos\tilde z = \pm\frac{1}{\operatorname{ch}\tilde w},\\
\operatorname{tg}\tilde z = \pm\operatorname{ch}\tilde w, \qquad \underline{\operatorname{th}\frac{i\tilde z}{2} = \operatorname{tg}\frac{i\tilde w}{2}},
\end{gathered}
$$

получим канонические представления

$$
(3.2_1)\quad
\begin{cases}
(\cos 2\tilde x) + (\cos 2\tilde y)(\operatorname{th}^2 \tilde p) + \left(-\dfrac{1}{\operatorname{ch}^2 \tilde p}\right) = 0,\\
(\cos 2\tilde x) + (\cos 2\tilde y)(\operatorname{cth}^2 \tilde q) + \left(-\dfrac{1}{\operatorname{sh}^2 \tilde q}\right) = 0,
\end{cases}
$$

или

$$
(3.2_2)\quad
\begin{cases}
(-\operatorname{ctg}^2 \tilde x)(\operatorname{ch} 2\tilde p) + (\operatorname{ch} 2\tilde q) + \left(\dfrac{1}{\sin^2 \tilde x}\right) = 0,\\
(-\operatorname{tg}^2 \tilde y)(\operatorname{ch} 2\tilde p) + (\operatorname{ch} 2\tilde q) + \left(-\dfrac{1}{\cos^2 \tilde y}\right) = 0.
\end{cases}
$$

Для равносильных между собой зависимостей нулевого жанра (см. (3.3) предыдущей работы)

$$\tilde{w} = \ln \operatorname{th} \tilde{z}, \qquad e^{\tilde{w}} = \operatorname{th} \tilde{z}, \qquad \operatorname{sh} 2\tilde{z} = -\frac{1}{\operatorname{sh} \tilde{w}},$$

$$(3.3) \qquad \tilde{w} + \frac{\pi i}{2} = \ln \operatorname{tg}(i\tilde{z}), \qquad \operatorname{th} 2\tilde{z} = \frac{1}{\operatorname{ch} \tilde{w}},$$

$$-2\tilde{z} = \ln \operatorname{th}\left(-\frac{\tilde{w}}{2}\right),$$

получим канонические представления

$$(3.3_1) \qquad \begin{cases} (-\operatorname{th}^2 \tilde{p})(\operatorname{ch} 4\tilde{x}) + (\operatorname{ch} 4\tilde{y}) + \left(\dfrac{1}{\operatorname{ch}^2 \tilde{p}}\right) = 0, \\ (-\operatorname{th}^2 \tilde{q})(\operatorname{ch} 4\tilde{x}) + (\operatorname{ch} 4\tilde{y}) + \left(-\dfrac{1}{ch^2 \tilde{q}}\right) = 0, \end{cases}$$

или

$$(3.3_2) \qquad \begin{cases} (-\operatorname{th}^2 2\tilde{x})(\operatorname{ch} 2\tilde{p}) + (\operatorname{ch} 2\tilde{q}) + \left(\dfrac{1}{\operatorname{ch}^2 2\tilde{x}}\right) = 0, \\ (-\operatorname{tg}^2 2\tilde{y})(\operatorname{ch} 2\tilde{p}) + (\operatorname{ch} 2\tilde{q}) + \left(-\dfrac{1}{\operatorname{ch}^2 2\tilde{y}}\right) = 0. \end{cases}$$

Во всех трех случаях теперь легко, следуя $(3.1_1')$, $(3.1_2')$, $(3.2_1')$, $(3.2_2')$, $(3.3_1')$, $(3.3_2')$ предыдущей работы, написать матрицы номограммы и прямо убедиться в проектной совместности по $\tilde{w}$ номограмм зависимостей (3.2) и (3.3) совершенно аналогично тому как это имело место для гауссовых зависимостей (3.2) и (3.3).

При помощи канонических ли представлений (3.1), (3.2) (3.3) или же применяя принцип перенесения к номограммам-матрицам (3.1′), (3.2′), (3.3′) предыдущей работы, мы получим следующие номограммы-матрицы соответствующих соотношениям (3.1), (3.2), (3.3) настоящей работы:

$$(3.1_1') \qquad \begin{Vmatrix} \dfrac{1}{\operatorname{ch} 2\tilde{x}} & 0 & \operatorname{sh}^2 2\tilde{p} & -\operatorname{sh}^2 2\tilde{q} \\ 0 & \dfrac{1}{\operatorname{ch} 2\tilde{y}} & -\operatorname{ch}^2 2\tilde{p} & \operatorname{ch}^2 2\tilde{q} \\ 1 & 1 & 1 & 1 \end{Vmatrix},$$

$$(3.1_2') \qquad \begin{Vmatrix} \dfrac{1}{\operatorname{ch} 4\tilde{p}} & 0 & \operatorname{sh}^2 \tilde{x} & -\operatorname{sh}^2 \tilde{y} \\ 0 & \dfrac{1}{\operatorname{ch} 4\tilde{q}} & -\operatorname{ch}^2 \tilde{x} & \operatorname{ch}^2 \tilde{y} \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

Таковы номограммы-матрицы (3.1), т. е. для соотношений равносильных $\tilde{z} = \ln \operatorname{th} \tilde{w}$; (3.1)

$$(3.2'_1)\qquad \begin{Vmatrix} \dfrac{1}{\cos 2\tilde{x}} & 0 & \mathrm{ch}^2\,\tilde{p} & \mathrm{sh}^2\,\tilde{q} \\ 0 & \dfrac{1}{\cos 2\tilde{y}} & \mathrm{sh}^2\,\tilde{p} & \mathrm{ch}^2\,\tilde{q} \\ 1 & 1 & 1 & 1 \end{Vmatrix},$$

$$(3.2'_2)\qquad \begin{Vmatrix} \dfrac{1}{\mathrm{ch}\,2\tilde{p}} & 0 & \cos^2\tilde{x} & -\sin^2\tilde{y} \\ 0 & \dfrac{1}{\mathrm{ch}\,2\tilde{q}} & -\sin^2\tilde{x} & \cos^2\tilde{y} \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

Для зависимостей равносильных зависимостям

$$(3.2)\qquad \tilde{w} = \ln \mathrm{tg}\left(\frac{\tilde{z}}{2} + \frac{\pi}{4}\right), \qquad \mathrm{tg}\,\frac{\tilde{z}}{2} = \mathrm{th}\,\frac{\tilde{w}}{2};$$

$$(3.3'_1)\qquad \begin{Vmatrix} \dfrac{1}{\mathrm{ch}\,4\tilde{x}} & 0 & \mathrm{sh}^2\,\tilde{p} & -\mathrm{sh}^2\,\tilde{q} \\ 0 & \dfrac{1}{\mathrm{ch}\,4\tilde{y}} & -\mathrm{ch}^2\,\tilde{p} & \mathrm{ch}^2\,\tilde{q} \\ 1 & 1 & 1 & 1 \end{Vmatrix},$$

$$(3.3'_2)\qquad \begin{Vmatrix} \dfrac{1}{\mathrm{ch}\,2\tilde{p}} & 0 & \mathrm{sh}^2\,2\tilde{x} & -\mathrm{sh}^2\,2\tilde{y} \\ 0 & \dfrac{1}{\mathrm{ch}\,2\tilde{q}} & -\mathrm{ch}^2\,2\tilde{x} & \mathrm{ch}^2\,2\tilde{y}, \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$$

для зависимостей равносильных соотношению

$$(3.3)\qquad \tilde{w} = \ln \mathrm{th}\,\tilde{z}.$$

Этим же способом, исходя из номограмм-матриц предыдущей работы, без труда получаются номограммы-матрицы для всех остальных зависимостей этой работы.

§ 4. Рассмотрим теперь соотношение нулевого жанра

$$(4.1)\qquad \tilde{w} = \ln \mathrm{tg}\,\tilde{z}.$$

При помощи прямого принципа перенесения (2.5) работы [1] получим из (4.3_1), (4.3_2) предыдущей работы, пользуясь таблицей (1.2″) настоящей работы, канонические представления

$$(4.1_1)\qquad \begin{cases} (\cos 4\tilde{x}) + (\cos 4\tilde{y})(-\mathrm{th}^2\,\tilde{p}) + \left(\dfrac{1}{\mathrm{ch}^2\,\tilde{p}}\right) = 0, \\ (\cos 4\tilde{x}) + (\cos 4\tilde{y})(-\mathrm{cth}^2\,\tilde{q}) + \left(\dfrac{1}{\mathrm{sh}^2\,\tilde{q}}\right) = 0, \end{cases}$$

или

$$
(4.1_2) \qquad \begin{cases} (-\mathrm{tg}^2 2\tilde{x})(\mathrm{ch}\, 2\tilde{p}) + (\mathrm{ch}\, 2\tilde{q}) + \left(\dfrac{1}{\cos^2 2\tilde{x}}\right) = 0, \\ (-\mathrm{tg}^2 2\tilde{y})(\mathrm{ch}\, 2\tilde{p}) + (\mathrm{ch}\, 2\tilde{q}) + \left(-\dfrac{1}{\cos^2 2\tilde{y}}\right) = 0. \end{cases}
$$

§ 5. Аналогично соображению § 5 предыдущей работы рассмотрим теперь зависимость нулевого жанра

$$
(5.1) \qquad \mathrm{cth}\, \tilde{w} = \mathrm{tg}\, \tilde{z}.
$$

Применяя прямой принцип перенесения к представлениям (5.4_1), (5.4_2) предыдущей работы, получим

$$
(5.1_1) \qquad \begin{cases} (\cos 4\tilde{x}) + (\cos 4\tilde{y})(\mathrm{th}^2 2\tilde{p}) + \left(-\dfrac{1}{\mathrm{ch}^2 2\tilde{p}}\right) = 0, \\ (\cos 4\tilde{x}) + (\cos 4\tilde{y})(\mathrm{cth}^2 2\tilde{q}) + \left(-\dfrac{1}{\mathrm{sh}^2 2\tilde{q}}\right) = 0, \end{cases}
$$

или

$$
(5.1_2) \qquad \begin{cases} (-\mathrm{ctg}^2 2\tilde{x})(\mathrm{ch}\, 4\tilde{p}) + (\mathrm{ch}\, 4\tilde{q}) + \left(\dfrac{1}{\sin^2 2\tilde{x}}\right) = 0, \\ (-\mathrm{tg}^2 2\tilde{y})(\mathrm{ch}\, 4\tilde{p}) + (\mathrm{ch}\, 4\tilde{q}) + \left(-\dfrac{1}{\cos^2 2\tilde{y}}\right) = 0. \end{cases}
$$

§ 6. Для зависимости второго жанра

$$
(6.1) \qquad \tilde{z} = \mathrm{sh}\, \tilde{w},
$$

канонические представления имеют вид (сравнить с § 6 предыдущей работы)

$$
(6.1_1) \qquad \begin{cases} \dfrac{\tilde{x}^2}{\sin^2 \tilde{p}} + \dfrac{\tilde{y}^2}{\cos^2 \tilde{p}} - 1 = 0, \\ \dfrac{\tilde{x}^2}{\cos^2 \tilde{q}} + \dfrac{\tilde{y}^2}{\sin^2 \tilde{q}} - 1 = 0. \end{cases}
$$

§ 7. Для зависимости второго жанра (см. § 7 предыдущей работы)

$$
(7.1) \qquad \tilde{z} = \mathrm{sh}\, \tilde{w},
$$

применяя принцип перенесения к соответствующим гауссовым представлениям (7.3), найдем представления

$$
(7.1_1) \qquad \begin{cases} \tilde{x}^2 - \tilde{y}^2(\mathrm{th}^2 \tilde{p}) + (-\mathrm{sh}^2 \tilde{p}) = 0, \\ \tilde{x}^2 - \tilde{y}^2(\mathrm{cth}^2 \tilde{p}) + (\mathrm{ch}^2 \tilde{q}) = 0. \end{cases}
$$

§ 8. Для зависимости

$$
(8.1) \qquad \tilde{z} = \ln \sin \tilde{w},
$$

применяя принцип перенесения к представлениям (8.2) предыдущей работы, найдем

$$
(8.1_1) \qquad \begin{cases} \left(-\dfrac{\sin^2 2\tilde{p}}{2}\right)(e^{-2\tilde{x}}) + (\mathrm{ch}\, 2\tilde{y}) + (\cos 2\tilde{p}) = 0, \\ \left(-\dfrac{\sin^2 2\tilde{q}}{2}\right)(e^{-2\tilde{x}}) + (\mathrm{ch}\, 2\tilde{y}) + (-\cos 2\tilde{q}) = 0. \end{cases}
$$

§ 9. Для зависимости второго жанра

$$\tilde{z} = \ln \operatorname{sh} \tilde{w}, \tag{9.1}$$

применяя прямой принцип перенесения к представлениям (9.3) предыдущей работы, найдем

$$\begin{cases} \left(-\dfrac{\operatorname{sh}^2 2\tilde{p}}{2}\right)(e^{-\widetilde{2x}}) + (\operatorname{ch} 2\tilde{y}) + (\operatorname{ch} 2\tilde{p}) = 0, \\ \left(-\dfrac{\operatorname{sh}^2 2\tilde{q}}{2}\right)(e^{-\widetilde{2x}}) + (\operatorname{ch} 2\tilde{y}) + (-\operatorname{ch} 2\tilde{q}) = 0. \end{cases} \tag{9.1$_1$}$$

§ 10. Для зависимости второго жанра

$$\tilde{z} = \tilde{w}^2, \tag{10.1}$$

с помощью принципа перенесения найдем согласно (10.2) предыдущей работы канонические представления

$$\begin{cases} (\tilde{x})(4\tilde{p}^2) + (-\tilde{y}^2) + (-4\tilde{p}^4) = 0, \\ (\tilde{x})(-4\tilde{q}^2) + (-1)(-\tilde{y}^2) + (4\tilde{q}^4) = 0. \end{cases} \tag{10.1$_1$}$$

§ 11. Для зависимости нулевого жанра

$$\tilde{w} = e^{\tilde{z}}, \tag{11.1}$$

с помощью принципа перенесения (прямого) и (11.2), (11.3) предыдущей работы найдем канонические представления

$$\begin{cases} \tilde{p}^2 - \tilde{q}^2 + (e^{\widetilde{2x}}) = 0, \\ \tilde{p}^2(-\operatorname{th}^2 \tilde{y}) + (-1)(-\tilde{q}^2) = 0, \end{cases} \tag{11.1$_1$}$$

или

$$\begin{cases} (2\tilde{p}^2)(e^{-\widetilde{2x}}) - (\operatorname{ch} 2\tilde{y}) + (-1) = 0, \\ (-2\tilde{q}^2)(e^{-\widetilde{2x}}) + (\operatorname{ch} 2\tilde{y}) + (-1) = 0. \end{cases} \tag{11.1$_2$}$$

§ 12. Для линейной зависимости (нулевой жанр)

$$\tilde{w} = (\tilde{m} + \tilde{n}\varepsilon)(\tilde{z} - \tilde{z}_0) + \tilde{w}_0, \tag{12.1}$$

при помощи принципа перенесения найдем [см. (12.2$_1$) и (12.2$_2$) предыдущей работы]

$$\begin{cases} \tilde{m}(\tilde{x} - \tilde{x}_0) + \tilde{n}(\tilde{y} - \tilde{y}_0) - (\tilde{p} - \tilde{p}_0) = 0, \\ \tilde{n}(\tilde{x} - \tilde{x}_0) + \tilde{m}(\tilde{y} - \tilde{y}_0) - (\tilde{q} - \tilde{q}_0) = 0, \end{cases} \tag{12.1$_1$}$$

или

$$\begin{cases} \tilde{m}(\tilde{p} - \tilde{p}_0) - \tilde{n}(\tilde{q} - \tilde{q}_0) - (\tilde{m}^2 - \tilde{n}^2)(\tilde{x} - \tilde{x}_0) = 0, \\ \tilde{n}(\tilde{p} - \tilde{p}_0) - \tilde{m}(\tilde{q} - \tilde{q}_0) + (\tilde{m}^2 - \tilde{n}^2)(\tilde{y} - \tilde{y}_0) = 0. \end{cases} \tag{12.1$_2$}$$

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ON SUBSTITUTION OF OPERATIONS IN SYSTEMS OF EQUATIONS OVER ALGEBRAS

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The mapping transforming a system of equations over an algebra into another algebra with different operations (from a different class of algebras) is frequently constructed for solving this system (for instance in operator calculus). A solution of the system in the second algebra is transformed back again into the first algebra and in individual cases it is proved that the transformed back solution is a solution of the initial system. The theorem which gives conditions for back transformation can be generalized and assumptions of it can be weakened.

The conceptions of the systems of equations over algebra and of the regularizer are taken from [1] and [2].

1.

By the symbol $\mathfrak{A} = (A, O_\Gamma)$ there is denoted the algebra with the set of generators A and the set of operations O_Γ. For each operation $o_\gamma \in O_\Gamma$ there exists an ordinal number k_γ — the so called arity of o_γ. By the symbol $\{a_\alpha, \alpha < k_\gamma\}$ it is denoted a sequence of the type k_γ formed by the elements a_α. Let $a_\alpha \in \mathfrak{A}$ for $\alpha < k$. The result of operation o_γ for elements $\{a_\alpha, \alpha < k_\gamma\}$ is denoted by $o_\gamma(a_\alpha, k_\gamma)$

Let $\mathscr{A}_x$ be the set of all expressions consisting of elements of the algebra $\mathfrak{A} = (A, O_\Gamma)$, of the O_Γ and of the set $X = \{x_\mu, \mu < s\}$ (where $X \cap A = \emptyset$, $X \cap O_\Gamma = \emptyset$) which would give elements of $\mathfrak{A}$ if the elements of X were replaced by elements of $\mathfrak{A}$; that is, expressions with the right number of elements (of $\mathfrak{A}$ or X) after each operation-symbol.

Let us introduce the equivalence ω on $\mathscr{A}_x$: an element $\tau \in \mathscr{A}_x$ is equivalent to element $\vartheta \in \mathscr{A}_x$, symbolically $\tau\,\omega\,\vartheta$ iff $\tau = \vartheta$ for each replacing elements of X by elements of $\mathfrak{A}$. We can introduce an operation $o_\gamma \in O_\Gamma$ for elements of $\mathscr{A}_x$:

$\tau\,\omega\,o_\gamma(\vartheta_\alpha, k_\gamma)$, $\vartheta_\alpha \in \mathscr{A}_x$ iff $\bar{\tau} = o_\gamma(\bar{\vartheta}_\alpha, k_\gamma)$ for arbitrary replacing elements of X by lements of $\mathfrak{A}$ (where $\bar{\vartheta}_\alpha$ is obtained from ϑ_α by replacing elements of X by elements of $\mathfrak{A}$).

It is clear that ω is a congruence relation on $\mathscr{A}_x$.

Definition 1. The factor algebra $\mathscr{A}_x/\omega$ is called *formal $\mathfrak{A}$-polynomial algebra* and is denoted by For $(\mathfrak{A}, X)$. Each element of For $(\mathfrak{A}, X)$ is called the *$\mathfrak{A}$-term* (or briefly the *term*).

Any term V of For $(\mathfrak{A}, X)$ generated only by the set $\{x_\mu, \mu < k\} \cup A$ and by operations O'_Γ, where $k \leqq s$, $O'_\Gamma \subseteq O_\Gamma$ is denoted by $V(x_\mu, k, O'_\Gamma)$.

Remark. $\mathfrak{A}$ is a subalgebra of For $(\mathfrak{A}, X)$ because $Y \subseteq X \Rightarrow$ For $(\mathfrak{A}, Y) \subseteq$ For $(\mathfrak{A}, X)$ and $\mathfrak{A} =$ For $(\mathfrak{A}, \emptyset)$.

Let $\mathfrak{A} = (A, O_\Gamma)$ be an algebra of the class Λ, $\mathfrak{B} = (B, O_H)$ be an algebra of the of the class Λ'.

Definition 2. The mapping φ of For $(\mathfrak{A}, X)$ into For $(\mathfrak{B}, X)$ is said to be the *S-mapping* iff the following conditions hold:

(i) the image of the $\mathfrak{A}$-term $o_\gamma(x_\mu, k_\gamma)$ is the $\mathfrak{B}$-term

$$V_\gamma(x_\mu, k_\gamma, O_H)$$

(ii) $\varphi|\mathfrak{A}$ is the mapping of $\mathfrak{A}$ into $\mathfrak{B}$

(iii) for each sequence $\{a_\mu, \mu < k\}$, $a_\mu \in \mathfrak{A}$ the identity $\varphi(o_\gamma(a_\mu, k_\gamma)) = V_\gamma(\varphi(a_\mu), k_\gamma, O_H)$ holds.

Then the operation $o_\gamma \in O_\Gamma$ is called *substitutable by the operation O_H* in the algebra $\mathfrak{B}$.

Remark. A special case of S-mapping is a homomorphic mapping (i.e. $\mathfrak{A}, \mathfrak{B} \in \Lambda$, $O_\Gamma = O_H$, $V_\gamma(x_\mu, k_\gamma, O_H') = o_\gamma(x_\mu, k_\gamma)$, $o_\gamma \in O_H$). If $\mathfrak{A}$ is a grupoid and $\mathfrak{B}$ is an algebrá with one binary operation and two suitable defined unary operations, then the conception of substitutability can be equal to the conception of isotopy.

Theorem 1. *The equivalence relation Φ induced by an S-mapping on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$.*

Proof. Let $\{a_\mu, \mu < k\}, \{b_\mu, \mu < k\}$ be sequences of elements of $\mathfrak{A}$, φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$ and $\langle a_\mu, b_\mu\rangle \in \Phi$ for each $\mu < k$, i.e. $\varphi(a_\mu) = \varphi(b_\mu)$. Then $\varphi(o_\gamma(a_\mu, k_\gamma)) = V_\gamma(\varphi(a_\mu) \ k_\gamma O_H') = V_\gamma(\varphi(b_\mu), k_\gamma, O_H') = \varphi(o_\gamma(b_\mu, k_\gamma))$, thus $\langle o_\gamma(a_\mu, k_\gamma), o_\gamma(b_\mu, k_\gamma)\rangle \in \Phi$ for each k_γ-ary operation $o_\gamma \in O_\Gamma$. Accordingly, Φ is a congruence relation on $\mathfrak{A}$.

EXAMPLES ON THE SUBSTITUTION OF OPERATIONS

1. Let $\mathfrak{A}$ be the Boolean algebra with n generators $\{a_\mu, \mu = 0, 1, \ldots, n-1\}$, $\mathfrak{B}$ be the Boolean ring with unity generated by the set of generators $\{b_\mu, \mu = 0, 1, \ldots, n-1\}$. Let φ be the mapping of For $(\mathfrak{A}, X)$ into For $(\mathfrak{B}, X)$ for which $\varphi(a_\mu) = b_\mu$ for $\mu < n$, and for arbitrary $x_0, x_1 \in X$ is

$$\varphi(x_0 \cup x_1) = \varphi(x_0) + \varphi(x_1) - \varphi(x_0) \cdot \varphi(x_1)$$

$$\varphi(x_0 \cap x_1) = \varphi(x_0) \cdot \varphi(x_1)$$

$$\varphi(\overline{x_0}) = 1 - \varphi(x_0).$$

Then φ is the S-mapping and operation, $\cup$ is substitutable in $\mathfrak{B}$ by $O_H = \{+, -, .\}$. Analogously for other Boolean operations. The inverse mapping φ^{-1} is the S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$. It follows directly from [8] and [3]. Then for example the operation $+$ of $\mathfrak{B}$ is substitutable in $\mathfrak{A}$ by the set $O_\Gamma = \{\cup, \cap, \overline{\ \ }\}$ of all operations of $\mathfrak{A}$ because

$$\varphi^{-1}(x_0 + x_1) = (\varphi^{-1}(x_0) \cap \overline{\varphi^{-1}(x_1)}) \cup (\overline{\varphi^{-1}(x_0)} \cap \varphi^{-1}(x_1)).$$

2. Let $\mathfrak{A}$ be the set of non-zero complex functions of the real variable t which have the continuous first derivative in the interval $\langle 0, \infty)$ and fulfil $|f(t)| \leqq Me^{St}$, where $M \geqq 0$, $S \geqq 0$ are constants, let the set of operations of $\mathfrak{A}$ be $O_\Gamma = \{+, -, ., :,$ derivative, integral$\}$. Let $\mathfrak{B}$ be the field of operators, where $O_H = \{+, -,$ operator product, operator quotient, multiplications by a constants$\}$. Then there exist various

S-mappings of $\mathfrak{A}$ into $\mathfrak{B}$, for example the Laplace transformation, the Garson-Laplace transformation, the Fourier transformation etc. Operations of $\mathfrak{A}$ are substitutable in $\mathfrak{B}$ by the operations of $\mathfrak{B}$.

2.

Let $\mathscr{A}$ be the set of elements of For $(\mathfrak{A}, X)$, $X = \{x_\mu, \mu < s\}$ and $\mathfrak{A} = (A, O_\Gamma)$.

Definition 3. The subset E of the Cartesian product $\mathscr{A} \times \mathscr{A}$ is said to be the *system of equations over* $\mathfrak{A}$, each pair $\langle \tau, \vartheta \rangle \in E$ of $\mathfrak{A}$-terms τ, ϑ is called the *equation*. Elements of X (resp. of A) generating $\mathfrak{A}$-terms τ, ϑ are called *unknowns* (resp. *parameters*) of the equation $\langle \tau, \vartheta \rangle \in E$.

Definition 4. A homomorphic mapping h of For $(\mathfrak{A}, X)$ into $\mathfrak{A}' = (A', O_\Gamma)$, where $\mathfrak{A}'$, $\mathfrak{A}$, For $(\mathfrak{A}, X)$ are of the same class of algebras, is called the *characteristic mapping* of the system E iff $h(\tau) = h(\vartheta)$ for each $\langle \tau, \vartheta \rangle \in E$ and $h(\text{For}(\mathfrak{A}, X)) = h(\mathfrak{A})$. If $h \mid \mathfrak{A}$ is an isomorphic mapping of $\mathfrak{A}$ into $\mathfrak{A}'$, the characteristic mapping h is said. to be *proper*. The congruence relation induced by h on For $(\mathfrak{A}, X)$ is called the *regularizer* of the system E. If h is proper, the regularizer induced by h is called *proper*.

By the symbol $\bar{\tau}$ we denote the $\mathfrak{A}$-term τ where all elements of X which generate τ are replaced by elements of $\mathfrak{A}$ and each x_μ is replaced by the same $a_\mu \in \mathfrak{A}$ in all places in τ.

Definition 5. Let E be the system of equations over $\mathfrak{A}$ and $\sim$ be a regularizer of E. The sequence $\{V_\mu(a_\alpha, k_\mu, O_\Gamma), \mu < s\}$, where $a_\alpha \in \mathfrak{A}$, is said to be the *solution of the system* E with the regularizer $\sim$ iff we obtain $\langle \bar{\tau}, \bar{\vartheta} \rangle \in \sim$ for each $\langle \tau, \vartheta \rangle \in E$ by replacing each element x_μ by the element $V_\mu(a_\alpha, k_\mu, O_\Gamma)$. If the regularizer $\sim$ is proper, the solution is called *proper*.

In [2] it is shown that the solution $\{V_\mu(a_\alpha, k_\mu, O_\Gamma), \mu < s\}$ is proper iff $\bar{\tau} = \bar{\vartheta}$ for each $\langle \tau, \vartheta \rangle \in E$. Accordingly, the proper solution is the solution in sense of the classical definition (see for example [6]). The definition 5 is, however, more general than that one.

3.

Let E be the system of equations over an algebra $\mathfrak{A} = (A, O_\Gamma)$, let φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B} = (B, O_H)$. The mapping φ maps each $\mathfrak{A}$-term τ onto $\mathfrak{B}$-term $\varphi(\tau)$ and each equation $\langle \tau, \vartheta \rangle$ of E onto $\mathfrak{B}$-equation $\langle \varphi(\tau), \varphi(\vartheta) \rangle$. Let us denote by $\varphi(E)$ the set of all $\langle \varphi(\tau), \varphi(\vartheta) \rangle$ for $\langle \tau, \vartheta \rangle \in E$.

We use frequently (for example in the operator calculus) the theorem on transforming of the solution of $\varphi(E)$ onto a solution of E. This theorem can be generalized for arbitrary algebras:

Theorem 2. *Let E be a system of equations over $\mathfrak{A}$, φ be an injective S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$ and ψ be an injective S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$. If $\{V_\mu(b_\alpha, k_\mu O_H), \mu < s\}$ is a solution of the system $\varphi(E)$ with regularizer $\sim$, then*

$$\{\psi(V_\mu(b_\alpha, k_\mu, O_H)), \mu < s\}$$

is a solution of the system E with regularizer $\sim_\varphi$ defined by the rule:

$$\langle a, b\rangle \in \sim_\varphi \text{ iff } \langle \varphi(a), \varphi(b)\rangle \in \sim \tag{P}$$

We can however, weaken the assumption of this theorem and extend the range of applications of it. The assumption of existence of „inverse" S-mapping ψ is too strong for the applications where the theorem can bring new results (for new transformations in the operator calculus, for modeling various systems etc.).

Lemma. *Let $\mathfrak{A} = (A, O_\Gamma)$, φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B} = (B, O_H)$. If $\sim$ is a congruence relation on $\mathfrak{B}$, the relation $\sim$a defined by* (P) *is the congruence relation on $\mathfrak{A}$.*

Proof. If is evident that $\sim_\varphi$ is an equivalence relation on $\mathfrak{A}$ because (P) implies the reflexivity, transitivity and symmetry of relation $\sim_\varphi$ for congruence $\sim$. Let $\sim_\varphi$ be not a congruence relation. Then there exists at least one sequence $\{\langle a_\mu, b_\mu\rangle, \mu < k\}$ and at least one k_γ-ary operation $o_\gamma \in O_\Gamma$ so that $\langle a_\mu, b_\mu \rangle \in \sim_\varphi$ for $\mu < k$ and $\langle o_\gamma(a_\mu, k_\gamma), o_\gamma(b_\mu, k_\gamma)\rangle \notin \sim_\varphi$. From it follows by (P):

$$\langle \varphi(o_\gamma(a_\mu, k_\gamma)), \varphi(o_\gamma(b_\mu, k_\gamma)) \rangle \notin \sim$$

which is a contradiction because $\sim$ is a congruence relation by the asumption and $\langle \varphi(o_\gamma(a_\mu, k_\gamma)), \varphi(o_\gamma(b_\mu, k_\gamma))\rangle = \langle V_\gamma(\varphi(a_\mu), k_\gamma, O'_H), V_\gamma(\varphi(b_\mu), k_\gamma, O'_H)\rangle \in \sim$.

Theorem 3. *Let $\mathfrak{A} = (A, O_\Gamma)$, $\mathfrak{B} = (B, O_H)$, φ be a S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$. Let E be a system of equations over $\mathfrak{A}$. If $\{V_\mu(b_\alpha, k_\mu, O'_H), \mu < s\}$ is a solution of the system $\varphi(E)$ with regularizer $\sim$ and W_μ is an arbitrary element of $\mathfrak{A}$ fulfilling $\varphi(W_\mu) = V_\mu$ for each $\mu < s$, then $\{W_\mu, \mu < s\}$ is a solution of E with the regularizer $\sim_\varphi$ given by* (P).

Proof. By the lemma $\sim_\varphi$ is a congruence relation on $\mathfrak{A}$. Let $\bar{\tau}$ be an $\mathfrak{A}$-term τ where each x_μ is replaced by $W_\mu \in \mathfrak{A}$, $\overline{\varphi(\tau)}$ be $\mathfrak{B}$-term $\varphi(\tau)$, where x_μ is replaced by $V_\mu \in \mathfrak{A}$ and let $\varphi(W_\mu) = V_\mu$, where $\{V_\mu, \mu < s\}$ is a solution of $\varphi(E)$.

By the condition (iii) od the definition 2 we obtain $\varphi(\bar{o}_\gamma) = \overline{\varphi(o_\gamma)}$, where $o_\gamma(x_\mu, k_\gamma)$ is an $\mathfrak{A}$-term. By the theorem 1 we obtain $\varphi(\bar{\tau}) = \overline{\varphi(\tau)}$ for an arbitrary $\mathfrak{A}$-term. Thus $\langle \overline{\varphi(\tau)}, \overline{\varphi(\vartheta)}\rangle \in \sim$ implies $\langle \varphi(\bar{\tau}), \varphi(\bar{\vartheta})\rangle \in \sim$ and by (P) we obtain $\langle \bar{\tau}, \bar{\vartheta}\rangle \in \sim_\varphi$. Accordingly, $\{W_\mu, \mu < s\}$ is really the solution of E with the regularizer $\sim_\varphi$.

The theorem 2 now follows from the theorem 3.

4.

It is possible that the system E has other solutions which can not be obtained from solu tions of $\varphi(E)$ hy the theorem 3. The solution of E obtained from the solution $\{V_\mu, \mu < s\}$ of $\varphi(E)$ by this theorem is called *induced by the solution* $\{V_\mu, \mu < s\}$. The solution of E induced by proper solution of $\varphi(E)$ need not be proper.

Theorem 4. *Let E be a system of equations over $\mathfrak{A}$, φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$. The solutio n of E induced by the solution $\{V_\mu, \mu < s\}$ of $\varphi(E)$ is proper iff:*

(1) $\{V_\mu, \mu < s\}$ *is a proper solution of* $\varphi(E)$

(2) $\varphi^{-1}[b] \cap \mathfrak{A}$ *is a one-element set for each* $b \in \mathfrak{B}$.

Proof. The sufficiency is evident. Necessity: Let $\{V_\mu, \mu < s\}$ be not proper solution of $\varphi(E)$, i.e. $\sim \not\equiv =$ and let the induced solution be proper. i.e. $\sim_\varphi \equiv =$.

Then from $\langle \bar{\tau}, \bar{\vartheta} \rangle \in \sim_\varphi$ we have $\bar{\tau} = \bar{\vartheta}$ and by (P) we obtain $\varphi(\bar{\tau}) = \varphi(\bar{\vartheta})$ for each $\langle \varphi(\tau), \varphi(\vartheta) \rangle \in \varphi(E)$. From it $\sim \equiv =$ and thus $\{V_\mu, \mu < s\}$ is proper which is a contradiction. Let $\varphi^{-1}[b] \cap \mathfrak{A}$ be not one-element set and $\sim \equiv =$, then $\varphi(\bar{\tau}) = \varphi(\bar{\vartheta})$. By the theorem 1, $\sim_\varphi$ is a congruence relation, but $\varphi^{-1}[b] \cap \mathfrak{A}$ is not one-element set, i.e. $\sim_\varphi \not\equiv =$ which is a contradiction with the assumptions of the proof again.

For applications the following sufficiency condition can be often use:

Corollary 5. *Let φ be an injective S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$, E be a system of equations over $\mathfrak{A}$. Then the solution of E induced by a proper solution of $\varphi(E)$ is proper.*

Proof. The assumptions of corollary fulfils (1) of the theorem 4. From injectivity of φ we obtain condition (2) of the theorem 4. By this theorem we obtain the assertion of corollary.

Theorem 3 and corollary 5 can be applied to the operator calculus if the transformation into the field of operators is injective but there does not exist substitutability for each operator operation into initial functional algebra (i.e. the inverse mapping of the S-mappint φ of $\mathfrak{A}$ into $\mathfrak{B}$ is not an S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$ for each operation of $\mathfrak{B}$).

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ÜBER ERWEITERUNGEN GEORDNETER MENGEN

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Sei G eine Menge. Unter einer *Ordnung* auf G verstehen wir, wie üblich, eine areflexive und transitive binäre Relation auf G; diese Relation wird $<$ oder $\prec$, eventuell noch mit Indices bezeichnet. Die Menge G mit der Ordnung $<$ heißt *geordnete Menge* und wird als $G(<)$ bezeichnet; wenn kein Mißverständnis droht, schreiben wir kurz G statt $G(<)$. Eine Untermenge einer geordneten Menge fassen wir immer auch als eine geordnete Menge mit der induzierten Ordnung. Für $x, y \in G$ schreiben wir $x \leqq y$, wenn $x < y$ oder $x = y$ gilt; die Relation $\leqq$ ist reflexiv, antisymmetrisch und transitiv. Wenn $x, y \in G$ und $x \leqq y$ oder $y \leqq x$ gilt, dann heissen die Elemente x, y *vergleichbar;* sonst nennen wir sie *unvergleichbar* und schreiben dafür $x \parallel y$. Eine Ordnung auf G heißt *linear,* wenn es keine unvergleichbare Elemente in G gibt; G heißt dann eine *linear geordnete Menge* oder eine *Kette.* Umgekehrt nennen wir eine geordnete Menge G *Antikette,* wenn es in G keine verschiedene vergleichbare Elemente gibt. Eine geordnete Menge G heißt *gerichtet,* wenn folgendes gilt: $x, y \in G \Rightarrow$ es existiert $z \in G$ so, daß $x \leqq z$, $y \leqq z$.

Da eine Ordnung auf G eine binäre Relation ist, also eine Untermenge des kartesischen Quadrats G^2, können wir verschiedene Ordnungen auf G durch die mengentheoretische Inklusion vergleichen. Wenn wir im weiteren zwei (oder mehr) Ordnungen auf G vergleichen, dann bedeutet es immer eine Vergleichung durch die Inklusion. Auch können wir zu Ordnungen auf G mengentheoretische Operationen anwenden. Während der Durchschnitt einer beliebigen Menge von Ordnungen auf G wieder eine Ordnung auf G ist, gilt dieses nicht für die Vereinigung. Im weiteren werden wir jedoch die folgende, übrigens allgemein bekannte Behauptung brauchen.

Lemma. *Sei* $\{<_i \mid i \in I\}$ *eine gerichtete Menge der Ordnungen auf* G. *Dann ist* $\bigcup_{i \in I} <_i$ *wieder eine Ordnung auf* G.

Beweis. Bezeichnen wir $\bigcup_{i \in I} <_i$ als ϱ. Die Relation ϱ ist offenkundig areflexiv. Seien $x, y, z \in G$, $x \varrho y$, $y \varrho z$. Dann existieren $i, j \in I$ so, daß $x <_i y$, $y <_j z$. Da $\{<_i \mid i \in I\}$ eine gerichtete Menge ist, existiert ein Index $k \in I$ so, daß $<_i \subseteq <_k$, $<_j \subseteq <_k$. Also $x <_k y$, $y <_k z$ und da $<_k$ transitiv ist, $x <_k z$. Daraus $x \varrho z$, ϱ ist transitiv und also eine Ordnung auf G.

Da eine Kette immer eine gerichtete Menge ist, bekommen wir die

Folgerung. *Sei* $\{<_i \mid i \in I\}$ *eine Kette der Ordnungen auf* G. *Dann ist* $\bigcup_{i \in I} <_i$ *wieder eine Ordnung auf* G.

Eine duale Relation zu einer Ordnung auf G ist wieder eine Ordnung auf G; G mit dieser Relation heißt *dual geordnet.* Das Element a einer geordneten Menge G heißt das *kleinste Element,* wenn $a \leqq x$ für jedes $x \in G$ gilt. $a \in G$ heißt *minimales Element,* wenn es kein $x \in G$ mit $x < a$ gibt. Dual dazu werden das *größte Element* und ein *maximales Element* definiert. Eine geordnete Menge heißt *wohlgeordnet*, wenn jede nicht leere Untermenge von G das kleinste Element besitzt. Eine wohlgeordnete Menge nennen wir eine *wachsende Kette;* eine dual wohlgeordnete Menge heißt eine *fallende Kette.*

Eine geordnete Menge G erfüllt die *Minimalbedingung*, wenn jede fallende Kette in G endlich ist. Äquivalent damit ist die Bedingung, daß jede nicht leere Untermenge von G ein minimales Element besitzt.

Im folgenden werden wir noch den Begriff der *Ordinalsumme* ([1]) brauchen. Seien G, H disjunkte geordnete Mengen.

Die Ordinalsumme $G \oplus H$ ist die Menge $G \cup H$ mit dieser Ordnungsrelation: $x, y \in G \cup H,\ x < y \Leftrightarrow x, y \in G$ und $x < y$ in G oder $x, y \in H$ und $x < y$ in H oder $x \in G,\ y \in H$.

Sei G eine Menge, seien $<$, $\prec$ Ordnungen auf G. Wenn $< \subseteq \prec$, dann nennen wir die Ordnung $\prec$ *Erweiterung* der Ordnung $<$. Auch $G\ (\prec)$ heißt dann eine Erweiterung von $G\ (<)$. $G(\prec)$ ist also eine Erweiterung von $G\ (<)$, wenn folgendes gilt: $x, y \in G,\ x < y \Rightarrow x \prec y$. Eine Erweiterung $\prec$ der Ordnung $<$ auf G heißt *linear*, wenn $G\ (\prec)$ eine linear geordnete Menge ist. Diese Erweiterung heißt *Wohlerweiterung*, wenn $G(\prec)$ eine wohlgeordnete Menge ist.

Die Existenz einer linearen Erweiterung für jede geordnete Menge wurde zuerst von E. Szpilrajn in [6] bewiesen. Seitdem sind viele andere Beweise dieses Satzes publiziert worden; siehe etwa [2], [3], [4], [5]. In Wirklichkeit hat E. Szpilrajn folgendes bewiesen: *Ist $G(<)$ eine geordnete Menge und sind x, y beliebige unvergleichbare Elemente in G, dann existiert eine linerare Erweiterung $\prec$ der Ordnung $<$ auf G, für die $x \prec y$ gilt.* Dieser Satz kann wie folgt verallgemeinert werden.

Satz. 1. *Sei $G\ (<)$ eine geordnete Menge, sei $H \subseteq G$. Ist $\prec$ eine beliebige Erweiterung der Ordnung $<$ auf H, dann existiert eine lineare Erweiterung der Ordnung $<$ auf G, die auch Erweiterung der Ordnung $\prec$ auf H ist.*

Beweis. Definieren wir eine Relation ϱ auf G folgenderweise:

für $x, y \in H$ setzen wir $x\varrho y$ dann und nur dann, wenn $x \prec y$ gilt

für $x \in H,\ y \in G - H$ setzen wir $x \varrho y$ dann und nur dann, wenn es ein solches $u \in H$ gibt, für das $x \underline{\underline{\prec}}\ u$ und $u < y$ gilt

für $x \in G - H,\ y \in H$ setzen wir $x\varrho y$ dann und nur dann, wenn es ein solches $u \in H$ gibt, für das $x < u$ und $u \underline{\underline{\prec}}\ y$ gilt

für $x, y \in G - H$ setzen wir $x\varrho y$ dann und nur dann, wenn entweder $x < y$ oder es solche $u, v \in H$ gibt, für die $x < u,\ u \prec v,\ v < y$ gilt.

Die Relation ϱ ist areflexiv: ist $x \in H$, dann $x\bar{\varrho}x$, denn $\prec$ ist areflexiv; ist $x \in G - H$ und wäre $x\varrho x$, dann entweder $x < x$ oder gäbe es $u, v \in H$ mit $x < u,\ u \prec v,\ v < x$. Der erste Fall ist unmöglich und der zweite gibt $v < x < u$, also $v < u$, was ein Widerspruch mit $u \prec v$ ist, denn $\prec$ ist eine Erweiterung von $<$ auf H.

Wir beweisen weiter, daß die Relation ϱ transitiv ist. Seien also $x, y, z \in G,\ x\varrho y,\ y\varrho z$. Es können folgende Möglichkeiten eintreten:

(1) $x, y, z \in H$. Dann $x \prec y,\ y \prec z$, also $x \prec z$ und auch $x\varrho z$.

(2) $x, y \in H,\ z \in G - H$. Dann $x \prec y$ und es gibt $u \in H$ so, daß $y \underline{\underline{\prec}}\ u,\ u < z$. Daraus folgt $x \prec u,\ u < z$ und also $x\varrho z$.

(3) $x \in H,\ y \in G - H,\ z \in H$. In diesem Fall gibt es $u \in H$ mit $x \underline{\underline{\prec}}\ u,\ u < y$ und $v \in H$ mit $y < v,\ v \underline{\underline{\prec}}\ z$. Aus $u < y,\ y < v$ folgt $u < v$, also auch $u \prec v$ und wir haben $x \underline{\underline{\prec}}\ u \prec v \underline{\underline{\prec}}\ z$, deshalb $x \prec z$ und auch $x\varrho z$.

(4) $x \in G - H,\ y, z \in H$. Dann existiert $u \in H$ mit $x < u,\ u \underline{\underline{\prec}}\ y$ und es gilt $y \prec z$. Also haben wir $u \prec z$ und $x < u,\ u \prec z$ impliziert $x\varrho z$.

(5) $x \in H, y, z \in G - H$. Dann gibt es $u \in H$ mit $x \leqq u$, $u < y$ und es gilt entweder $y < z$ oder existieren $v, w \in H$ mit $y < v$, $v \prec w$, $w < z$. Im ersten Fall $x \leqq u$, $u < z$, also $x\varrho z$; im zweiten haben wir $u < v$, also auch $u \prec v$, woraus $u \prec w$ folgt. Im ganzen gilt es $x \leqq u$, $u \prec w$, $w < z$, also $x \prec w$, $w < z$ und $x\varrho z$.

(6) $x \in G - H$, $y \in H$, $z \in G - H$. In diesem Fall existiert $u \in H$ mit $x < u$, $u \leqq y$ und $v \in H$ mit $y \leqq v$, $v < z$. Ist $u = v$, dann $x < u$, $u < z$ und $x < z$. Ist $u \prec v$, dann $x < u$, $u \prec v$, $v < z$. In beiden Fällen haben wir $x\varrho z$.

(7) $x, y \in G - H$, $z \in H$. Dann gibt es $w \in H$ mit $y < w$, $w \leqq z$ und entweder $x < y$ oder existieren $u, v \in H$ so, daß $x < u$, $u \prec v$, $v < y$. Im ersten Fall gilt $x < w$, $w \leqq z$, woraus $x\varrho z$; im zweiten haben wir $v < w$, also auch $v \prec w$ und $u \prec v$, $v \prec w$, $w \leqq z \Rightarrow u \prec z$. Dann $x < u$, $u \prec z$ und das impliziert $x\varrho z$.

(8) $x, y, z \in G - H$. Dann $x < y$ oder gibt es $u, v \in H$ mit $x < u$, $u \prec v$, $v < y$ und $y < z$ oder gibt es $w, t \in H$ mit $y < w$, $w \prec t$, $t < z$. Gilt $x < y$ und $y < z$, dann $x < z$ und $x\varrho z$. Gilt $x < y$ und gibt es $w, t \in H$ mit $y < w$, $w \prec t$, $t < z$, dann haben wir $x < w$, $w \prec t$, $t < z$, also $x\varrho z$. Gilt $y < z$ und gibt es $u, v \in H$ mit $x < u$, $u \prec v$, $v < y$, dann haben wir $x < u$, $u \prec v$, $v < z$, also $x\varrho z$.

Im letzten Fall gibt es $u, v, w, t \in H$ mit $x < u$, $u \prec v$, $v < y$, $y < w$, $w \prec t$, $t < z$. Dann $v < w$, also auch $v \prec w$ und aus $u \prec v$, $v \prec w$, $w \prec t$ folgt $u \prec t$. Daraus $x < u$, $u \prec t$, $t < z$ und $x\varrho z$.

Die Relation ϱ auf G ist also auch transitiv und deshalb ist sie eine Ordnung auf G. Aus ihrer Definition folgt gleich, daß sie eine Erweiterung der Ordnung $<$ auf G ist und daß sie auf der Menge H mit $\prec$ übereinstimmt. Wenn wir also jetzt eine beliebige lineare Erweiterung der Ordnung ϱ auf G konstruieren, dann hat diese lineare Erweiterung die erwünschten Eigenschaften.

In [4] wird es bewiesen: *Eine geordnete Menge hat eine Wohlerweiterung dann und nur dann, wenn sie die Minimalbedingung erfüllt.* Im Zusammenhang damit entsteht folgende Frage: Welche ist die notwendige und hinreichende Bedingung dafür, daß jede lineare Erweiterung der geordneten Menge G eine Wohlerweiterung ist? Die Antwort gibt folgender Satz.

Satz 2. *Sei G eine geordnete Menge. Jede lineare Erweiterung der Menge G ist eine Wohlerweiterung dann und nur dann, wenn G die Minimalbedingung erfüllt und wenn jede Antikette in G endlich ist.*

Beweis. 1. Seien beide Bedingungen für die Menge G erfüllt. Nehmen wir an, daß es eine lineare Erweiterung $\prec$ der Ordnung $<$ auf G gibt, die keine Wohlerweiterung ist. Also gibt es in $G(\prec)$ eine unendliche fallende Kette $x_1 \succ x_2 \succ \ldots \succ x_n \succ \ldots$ Wir bezeichnen $C = \{x_1, x_2, \ldots, x_n, \ldots\}$. Die geordnete Menge $C(<)$ erfüllt die Minimalbedingung, also hat sie ein minimales Element x_{n_1}. Da wir $x_n \prec x_{n_1}$ für jedes $n > n_1$ haben und da $\prec$ eine Erweiterung von $<$ ist, gilt es $x_n < x_{n_1}$ oder $x_n \parallel x_{n_1}$*) für alle solche n. Die erste Möglichkeit ist ausgeschlossen, denn x_{n_1} ist ein minimales Element in $C(<)$. Also $x_n \parallel x_{n_1}$ für jedes $n > n_1$. Bezeichnen wir jetzt $C_1 = \{x_{n_1+1}, x_{n_1+2}, \ldots\}$. Die Menge $C_1(<)$ erfüllt die Minimalbedingung und hat also ein minimales Element $x_{n_2} (n_2 > n_1)$. Aus demselben Grund wie oben gilt $x_n \parallel x_{n_2}$ für jedes $n > n_2$. Bezeichnen wir jetzt $C_2 = \{x_{n_2+1}, x_{n_2+2}, \ldots\}$ und finden ein minimales Element x_{n_3} in $C_2(<)$. Durch die vollständige Induktion können wir ein Element x_{n_k} für jedes natürliche k so finden, daß $x_{n_k} \parallel x_n$ für alle $n > n_k$ gilt. Dann ist $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots\}$ eine unendliche Antikette in $G(<)$, was ein Widerspruch ist.

*) Das Symbol $\parallel$ bedeutet hier die Unvergleichbarkeit im Bezug auf die Ordnung $<$.

2. Seien die Bedingungen des Satzes für eine geordnete Menge $G\ (<)$ nicht erfüllt. Erfüllt $G\ (<)$ nicht die Minimalbedingung, dann ist keine lineare Erweiterung von $G\ (<)$ eine Wohlerweiterung. Enthält $G\ (<)$ eine unendliche Antikette H (wir können voraussetzen, daß H abzählbar ist), dann wählen wir auf H eine beliebige lineare Ordnung $\prec$, die keine Wohlordnung ist (z. B. eine Ordnung in den Typ η aller rationalen Zahlen). Nach dem Satz 1. gibt es eine lineare Erweiterung der Ordnung $<$ auf G, die zugleich eine Erweiterung der Ordnung $\prec$ auf H ist; diese lineare Erweiterung ist dann keine Wohlerweiterung der Menge $G\ (<)$.

In [4] wird eine Konstruktion beschrieben, die die *Konstruktion* (K) genannt wird. Diese Konstruktion ist folgende: Sei $G\ (<)$ eine geordnete Menge. Schreiben wir diese Menge in der Form einer wachsenden Folge des Typs β (β eine Ordnungszahl): $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \beta\}$ und bezeichnen $G_\alpha = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \alpha\}$ für jedes $\alpha \leqq \beta$. Auf G_α definieren wir jetzt eine Ordnung $\prec$ durch die transfinite Induktion so:

G_0 ist leer, also geordnet. Wenn die Relation $\prec$ auf jeder Menge $G_\lambda\ (\lambda < \alpha)$ definiert wird, dann setzen wir:

1) ist α eine isolierte Ordnungszahl dann

$$g_{\alpha-1} \prec g_\lambda (\lambda < \alpha - 1) \Leftrightarrow \text{es existiert}$$

eine Ordnungszahl $\lambda_{\alpha-1} < \alpha - 1$ mit $g_{\alpha-1} < g_{\lambda_{\alpha-1}}$ und $g_{\lambda_{\alpha-1}} \leqq g_\lambda$ in $G_{\alpha-1}\ (\prec)$

$$g_\lambda \prec g_{\alpha-1} (\lambda < \alpha - 1) \quad \text{anders}$$

$$g_\lambda \prec g_\mu (\lambda, \quad \mu < \alpha - 1) \Leftrightarrow g_\lambda \prec g_\mu \quad \text{in } G_{\alpha-1}(<).$$

2) Ist α eine Limeszahl und sind $g_\lambda, g_\mu \in G_\alpha\ (\lambda, \mu < \alpha)$, dann $g_\lambda \prec g_\mu \Leftrightarrow g_\lambda \prec g_\mu$ in $G_\nu\ (\prec)$ für ein passendes $\nu < \alpha$, $\nu > \lambda$, $\nu > \mu$.

Zugleich wird in [4] bewiesen: *Ist $G\ (<)$ eine geordnete Menge und $G_\beta\ (\prec)$ eine Kette, die man aus $G\ (<)$ durch die Konstruktion (K) bekommt, dann is $G_\beta\ (\prec)$ eine lineare Erweiterung von $G\ (<)$. Wenn überdies $G\ (<)$ die Minimalbedingung erfüllt, dann ist $G_\beta\ (\prec)$ eine Wohlerweiterung von $G\ (<)$.* Dieser Satz kann wie folgt ergänzt werden:

Satz 3. *Sei G eine geordnete Menge, die die Minimalbedingung erfüllt. Jede Wohlerweiterung der Menge G kann man durch die Konstruktion (K) bekommen.*

Beweis. Sei $\prec$ eine beliebige Wohlerweiterung der Ordnung $<$ auf G, also $G = \{g_0 \prec g_1 \prec \ldots \prec g_\lambda \prec \ldots \mid \lambda < \beta\}$. Wenden wir die Konstruktion (K) zu $G\ (<)$ an und zwar so, daß wir G in der Form der oben gewähnten Folge schreiben, d. h. $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \beta\}$. Sei $G_\alpha = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \alpha\}$ $(\alpha \leqq \beta)$ und bezeichnen wir die durch diese Konstruktion bekommene Wohlerweiterung als ϱ. Wir beweisen durch die transfinite Induktion, daß die Ordnungen $\prec$ und ϱ auf G_α für jedes $\alpha \leqq \beta$ identisch sind. Für $\alpha = 0$ ist die Behauptung klar. Sei $\alpha > 0$, $\alpha \leqq \beta$ und nehmen wir an, daß diese Behauptung für jedes $\lambda < \alpha$ richtig ist. Wenn α eine isolierte Ordnungszahl ist, dann sind $\prec$ und ϱ identisch auf $G_{\alpha-1}$. Nehmen wir an, daß sie auf G_α nicht identisch sind. Also ist $g_{\alpha-1}$ nicht das größte Element in G_α (ϱ), d. h. $g_{\alpha-1} \varrho g_\lambda$ für ein $\lambda < \alpha - 1$. Das bedeutet, daß es ein $\lambda_{\alpha-1} < \alpha - 1$ gibt mit $g_{\alpha-1} < g_{\lambda_{\alpha-1}}$ und $g_{\lambda_{\alpha-1}} \varrho g_\lambda$ oder $\lambda_{\alpha-1} = \lambda$. Daraus $g_{\alpha-1} \prec g_{\lambda_{\alpha-1}}$ und da die Ordnung $\prec$ mit der Ordnung der Indices übereinstimmt, $\alpha - 1 < \lambda_{\alpha-1}$, was ein Widerspruch ist. Wenn α eine Limeszahl ist, dann sind $\prec$ und ϱ identisch auf G_λ für jedes $\lambda < \alpha$. Seien $g_\mu, g_\nu \in G_\alpha$, d. h. $\mu < \alpha$, $\nu < \alpha$. Es existiert ein $\lambda < \alpha$ mit $\mu < \lambda$, $\nu < \lambda$. Dann $g_\mu, g_\nu \in G_\lambda$ und deshalb $g_\mu \prec g_\nu \Leftrightarrow g_\mu\ \varrho\ g_\nu$. Die Ordnungen $\prec$ und ϱ sind also identisch auch auf G_α. Für $\alpha = \beta$ bekommen wir die Behauptung des Satzes.

In [5] beschreibt V. Sedmak eine andere Konstruktion einer linearen Erweiterung. Wir geben hier diese Konstruktion in einer ein wenig modifizierten Form an und werden sie *Konstruktion* (*S*) nennen.

Ist G eine Menge mit einer beliebigen Ordnung $<$ und $a \in G$, dann setzen wir A (a) $= \{x \in G \mid x \leqq a\}$ und bezeichnen $<_a$ die Ordnung, die zu der Ordinalsumme $A(a) \oplus [G - A(a)]$ zugehört. Es ist klar, daß diese Ordnung eine Erweiterung der Ordnung $<$ ist und daß das Element a vergleichbar mit allen $x \in G$ in $<_a$ ist. Die Konstruktion (S) ist jetzt folgende.

Sei G ($<$) eine geordnete Menge. Schreiben wir G in der Form einer wachsenden Folge des Typs β: $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \beta\}$ und definieren wir die Ordnung $<_\lambda$ für jedes $\lambda < \beta$ durch die transfinite Induktion so: $<_0 = <_{g_0}$ und $<_\lambda = (\bigcup\limits_{\mu<\lambda} <_\mu)g_\lambda$ für $\lambda > 0$. Aus der Definition folgt gleich, daß $<_\lambda$ Erweiterung von $<_\mu$ ist, wenn $\mu < \lambda$ gilt. Die konstruierende lineare Erweiterung ist dann $\prec = \bigcup\limits_{\lambda<\beta} <_\lambda$. V. Sedmak hat bewiesen, daß die Konstruktion (S) eine lineare Erveiterung liefert. Sein Resultat kann wie folgt ergänzt werden.

Satz 4. *Sei G eine geordnete Menge, die die Minimalbedingung erfüllt. Dann ist jede mit der Konstruktion (S) gewonnene lineare Erweiterung eine Wohlerweiterung der Menge G und man kann jede Wohlerweiterung von G mit der Konstruktion (S) gewinnen.*

Beweis. Schreiben wir G in der Form $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \beta\}$ und bezeichnen die durch die Konstruktion (S) bekommene lineare Erweiterung als $\prec$. Nehmen wir an, daß sie keine Wohlerweiterung ist, d. h. daß es eine unendliche fallende Kette $x_1 > x_2 > \ldots > x_n > \ldots$ in G ($<$) gibt. Für jedes n existiert ein Index λ_n so, daß $x_n = g_{\lambda_n}$. In der Folge $\{\lambda_n\}$ gibt es eine wachsende Teilfolge; also existieren in G ($\prec$) unendliche fallende Ketten $\{g_{\lambda_n}\}$, für die $\{\lambda_n\}$ eine wachsende Folge ist. Wählen wir eine solche Kette folgenderweise: λ_1 ist die kleinste Ordnungzsahl mit der Eigenschaft: in G ($\prec$) gibt es eine unendliche fallende Kette der Elemente, deren Indices eine wachsende Folge bilden und mit dem größten Element g_{λ_1}; $\lambda_2 > \lambda_1$ ist die kleinste Ordnungszahl mit der Eigenschaft: in G ($\prec$) gibt es eine unendliche fallende Kette der Elemente, deren Indices eine wachsende Folge bilden und mit dem größten und zweitgrößten Elemente g_{λ_1}, g_{λ_2} usw. Wir haben also $g_{\lambda_1} \succ g_{\lambda_2} \succ \ldots \succ g_{\lambda_n} \succ \ldots$ und da $\prec$ eine Eeweiterung von $<$ ist, gilt $g_{\lambda_n} > g_{\lambda_{n+1}}$ oder $g_{\lambda_n} \parallel g_{\lambda_{n+1}}$ für jedes n. Es ist unmöglich $g_{\lambda_n} > g_{\lambda_{n+1}}$ für jedes n, denn G ($<$) erfüllt die Minimalbedingung. Deshalb existiert die kleinste natürliche Zahl k mit $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$. Da $g_{\lambda_k} \succ g_{\lambda_{k+1}}$, existiert die kleinste Ordnungszahl $\mu < \beta$, für die $g_{\lambda_{k+1}} <_\mu g_{\lambda_k}$; dabei ist $\mu \leqq \lambda_k$, denn in $<_{\lambda_k}$ ist g_{λ_k} vergleichbar mit allen Elementen aus G.

(1) Sei $\mu = \lambda_k$. Dann ist $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$ in $<_\nu$ für jedes $\nu < \lambda_k$, also auch $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$ in $\bigcup\limits_{\nu<\lambda_k} <_\nu$. Daraus folgt $g_{\lambda_k} <_{\lambda_k} g_{\lambda_{k+1}}$, denn $g_{\lambda_{k+1}} \in G - A(g_{\lambda_k})$ in G $(\bigcup\limits_{\nu<\lambda_k} <_\nu)$ und das ist ein Widerspruch.

(2) Sei $\mu < \lambda_k$. Wir haben wieder $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$ in $<_\nu$ für jedes $\nu < \mu$, also auch $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$ in $\bigcup\limits_{\nu<\mu} <_\nu$. Wenn jetzt $g_{\lambda_k}, g_{\lambda_{k+1}} \in A(g_\mu)$ oder $g_{\lambda_k}, g_{\lambda_{k+1}} \in G - A(g_\mu)$ in $G(\bigcup\limits_{\nu<\mu} <_\nu)$ wäre, dann $g_{\lambda_k} \parallel g_{\lambda_{k+1}}$ in $<_\mu$, was ein Widerspruch ist. Ebenso ist unmöglich $g_\lambda \in A(g_\mu)$, $g_{\lambda_{k+1}} \in G - A(g_\mu)$, denn dann $g_{\lambda_k} <_\mu g_{\lambda_{k+1}}$, ein Widerspruch. Es muß also $g_{\lambda_k} \in G - A(g_\mu)$, $g_{\lambda_{k+1}} \in A(g_\mu)$ in $G(\bigcup\limits_{\nu<\mu} <_\nu)$ gelten, woraus $g_{\lambda_{k+1}} <_\mu g_\mu$,

$g_\mu <_\mu g_{\lambda_k}$, also auch $g_{\lambda_{k+1}} \prec g_\mu$, $g_\mu \prec g_{\lambda_k}$ folgt. Finden wir das größte n mit $\lambda_n < \mu$, dann ist $n \leqq k - 1$ und $g_{\lambda_1} \succ \ldots \succ g_{\lambda_n} \succ g_\mu \succ g_{\lambda_{k+1}} \succ \ldots$ ist eine unendliche fallende Kette mit der wachsenden Folge der Indices. Das ist aber ein Widerspruch mit der Minimalität von λ_{n+1}. Sei jetzt $\prec$ eine beliebige Wohlerweiterung von $<$ auf G, also $G = \{g_0 \prec g_1 \prec \ldots \prec g_\lambda \prec \ldots \mid \lambda < \beta\}$. Wenden wir die Konstruktion (S) zu $G\,(<)$ an und zwar so, daß wir setzen $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \alpha\}$. Wir zeigen, daß $\prec$ eine Erweiterung von $<_\lambda$ für jedes $\lambda < \beta$ ist.

Der Beweis wird mit der transfiniten Induktion durchgeführt.

Sei $\lambda = 0$. Das Element g_0 muß minimal in $G\,(<)$ sein; also $A(g_0) = \{g_0\}$ und g_0 ist das kleinste Element in $G\,(<_0)$; zugleich $(G - \{g_0\})\,(<_0) = (G - \{g_0\})\,(<)$. Da $\prec$ eine Erweiterung von $<$ auf G, also auch auf $G - \{g_0\}$ ist, folgt daraus, daß sie auch eine Erweiterung von $<_0$ ist.

Sei $\lambda > 0$, $\lambda < \beta$ und nehmen wir an, daß die Behauptung für jedes $\mu < \lambda$ richtig ist. Also ist $\prec$ eine Erweiterung von $<_\mu$ für jedes $\mu < \lambda$ und deshalb ist sie auch eine Erweiterung von $\bigcup\limits_{\mu<\lambda} <_\mu$. Bezeichnen wir die Ordnung $\bigcup\limits_{\mu<\lambda} <_\mu$ als ϱ. Setzen wir jetzt voraus, daß $\prec$ keine Erweiterung von $<_\lambda$ ist. Dann existieren ν, $\varkappa < \beta$, $\nu < \varkappa$ so, daß $g_\varkappa <_\lambda g_\nu$. Wenn g_ν, $g_\varkappa \in A(g_\lambda)$ in $G(\varrho)$ wäre, dann häten wir einen Widerspruch, denn $<_\lambda$ ist identisch mit ϱ auf $A(g_\lambda)$; ebenso ist unmöglich g_ν, $g_\varkappa \in G - A(g_\lambda)$. Also $g_\varkappa \in A(g_\lambda)$, $g_\nu \in G - A(g_\lambda)$. Daraus folgt $g_\varkappa \varrho g_\lambda$ oder $g_\varkappa = g_\lambda$, also auch $g_\varkappa \underline{\preceq} g_\lambda$, d. h. $\varkappa \leqq \lambda$. Ist jetzt $g_\lambda \varrho g_\nu$, dann $g_\lambda \prec g_\nu$ und $\lambda < \nu$. Ist $g_\lambda \parallel g_\nu$ in ϱ, dann auch $\lambda < \nu$, denn aus $\nu < \lambda$ folgt, daß die Elemente g_ν, g_λ vergleichbar in $<_\nu$, also auch in $\varrho = \bigcup\limits_{\mu<\lambda} <_\mu$ sind. In jedem Fall gilt also $\lambda < \nu$, sodaß $\varkappa \leqq \lambda < \nu \Rightarrow$ $\Rightarrow \varkappa < \nu$, was ein Widerspruch mit der Voraussetzung $\nu < \varkappa$ ist.

Die Ordnung $\prec$ ist also eine Erweiterung von $<_\lambda$ für jedes $\lambda < \beta$, also auch eine Erweiterung von $\bigcup\limits_{\lambda<\beta} <_\lambda$. Da $\bigcup\limits_{\lambda<\beta} <_\lambda$ linear ist, gilt $\prec = \bigcup\limits_{\lambda<\beta} <_\lambda$ und der Satz wird bewiesen.

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LATTICE-ORDERED GROUPS WITH MINIMAL PRIME SUBGROUPS SATISFYING A CERTAIN CONDITION

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In this paper one problem of P. Conrad's book [2] is partially solved in connection with one problem of F. Šik. There is proved (Theorem 1) that the set of all cardinal summands of an l-group G is equal to the set of all polars of this group if and only if G is projectable and satisfies a certain property. Further a connection between minimal prime subgroups and cardinal summands and also a connection between minimal prime subgroups and polars is shown here.

Let $G = [G, +, \vee]$ be an l-group. For $x \in G$ we shall denote $| x | = x \vee -x$. If $| a | \wedge | b | = 0$, then elements $a, b \in G$ will be called *disjoint*. If $\varnothing \neq A \subseteq G$, then we denote $A' = \{x \in G: | x | \wedge | a| = 0$ for each $a \in A\}$. Now $A \subseteq G$ is called *polar* if $A'' = A$.(A'' denotes $(A')'$.) Instead of $\{a\}'$, $\{a\}''$ we write a', a'', respectively. It is known that any polar is a convex l-subgroup of G. The set of all polars of G will be denoted by $\Gamma = \Gamma(G)$. If $B \in \Gamma$, then B, B' are called complementary polars.

The following theorem has been proved by F. Šik in [3] (Teorema 1):

Theorem A. (1) *Polars of an l-group form a complete Boolean algebra Γ (ordered by inclusion, an infimum is formed by an intersection).*

(2) *Polars that are l-ideals form a closed subalgebra Γ_1 of Γ.*

(3) *Cardinal summands of G form a subalgebra Γ_2 of Γ_1 (not always complete), where a supremum is formed by a sum of summands.*

It holds that for $B \in \Gamma_2(G)$ it is $G = B \oplus B'$. An l-group G is called an r-group if it is isomorphic to a subdirect product of totally ordered groups. By [4], an l-group is an r-group if and only if each its polar is an l-ideal. A convex l-subgroup P is called *prime* if the following is satisfied:

(i) *If $x \notin P$, then $x' \subseteq P$.*

(i) and the following conditions are equivalent:

(ii) *P contains at least one of polars a'', a' ($a \in G$).*

(iii) *P contains at least one of complementary polars.*

Any prime subgroup contains at least one minimal prime subgroup. In $G \neq \{0\}$, minimal prime subgroups are characterized among convex l-subgroups as: $a \notin P$ iff $a' \subseteq P$. A convex l-subgroup Z is a *z-subgroup* if from $x \in Z$ and $y' = x'$ it follows $y \in Z$. It is known that every polar and every minimal prime subgroup is a z-subgroup. An l-group is called *projectable* if $G = g' \oplus g''$ for each $g \in G$. Clearly any projectable l-group is an r-group.

The following theorem is proved in [1] (Théorème 3.1):

Theorem B. *An l-group G is projectable if and only if any proper prime subgroup contains exactly one prime z-subgroup.*

The problem how to characterize those l-groups for which $\Gamma_2(G) = \Gamma(G)$ has been given by P. Conrad in the book [2, p. 2.8]. Clearly any such l-group will be projectable.

Note. This problem has been solved by F. Šik in [3, p. 8] yet. He has proved that for an l-group the following are equivalent:

(1) *An arbitrary polar is a direct summand.*
(2) *A sum of two arbitrary polars is also a polar.*
(3) *A sum of an arbitrary pair of complementary polars is also a polar.*
(4) *Any pair of complementary polars forms a direct decomposition of this l-group.*

Another characterization is given in [4, Satz 13].

Further denote the following condition:

(*) *For each minimal prime subgroup A of an l-group G and for each polar K of G it is satisfied: $K \subseteq A$ iff $K' \nsubseteq A$.*

F. Šik has proposed (in a letter) the problem how to characterize l-groups with the property (*).

The following theorem shows a certain connection between both problems.

Theorem 1. *For an l-group $G \neq \{0\}$ it holds $\Gamma(G) = \Gamma_2(G)$ if and only if G is projectable and possesses the property (*).*

Proof. a) Let $\Gamma(G) = \Gamma_2(G)$ and let A be a minimal prime subgroup of G. Let $K \in \Gamma(G)$, $K, K' \subseteq A$. Since $K \oplus K' = G$, $A = A + A \supseteq K + K' = G$. If $G \neq \{0\}$, then by [5, Folgerung 7.3] $A \neq G$, a contradiction. But since A is a prime subgroup, it contains K or K'. Thus G satisfies (*).

b) Let G be projectable and have the property (*). Let $K \in \Gamma(G)$ such that $K \oplus K' \neq G$. Let P be a proper prime subgroup of G such that $K \oplus K' \subseteq P$. Let us remind yet that the filet of an element $x \in G$ is $\bar{x} = \{y \in G\colon y' = x'\}$ and the set of all filets $\mathscr{F}(G)$ form a distributive lattice. Denote thus $\Phi = \{\bar{x} : x \notin P\}$. Evidently Φ is a filter of $\mathscr{F}(G)$. For each $y \in K \cup K'$ it holds $\bar{y} \notin \Phi$. (If, namely, $y \in K$, $\bar{y} \in \Phi$, then $y'' = a''$ for some $a \notin P$ thus $y'' \nsubseteq P$; but $y'' \subseteq K$, and we have a contradiction. Similarly for $z \in K'$.)

Now if $x \in K \cup K'$, then denote a maximal filter of $\mathscr{F}(G)$ that contains Φ and does not $\bar{x}$ by Φ^x. It holds Φ^x is a prime filter. Therefore $Z^x = \{u \in G\colon \bar{u} \notin \Phi^x\}$ is a prime z-subgroup of G and clearly $Z^x \subseteq P$. Since G is projectable, all prime z-subgroups contained in $P \neq G$ are (by Theorem B) identical, thus for each $x_1, x_2 \in K \cup K'$ $Z^{x_1} = Z^{x_2}$. Further $\Phi^{x_1} = \Phi^{x_2}$ iff $\{u : \bar{u} \notin \Phi^{x_1}\} = \{v : \bar{v} \notin \Phi^{x_2}\}$ and this holds iff $Z^{x_1} = Z^{x_2}$. Thus for each $x_1, x_2 \in K \cup K'$ $\Phi^{x_1} = \Phi^{x_2}$ and therefore $\Psi = \bigcap_{x \in K \cup K'} \Phi^x = \Phi^x$ for each $x \in K \cup K'$. Hence Ψ is a prime filter of $\mathscr{F}(G)$ and $Z = \{w : \bar{w} \notin \Psi\}$ is a prime z-subgroup of G such that $Z \subseteq P$. Consequently, by [1, Proposition 3.1 and its proof], $Z = \bigcup_{a \notin P} a'$.

For each $x \in K \cup K'$ $x \in Z$, therefore $K \subseteq Z$, $K' \subseteq Z$ and this contradicts the assumption that G satisfies (*).

Now, it is easy to prove the further

Theorem 2. *For a projectable l-group G the following conditions are equivalent:*

(1) *Any polar of G is a cardinal summand of G. (Thus $\Gamma(G) = \Gamma_2(G)$.)*
(2) *G satisfies the property (*).*
(3) *The algebra $\Gamma_2(G)$ is a $\vee$-closed subalgebra of $\Gamma(G)$.*
(4) *The algebra $\Gamma_2(G)$ is a $\wedge$-closed subalgebra of $\Gamma(G)$.*

Proof. (3) $\Rightarrow$ (1): Let $K \in \Gamma(G)$. It holds $K = \bigvee_{\substack{\Gamma \\ a \in K}} a''$ and $a'' \in \Gamma_2(G)$ implies by (3), $K \in \Gamma_2(G)$.

(4) $\Rightarrow$ (1) : If $K \in \Gamma(G)$, then $K = \bigwedge_{\substack{\Gamma \\ b \in K'}} b'$. We have $b' \in \Gamma_2(G)$, thus by (4), $K \in \Gamma_2(G)$.

If H is a prime subgroup of an l-group G, then we say H has the property (**) if it holds:

(**) *If* $K \in \Gamma(G)$ *then* $K \subseteq H$ *iff* $K' \nsubseteq H$.

Further we say $\varnothing \neq A \subseteq G$ is *dense* in G if $A' = \{0\}$.

We get

Theorem 3. *A prime subgroup H of an l-group G is either a polar in G or it is dense in G.*

Proof. Let H not be dense. Then $\{0\} \neq H' \nsubseteq H$. Therefore $H'' \subseteq H$ i.e. H is a polar.

The following theorem is a consequence of Theorems 3 and 1.

Theorem 4. *If a projectable l-group G satisfies* (*) *then each minimal prime subgroup of G is a cardinal summand or it is dense in G.*

Denote now the set of all z-subgroups of an l-group G by $\mathscr{Z}(G)$. It is known (see [1, Proposition 2.3)] $\mathscr{Z}(G)$ forms a complete distributive lattice. It holds $\Gamma(G) \subseteq \mathscr{Z}(G)$ but generally $\Gamma(G)$ need not be a sublattice of $\mathscr{Z}(G)$.

We get

Theorem 5. *Let G be an l-group and* $\Gamma(G)$ *a closed sublattice of* $\mathscr{Z}(G)$. *Then a proper prime subgroup H of G has the property* (**) *if and only if H is a polar.*

Proof. If $H \in \mathscr{Z}(G)$, then (by [1, Proposition 2.1)] $H = \bigcup_{a \in H} a'' = \bigvee_{\substack{\mathscr{Z} \\ a \in H}} a''$. By the assumption $\bigvee_{\substack{\Gamma \\ a \in H}} a'' = \bigvee_{\substack{\mathscr{Z} \\ a \in H}} a''$ thus H is a polar. The converse is evident.

Therefore it holds also

Theorem 6. a) *Let an l-group G satisfy* (*) *and let* $\Gamma(G)$ *be a closed sublattice of* $\mathscr{Z}(G)$. *Then each minimal prime subgroup of G is a polar in G.*

b) *Let, in addition, G be projectable. Then each minimal prime subgroup is a cardinal summand of G.*

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ON PLAIN ABSOLUTE EQUILIBRIUM POINTS IN GENERAL NON-ORDERED GAMES WITH PERFECT INFORMATION [I]

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An essential generalization of Berge's variant of the Zermelo—von Neumann theorem (and of the original result of Zermelo) is proved. Our theorem concerns quite general non-ordered games with perfect information and with chain-valued pay-off functions, and it admits infinite plays. Important particular cases are considered. A further generalization (poset-valued pay-offs) is shown.

§ 0. INTRODUCTION

We shall consider quite general non-ordered games with perfect information, but without chance moves. ("Non-ordered games" means "non-initial games on oriented graphs". The considered games may have infinitely many positions or players, infinite plays are admitted, and the pay-off functions are chain-valued.)

Under our conception, at any moment of every play (of a game with perfect information) the moving player knows the preceding course (including the momentary position) of the play. In a contradistinction to the usually investigated games on (finite rooted) trees, in the considered games position need not "involve" the preceding course of play; consequently, at defining the general (pure) strategies it is necessary to introduce certain auxiliary notions ("segments" etc., see § 5). Nevertheless, the case if players use only the knowledge of momentary position is to be considered as the most important; the strategies corresponding to this case are called plain (§§ 2.6, 5.6.3, 5.7.6).

The notion of (pure) equilibrium point in a position (of a game with perfect information) can be introduced in the usual way; a system of strategies which is an equilibrium point in each position of a game is called an absolute equilibrium point of the game. An equilibrium point is called plain if it consists only of plain strategies. (Cf. §§ 2 and 5.)

The basic known result – the so-called Zermelo – von Neumann theorem – concerns only the games (with perfect information; chance moves are admitted) on finite rooted trees. This theorem was obtained by H. W. Kuhn ([7], § 4) as a corollary of his considerations on the decomposition of certain games; of course, it can be proved directly, by induction (cf., e.g., McKinsey's book [10], ch. VI, § 2, Th. 6.1). Nevertheless, the original result of Zermelo (in [14]) concerns (chess and similar) games which admit also infinite plays.

C. Berge has introduced and investigated (cf. [1], Ch. 1 of [2], and Ch. 6 and Appendix 1 of [3]) games (with perfect information) which are more natural and more general than the usually considered games on finite rooted trees, namely non--ordered games with perfect information. (Our conception of the latter games is somewhat more general than that of Berge. Cf. § 2.) The Berge equilibrium point theorem (it involves – after the elimination of chance moves – the above mentioned variant of the Zermelo – von Neumann theorem as a special case see [2], Ch. I, § 7 – the fundamental theorem, and [3], Ch. 6 – the Zermelo – von Neumann theorem) says that if a Bergean game with perfect information is locally finite (i.e., it has no infinite play) and each of its evaluation functions is finite-valued, then the game has (speaking in the terminology of the present paper) a plain absolute equilibrium point (but cf. §§ 3.6, 3.5, 2.8 – 9!). (Berge considers very special pay-off functions, namely those corresponding to "active" or "passive" players. Cf. § 2.4.0.) The proof can be performed in a natural way, but by means of transfinite induction (starting from the end-positions).

The original result of Zermelo (see [14]) concerns very special antagonistic (see § 2.11) games with perfect information. This result can be generalized in several ways; I proved several such theorems even for a somewhat more general kind of non-ordered antagonistic games (namely antagonistic complete games; the results of this character are based mainly on theorems 6.25/1 – 3 and 3.11 of [4]; they will be published in some of the following parts of [4], their preliminary variant is presented in [5]).

The main purpose of this paper is to generalize Berge's variant of the Zermelo – von Neumann theorem. Our main theorem gives an essential generalization in four ways: the notion of plain absolute equilibrium point is "stronger" and more natural than Berge's notion of absolute equilibrium point (cf. §§ 3.6, 2.8 – 9); the class of considered game structures (i.e. games without respecting pay-off functions) is somewhat richer (cf. §2); the class of pay-off functions (those satisfying the sufficient condition of the theorem) is much richer, also after restriction to the locally finite case; there exist games having infinite plays and satisfying the sufficient condition of the theorem. The latter way of generalization is to be considered as the most principal, since the usual proof idea is quite inapplicable if infinite plays are possible. (Cf. § 5.9.) Naturally, that sufficient condition is sizably strong (e.g., for each player of a game, all the infinite plays give the same pay-off) – the existence of a plain absolute equilibrium point is a "very strong property" (cf. § 5.8.1-2).

A certain part of the proof method (for the main theorem) is taken from the proof of the equilibrium point theorem in [6] (§ 3.13; this theorem concerns a class of finite complete (two-player) games), but the fact that infinite games are admitted in the present paper has led to essential complications (connected, among others, with the necessity of the use of transfinite induction), while the simpler structure of games with perfect information (in comparison with complete games) made possible to use some simplifications (cf. § 5.11); there are some other disparities.

Of course, we wish to obtain a theorem which sufficiently utilizes the new proof idea. This aim has led, especially, to the introduction of two auxiliary "technical" conditions ((A) and (B)) in the main theorem. There is a number of various particular cases (having self-contained meanings) in which the satisfaction of the whole sufficient condition is to be seen immediately; among others, the Berge theorem and the original result of Zermelo belong to these particular cases. (Cf. § 3.)

In this paper we consider chain-valued pay-off functions. Nevertheless, it is possible to introduce poset-valued pay-off functions, and to obtain some generalization of our results (§ 3) in a simple way; cf. §§ 5.0-5.5.

§ 1. ORIENTED GRAPHS. PSEUDOLENGTHS. SPECIAL PAY-OFF FUNCTIONS

1.0. Preliminaries. We shall use the accepted logical and set theoretical denotations and notions ($\neg$, $\wedge$, $\vee$, $\Rightarrow$; "iff" is to be read "if and only if"; $\emptyset$ means the empty set, X denotes the general cartesian product, etc.). For a set A, card A denotes the cardinal number of A, and we write $\exp A = \{B;\ B \subseteq A\}$ (the *Boolean* of A). Under a *binary relation* we mean a set of ordered pairs. *Mappings* are considered as special binary relations: $f = \{(f(x), x);\ x \in \operatorname{dom} f\}$ for any mapping f (where "dom" is the domain), and we write $f = (f(x);\ x \in \operatorname{dom} f)$, while $\{f(x);\ x \in \operatorname{dom} f\} = \operatorname{im} f$. There is exactly one mapping with empty domain (the *empty mapping*), namely $\emptyset$. For a mapping f and a set $A \subseteq \operatorname{dom} f$, the *restriction* of f to A is denoted by $f \mid A$ $(= (f(x);\ x \in A) = (\operatorname{im} f \times A) \cap f)$; of course, if $g \subseteq f$, then $g = f \mid \operatorname{dom} g$. At mappings denoted by Greek letters, sometimes we do not write parentheses.

A *partially ordered set* (*poset*) is a pair $\mathscr{V} = (V, \leqq)$, where V is a set and $\leqq\ \subseteq V \times V$ is a binary relation which is reflexive (on V), antisymmetric, and transitive; $\mathscr{V}$ is said to be a *chain* (or a *totally* ordered set, or a *linearly* ordered set) iff, moreover, $\leqq$ is full on V. (We say that $\rho \subseteq V \times V$ is *full* on V iff $V \times V = \rho^{-1} \cup \rho$.) We shall use the accepted elementary notions, denotations, and conventions for posets; especially, if we consider several posets or a system of posets, we often use the symbols $\leqq$, $>$, sup, min etc. *without index* (whenever no misunderstanding can arise by it).

1.1. Chains. Let $\mathscr{V} = (V, \leqq)$ be a chain. $\mathscr{V}$ is said to be *complete* iff sup A and inf A exist for any $A \subseteq V$ (i.e., iff the chain $\mathscr{V}$ is a complete lattice). $\mathscr{V}$ is said to be *well-ordered* [*inversely well-ordered*] iff min A [max A] exists for any nonempty $A \subseteq V$. It is easy to prove that *$\mathscr{V}$ is well-ordered and* (*at the same time*) *inversely well-ordered iff V is finite.*

1.2. Preference relations. Under a *preference relation* on a set X we mean a binary relation $\mathring{\leqq}\ \subseteq X \times X$ which is reflexive (on X), full (on X), and transitive.

Let X be a set, $\mathring{\mathscr{V}} = (V, \mathring{\leqq})$ be a set with a preference relation, let $f : X \to V$. Then the relation $\mathring{\leqq}_{(f,\mathring{\mathscr{V}})} = \{(x_1, x_2);\ x_1, x_2 \in X,\ f(x_1) \mathring{\leqq} f(x_2)\}$ is a preference relation on X.

On the other hand, if X is a set and $\mathring{\leqq}$ is a preference relation on X, then there exist a chain $\mathscr{V} = (V, \leqq)$ and a mapping $f : X \to V$ such that $\mathring{\leqq} = \mathring{\leqq}_{(f,\mathring{\mathscr{V}})}$. [It is sufficient to choose $V = X/(\mathring{\leqq}^{-1} \cap \mathring{\leqq}) = \{\{y;\ y \in X,\ y \mathring{\leqq} x,\ x \mathring{\leqq} y\};\ x \in X\}$ (the decomposition corresponding to the equivalence relation $\mathring{\leqq}^{-1} \cap \mathring{\leqq}$), $f = (\{y;\ y \in X, y \mathring{\leqq} x, y \mathring{\leqq} y\}; x \in X)$ (the natural surjection of X onto V), and $\leqq\ = \{(f(x_1), f(x_2));\ x_1, x_2 \in X,\ x_1 \mathring{\leqq} x_2\}$. We shall denote $\mathscr{V}_{\mathring{\leqq}} = \mathscr{V}$ and $f_{\mathring{\leqq}} = f$ for those $\mathscr{V}$ and f.]

1.3. Quasiorderings. Under a *quasiordering* on a set X we mean a binary relation (in X) which is reflexive (on X) and transitive. It is easy to see that it is admissible to re-formulate § 1.2 in the following way: "quasiordering" is to be written instead of "preference relation", "poset" is to be written instead of "chain", and "full (on X), " is to be omitted.

1.4. Let X be a set, let $\mathscr{V}_k = (V_k, \leqq_k)$ be chains and $f_k : X \to V_k$ for $k = 1, 2$. We say that f_1 with $\mathscr{V}_1$ and f_2 with $\mathscr{V}_2$ *express the same preference* iff $\mathring{\leqq}_{(f_1,\mathscr{V}_1)} = \mathring{\leqq}_{(f_2,\mathscr{V}_2)}$. We say that f_1 with $\mathscr{V}_1$ and f_2 with $\mathscr{V}_2$ *express antagonistic preferences* iff $\mathring{\leqq}_{(f_1,\mathscr{V}_1)} = (\mathring{\leqq}_{(f_2,\mathscr{V}_2)})^{-1}$.

1.5. Definition, remarks. Let $\mathring{\leqq}$ be a preference relation on a set X. We introduce binary relations $\mathring{\leqq}^{\varepsilon}$ ($\varepsilon = +, -$) on $(\exp X) \setminus \{\emptyset\}$ in such a way: for $\emptyset \neq A, B \subseteq X$

$A \mathring{\leqq}^{+} B \Leftrightarrow$ for each $a \in A$ there exists $b \in B$ such that $a \mathring{\leqq} b$;

$A \mathring{\leqq}^{-} B \Leftrightarrow$ for each $b \in B$ there exists $a \in A$ such that $a \mathring{\leqq} b$.

These two relations are preference relations on $(\exp X) \setminus \{\emptyset\}$. [Evidently, they are reflexive and transitive. If $\emptyset \neq A,\ B \subseteq X,\ \neg B \mathring{\leqq}^{+} A$, then there exists $b_0 \in B$ such that $\neg b_0 \mathring{\leqq} a$ for each $a \in A$, but then $a \mathring{\leqq} b_0$ for each $a \in A$, and, therefore, $A \mathring{\leqq}^{+} B$. The fullness of $\mathring{\leqq}^{-}$ can be proved analogously.]

In particular, if $\mathring{\leqq} = \mathring{\leqq}_{(f,\mathscr{V})}$ for some complete chain $\mathscr{V} = (V, \leqq)$ and a suitable mapping $f : X \to V$, then (for $\varepsilon = +, -$) there holds: $\mathring{\leqq}^{\varepsilon} = \mathring{\leqq}_{(f^{\varepsilon},\mathscr{V})}$, where the mapping $f^{\varepsilon} : (\exp X) \setminus \{\emptyset\} \to V$ is defined in such a way:

$$f^{+}(A) = \sup \{f(a);\ a \in A\}, \quad f^{-}(A) = \inf \{f(a);\ a \in A\},$$

where $\emptyset \neq A \subseteq V$, and sup and inf are taken in $\mathscr{V}$. (The proof is simple.)

The motivation of the introduction of $\mathring{\leqq}^{+}$ and $\mathring{\leqq}^{-}$ is connected with preference relations and pay-off functions in Bergean games with perfect information. (See § 2.4.1.)

1.6. Graphs. Under an *oriented graph* (or only: a *graph*; we shall consider only oriented graphs) we undersand a mapping Γ such that

$$\operatorname{im} \Gamma \subseteq \exp \operatorname{dom} \Gamma$$

(this definition conforms to Berge's conception of oriented graphs; we do not denote a graph by (Γ, X) with $\Gamma : X \to \exp X$, as $X = \operatorname{dom} \Gamma$ is given by Γ).

1.7. Convention. (Positions.) If a fixed graph Γ is considered, we use symbols P, P_0, Z in the following sense:

$$P = \operatorname{dom} \Gamma, \quad P_0 = \{x;\, x \in P, \Gamma x = \emptyset\}, \quad Z = P \setminus P_0;$$

the elements of P (i.e., the vertices of the graph Γ) $[P_0;\, Z]$ will be called *positions* [*final positions*, or *terminal positions; nonfinal positions*, or *nonterminal positions*] (respectively).

1.8 Our **interpretation of graphs** corresponds to that of Berge: if Γ is a graph, $x \in P$, then $y \in \Gamma x$ occurs iff "Γ contains an *edge* which goes from x to y"; therefore, a *vertex* (position) is terminal iff there does not exist an edge going from it. In our considerations (in the following §§), Γ will usually be the *graph of a game*, and then Γx means the set of (all) positions which can follow immediately after x; any *play* (cf. § 1.13) of such a game is performed in the following way: at a nonfinal position x, the moving player chooses an element $y \in \Gamma x$ as the next (following) position, while at any final position the play terminates.

1.9. Transformations. Let Γ be a graph. The set

$$\mathsf{T}(\Gamma) = \bigcup_{Y \subseteq Z} \mathop{\times}_{z \in Y} \Gamma z$$

is said to be the set of *plain* Γ-*transformations*; we shall say "Γ-*transformation*", too (as the general Γ-transformations are not considered in the main parts of this paper; cf. § 5.6.1). E.g., $\emptyset$ is a Γ-transformation (the *empty* Γ-*transformation*). $\sigma \in \mathsf{T}(\Gamma)$ is said to be *full* iff $\operatorname{dom} \sigma = Z$. $\mathsf{T}_F(\Gamma)$ will denote the set of full Γ-transformations. Under a *conservative* Γ-*transformation* we mean $\sigma \in \mathsf{T}(\Gamma)$ such that $\operatorname{im} \sigma \subseteq P_0 \cup \cup \operatorname{dom} \sigma$; of course, the empty Γ-transformation and also all full Γ-transformations are conservative. Clearly, a subset of a Γ-transformation is a Γ-transformation, too.

1.10. Denotation. In the whole paper, we denote

$$W = \{0, 1, 2, \ldots\} \cup \{\infty\}$$

and, for any $l \in W$,

$$W_l = \{k;\, k \in W, k < 1 + l\} \left(= \begin{Bmatrix} \{0, 1, \ldots, l\} \\ W \setminus \{\infty\} \end{Bmatrix} \quad \text{if} \quad l \begin{Bmatrix} < \\ = \end{Bmatrix} \infty \right),$$

$(1 + \infty = \infty)$. $\mathscr{W}$ will denote W with the natural ordering.

1.11. Definition. Let $l \in W$, let $\mathbf{y} = (y_k;\, k \in W_l)$ (be a mapping of W_l), and let x be an element. Then we put $x \oplus \mathbf{y} = (x_k;\, k \in W_{1+l})$, where $x_0 = x$ and $x_{k+1} = y_k$ for each $k \in W_l$.

1.12. Definition. Let $l \in W$, let $\boldsymbol{x} = (x_k;\ k \in W_l)$ (be a mapping of W_l), let $m \in W_l$. Then we put $\boldsymbol{x}^{[m]} = (x_{k+m};\ k \in W_{l-m})$ $(\infty - m = \infty)$; $\boldsymbol{x}^{[m]}$ is called the mth *remainder* of $\boldsymbol{x}$.

1.13. Definitions, remarks. (Plays.) Let Γ be a graph. We say that $\boldsymbol{x}$ is a Γ-*play* (or only "a *play*", if Γ is fixed) iff $\boldsymbol{x} = (x_k;\ k \in W_l)$ for some $l \in W$ and some elements $x_k \in P$ such that $x_{k+1} \in \Gamma x_k$ for each $k < l$, and $x_k \in P_0$ if $l < \infty$ (cf. § 1.10). $\boldsymbol{X}_\Gamma$ (or only $\boldsymbol{X}$) denotes *the set of all* Γ-*plays*. We say that $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ *starts from* x iff $x_0 = x$ $\boldsymbol{\Gamma}x$ will denote *the set of all plays which start from* $x \in P$, and $\boldsymbol{\Gamma}$ is considered as the corresponding mapping, i.e. $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}x;\ x \in P)$. Clearly, $\boldsymbol{X} = \bigcup_{x \in P} \boldsymbol{\Gamma}x$, and $\boldsymbol{\Gamma}x \cap \boldsymbol{\Gamma}y = \emptyset$ if $x, y \in P$, $x \neq y$; it is easy to see that $\boldsymbol{\Gamma}x \neq \emptyset$ for each $x \in P$. If $x \in P_0$, then $\boldsymbol{\Gamma}x$ contains exactly one element, namely $\boldsymbol{\Gamma}x = \{(x_k;\ k = 0)\}$, where $x_0 = x$; we shall denote this Γ-play by (x). We say that $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ *passes in* $Y \subseteq P$ iff $\{x_k;\ k \in W_l\} \subseteq Y$.

For $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ we denote $\mathrm{L}(\boldsymbol{x}) = l$ (the *length* of $\boldsymbol{x}$; $\boldsymbol{x}$ is said to be *infinite* iff $\mathrm{L}(\boldsymbol{x}) = \infty$). L (or L_Γ) itself is considered as the corresponding mapping ($\mathrm{L}_\Gamma = \mathrm{L} = (\mathrm{L}(\boldsymbol{x})\ \ \boldsymbol{x} \in \boldsymbol{X}) : \boldsymbol{X} \to W$), and it is called ***the (natural) length on*** Γ.

Γ is said to be *locally finite* (or *progressively finite*; cf. [2], ch. I, § 7, or [3], ch. 3) iff $\mathrm{L}(\boldsymbol{x}) < \infty$ for each $\boldsymbol{x} \in \boldsymbol{X}$ (i.e. iff Γ has no infinite play). Of course, it may happen that Γ is finite (i.e. P is finite) and is not locally finite, or conversely.

Evidently, if $\boldsymbol{x} \in \boldsymbol{X}$ and $m \in W_{\mathrm{L}(\boldsymbol{x})}$, then $\boldsymbol{x}^{[m]} \in \boldsymbol{X}$. If $x, y \in P$, $\boldsymbol{y} \in \boldsymbol{\Gamma}y$, then $x \oplus \boldsymbol{y} \in \boldsymbol{X}$ iff $y \in \Gamma x$.

Supposition. In the remainder of § 1, let Γ be a fixed graph, $\boldsymbol{X} = \boldsymbol{X}_\Gamma$.

1.14. Transformations and plays. Let $\sigma \in \mathrm{T}(\Gamma)$, $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$. We say that $\boldsymbol{x}$ *complies with* σ iff

$$k < l,\ x_k \in \operatorname{dom} \sigma \Rightarrow x_{k+1} = \sigma x_k .$$

It is easy to see that if σ is a *conservative* Γ-transformation and $x \in P_0 \cup \operatorname{dom} \sigma$, then there exists *exactly one* $\boldsymbol{x} \in \boldsymbol{\Gamma}x$ which complies with σ; this Γ-play will be denoted by $\boldsymbol{p}(x, \sigma)$.

Clearly, if $x \in P$, $y \in \Gamma x$, $\boldsymbol{y} \in \boldsymbol{\Gamma}y$, $\sigma \in \mathrm{T}(\Gamma)$, and if $\boldsymbol{y}$ complies with σ, then $x \oplus \boldsymbol{y}$ complies with σ iff $y = \sigma x$.

1.15. Plain sets of plays. $\boldsymbol{Y} \subseteq \boldsymbol{X}$ is said to be *plain* iff there holds: if $\boldsymbol{x} = (x_k;\ k \in W_r) \in \boldsymbol{Y}$, $\boldsymbol{y} = (y_k;\ k \in W_s) \in \boldsymbol{Y}$, $m < r$, $n < s$, $x_m = y_x$, then $x_{m+1} = y_{n+1}$.

It is easy to see that $\boldsymbol{Y} \subseteq \boldsymbol{X}$ is plain iff there exists $\sigma \in \mathrm{T}_\mathrm{F}(\Gamma)$ such that any $\boldsymbol{x} \in \boldsymbol{Y}$ complies with σ.

1.16. Pay-off functions. A (general) *pay-off function* on Γ is given by a chain $\mathscr{V} = (V, \leqq)$ and a mapping $\boldsymbol{f} : \boldsymbol{X} \to V$ (but usually $\boldsymbol{f}$ is called "pay-off function", while $\mathscr{V}$ is to be given separately); the pay-off function is said to be *real-valued* iff the chain of real numbers may be taken as that $\mathscr{V}$.

1.17. Pseudolengths. A ***pseudolength*** on Γ is given by a chain $\mathscr{W}^* = (W^*, \leqq^*)$ having the greatest element (the latter will be denoted by ∞^*) and by a mapping $L^* : \boldsymbol{X} \to W^*$ such that the following conditions are satisfied (for any $\boldsymbol{x} \in \boldsymbol{X}$, $x \in P$):

D(1) $L(\boldsymbol{x}) = \infty \Rightarrow L^*(\boldsymbol{x}) = \infty^*$

D(2) $y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y,\ L^*(\boldsymbol{y}_1) \leqq^* L^*(\boldsymbol{y}_2) \Rightarrow L^*(x \oplus \boldsymbol{y}_1) \leqq^* L^*(x \oplus \boldsymbol{y}_2)$

D(3) $y \in \Gamma x,\ \boldsymbol{y} \in \boldsymbol{\Gamma} y,\ L^*(x \oplus \boldsymbol{y}) <^* \infty^* \Rightarrow L^*(x \oplus \boldsymbol{y}) >^* L^*(\boldsymbol{y})$;

of course, in such a case there holds

D′(2) $y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y,\ L^*(\boldsymbol{y}_1) = L^*(\boldsymbol{y}_2) \Rightarrow L^*(x \oplus \boldsymbol{y}_1) = L^*(x \oplus \boldsymbol{y}_2)$

D′(3) $y \in \Gamma x,\ \boldsymbol{y} \in \boldsymbol{\Gamma} y \Rightarrow L^*(x \oplus \boldsymbol{y}) \geqq^* L^*(\boldsymbol{y})$.

E.g., L (with $\mathscr{W}$) is a pseudolength. Further, the constant mapping of $\boldsymbol{X}$ into $\{\infty\}$ (with the one-element chain containing ∞) is a pseudolength; it will be called the *trivial pseudolength.* (We have not said "...onto $\{\infty\}$", as it may happen $\Gamma = \emptyset$, then $\boldsymbol{X} = \emptyset$, and the trivial pseudolength is the empty mapping.)

1.18. Qualitative pay-off functions. Let L^* with a chain $\mathscr{W}^*$ be a pseudolength on Γ. Under an ***L^*-qualitative*** (or, more exactly speaking, (L^*, $\mathscr{W}^*$)-qualitative) ***pay-off function*** (on Γ) we mean a real-valued pay-off function $\boldsymbol{f}^*$ on Γ such that the following conditions are satisfied (for any $\boldsymbol{x} \in \boldsymbol{X}$):

D(o) $$\boldsymbol{f}^*(\boldsymbol{x}) \in \{-1, 0, +1\}$$

D(i) $$L(\boldsymbol{x}) = \infty \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = 0$$

D(ii) $$\boldsymbol{f}^*(\boldsymbol{x}) = 0 \Rightarrow L^*(\boldsymbol{x}) = \infty^*$$

D(iii) $$0 < L(\boldsymbol{x}) \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = \boldsymbol{f}^*(\boldsymbol{x}^{[1]});$$

of course, in such a case there holds (for $\boldsymbol{x} = (x_k;\ k \in W_l)$)

D′(iii) $$L(\boldsymbol{x}) < \infty \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = \boldsymbol{f}^*((x_{L(\boldsymbol{x})}))\ (= \boldsymbol{f}^*(\boldsymbol{x}^{[L(\boldsymbol{x})]})),$$

i.e., the pay-off for a finite play is determined by the terminal position of the play, while (cf. D(i)) any infinite play gives 0 as the pay-off.

E.g., the constant mapping of $\boldsymbol{X}$ into $\{0\}$ is an L^*-qualitative pay-off function.

1.19. Quasiqualitative pay-off functions. Let L^* with a chain $\mathscr{W}^*$ be a pseudolength on Γ, let $\boldsymbol{f}^*$ be an L^*-qualitative pay-off function. Let $\boldsymbol{f}$ with a chain $\mathscr{V}$ be a pay-off function on Γ. We say that $\boldsymbol{f}$ is an ***L^*-quasiqualitative pay-off function complying***

with $\boldsymbol{f^*}$ iff the following conditions are satisfied (for any $\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbf{X}$, $x \in P$; cf. § 1.0):

$$\mathrm{D}(\bar{1}) \qquad y \in \Gamma x, \mathbf{y}_1, \mathbf{y}_2 \in \boldsymbol{\Gamma} y, \boldsymbol{f}(\mathbf{y}_1) \leqq \boldsymbol{f}(\mathbf{y}_2) \Rightarrow \boldsymbol{f}(x \oplus \mathbf{y}_1) \leqq \boldsymbol{f}(x \oplus \mathbf{y}_2)$$

$$\mathrm{D}(\bar{2}) \qquad \boldsymbol{f}^*(\mathbf{x}) < \boldsymbol{f}^*(\mathbf{y}) \Rightarrow \boldsymbol{f}(\mathbf{x}) \leqq \boldsymbol{f}(\mathbf{y})$$

$$\mathrm{D}(\bar{3}) \qquad \boldsymbol{f}^*(\mathbf{x}) = \boldsymbol{f}^*(\mathbf{y}) = \begin{Bmatrix} +1 \\ -1 \end{Bmatrix}, \quad \mathrm{L}^*(\mathbf{x}) < \mathrm{L}^*(\mathbf{y}) \Rightarrow \boldsymbol{f}(\mathbf{x}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} \boldsymbol{f}(\mathbf{y}),$$

$$\mathrm{D}(\bar{4}) \qquad \boldsymbol{f}^*(\mathbf{x}) = \boldsymbol{f}^*(\mathbf{y}) \geqq 0, \ \ \mathrm{L}^*(\mathbf{x}) = \infty^* = \mathrm{L}^*(\mathbf{y}) \Rightarrow \boldsymbol{f}(\mathbf{x}) = \boldsymbol{f}(\mathbf{y});$$

of course, $\mathrm{D}(\bar{4})$ can be expressed equivalently in such a way:

$$\mathrm{D}(\overline{4.1}) \qquad \boldsymbol{f}^*(\mathbf{x}) = \boldsymbol{f}^*(\mathbf{y}) = 1, \ \ \mathrm{L}^*(\mathbf{x}) = \infty^* = \mathrm{L}^*(\mathbf{y}) \Rightarrow \boldsymbol{f}(\mathbf{x}) = \boldsymbol{f}(\mathbf{y}),$$

$$\mathrm{D}(\overline{4.2}) \qquad \boldsymbol{f}^*(\mathbf{x}) = \boldsymbol{f}^*(\mathbf{y}) = 0 \Rightarrow \boldsymbol{f}(\mathbf{x}) = \boldsymbol{f}(\mathbf{y})$$

(cf. D(o), D(ii)).

E.g., the function $\boldsymbol{f}^*$ itself (with the chain of real numbers) is an L*-quasiqualitative pay-off function complying with $\boldsymbol{f}^*$ (namely, then D(1) follows from D(iii)).

1.20. The interpretation (cf. §§ 1.17 – 19). Let L* with a chain $\mathscr{W}^*$ be a pseudolength, let $\boldsymbol{f}^*$ be an L*-qualitative pay-off function, let $\boldsymbol{f}$ be an L*-quasiqualitative pay-off function complying with $\boldsymbol{f}^*$.

L* can be interpreted as a certain *criterion of the continuance of plays*; ∞^* means the "*very long*" continuance. The conditions D(1) – D(3) are natural and their meanings are clear.

+1 [0; −1] means (as a value of $\boldsymbol{f}^*$) "*win*" ["*draw*"; "*loss*"]; therefore, *any infinite play is drawn* (under $\boldsymbol{f}^*$, see D(ii)), and the "*qualitative result*" *of any finite play* $\mathbf{x}$ *is equal to the natural evaluation* $\boldsymbol{f}^*((x))$ *of its terminal position* x (cf. Di(iii)). The condition D(ii) has a special character; *only* "*very long*" *plays may be drawn.*

Note that the important condition (of a certain *monotony* at one-move extensions of plays) expressed by D(2) for L* and by $\mathrm{D}(\bar{1})$ for $\boldsymbol{f}$ is not so strong as could be expected; cf. § 1.25.1.

The meanings of the other conditions are clear: $\mathrm{D}(\bar{2})$ expresses the *compliance of* $\boldsymbol{f}$ *with* $\boldsymbol{f}^*$; $\mathrm{D}(\bar{3})$ says that "*more rapidly*" (in the sense given by L*) *won* [*lost*] Γ-*plays are (nonstrongly) better* [*worse*]: $\mathrm{D}(\overline{4.1})$ says that *all* "*very long*" *won plays have the same value* (under $\boldsymbol{f}$); similarly, $\mathrm{D}(\overline{4.2})$ says that *all drawn plays* (they are "very long", see D(ii)) *have the same value.*

1.21. Lemma. *Let Γ be locally finite. Let* $\boldsymbol{f}$ *with a chain* $\mathscr{V}$ *be a pay-off function on Γ. Then the following statements are equivalent*:

(A) *There exists a pseudolength* L* (*on* Γ) *with a chain* $\mathscr{W}^*$ *and an* L*-*qualitative pay-off function* $\boldsymbol{f}^*$ (*on* Γ) *such that* $\boldsymbol{f}$ *is an* L*-*quasiqualitative pay-off function complying with* $\boldsymbol{f}^*$.

(B) *f is an* L^0*-quasiqualitative pay-off function complying with* f^0 *where* L^0 *is the trivial pseudolength on* Γ, *and (the* L^0*-qualitative pay-off function)* f^0 *is the mapping of* $\boldsymbol{X}$ *into* $\{-1\}$.

(C) *f satisfies the condition* $D(\bar{1})$.

Proof. Evidently, f^0 is an L^0-qualitative pay-off function. Of course, (B) $\Rightarrow$ (A), and (A) $\Rightarrow$ (C). If (C) is satisfied, then D(o)$-$(iii) and $D(\bar{2})-(\bar{4})$ with $L^* = L^0$, $f^* = f^0$ are satisfied in a trivial way, while $D(\bar{1})$ is satisfied by the supposition; hence, (B) holds. Therefore, (C) $\Rightarrow$ (B).

1.22. Lemma. *Let f be a real-valued pay-off function on* Γ. *Let the following conditions be satisfied (for any* $\mathbf{x} \in \boldsymbol{X}$, $x \in P$);

$$(\alpha) \qquad L(\mathbf{x}) = \infty \Rightarrow f(\mathbf{x}) = 0$$

$$(\beta) \qquad y \in \Gamma x,\ \mathbf{y}_1, \mathbf{y}_2 \in \Gamma y,\ |f(\mathbf{y}_1)| \leqq |f(\mathbf{y}_2)| \Rightarrow |f(x \oplus \mathbf{y}_1)| \leqq |f(x \oplus \mathbf{y}_2)|$$

$$(\gamma) \qquad L(\mathbf{x}) > 0, \qquad f(\mathbf{x}^{[1]}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} 0 \Rightarrow f(\mathbf{x}^{[1]}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} f(\mathbf{x}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} 0.$$

Let $\mathscr{W}^* = (W^*, \leqq^*)$ *be the chain with* $W^* = [0, \infty] \times [0, \infty]$ *and with the lexicographic ordering as* $\leqq^*$, *let* $\infty^* = (\infty, \infty)$ *(the greatest element of* $\mathscr{W}^*$).

Let mappings f^* *and* L^* *of* $\boldsymbol{X}$ *be defined in the following way:*

$$\mathbf{f}^*(\mathbf{x}) = \begin{Bmatrix} \operatorname{sgn} \mathbf{f}(\mathbf{x}^{[L(\mathbf{x})]}) \\ 0 \end{Bmatrix} \quad \text{if} \quad L(\mathbf{x}) \begin{Bmatrix} < \\ = \end{Bmatrix} \infty,$$

$$L^*(\mathbf{x}) = \begin{Bmatrix} \left(\dfrac{1}{|\mathbf{f}(\mathbf{x})|},\ L(\mathbf{x}) \right) \\ \infty^* \end{Bmatrix} \quad \text{if} \quad f(\mathbf{x}) \begin{Bmatrix} + \\ = \end{Bmatrix} 0,$$

(for any $\mathbf{x} \in \boldsymbol{X}$), *where* $\operatorname{sgn} a = 1\ [0:\ -1]$ *if* $a > 0$ $[a = 0:\ a < 0]$.

Then L^* *with* $\mathscr{W}^*$ *is a pseudolength (on* Γ), f^* *is an* L^**-qualitative pay-off function, and f is an* L^**-quasiqualitative pay-off function complying with* f^*.

Proof. There holds;

$$(\beta') \quad [y \in \Gamma x,\ \mathbf{y}_1, \mathbf{y}_2 \in \Gamma y, f(\mathbf{y}_1) \leqq f(\mathbf{y}_2) \Rightarrow f(x \oplus \mathbf{y}_1) \leqq f(x \oplus \mathbf{y}_2)] \ (\equiv D(\bar{1}))$$

$$(\gamma') \qquad L(\mathbf{x}) > 0 \Rightarrow |f(\mathbf{x}^{[1]})| \geqq |f(\mathbf{x})|$$

$$(\gamma'') \qquad |f(\mathbf{x})| \begin{Bmatrix} \leqq |f(\mathbf{x}^{[L(\mathbf{x})]})| \\ = 0 \end{Bmatrix} \quad \text{if} \quad L(\mathbf{x}) \begin{Bmatrix} < \\ = \end{Bmatrix} \infty,$$

$$(\gamma''') \qquad f^*(\mathbf{x}) = \operatorname{sgn} f(\mathbf{x})$$

[see: (β) and (γ); (γ); (γ') and (α); (γ) and (α) (respectively)].

The satisfaction of DD(o), (i), (iii), $(\bar{4})$ follows from the definitions given by the lemma. The satisfaction of DD(1), (2), (3), (ii), $(\bar{1})$, $(\bar{2})$, $(\bar{3})$ follows from: (α); (β); (γ'); (γ''); (β'); (γ'''); (γ'') (respectively). Thus, the lemma is proved.

1.23. Lemma. *Let* $0 \neq T \subseteq [0, \infty)$, *let* $\varphi : T \times \{0, 1, 2, \ldots\} \to [0, \infty)$ *be such that for any* $t, t_1, t_2 \in T$, $l_1, l_2 \in \{0, 1, 2, \ldots\}$ *there holds*;

(a.1) $$t_1 < t_2 \Rightarrow \varphi(t_1, l_1) < \varphi(t_2, l_2)$$

(a.2) $$l_1 < l_2 \Rightarrow \varphi(t, l_1) \geqq \varphi(t, l_2)$$

(b) $$\varphi(t, l_1) = \varphi(t, l_1 + 1) \Rightarrow \varphi(t, l_2) = \varphi(t, l_1) \quad \textit{for each } l_2 \geqq l_1 .$$

Let h *be a (real-valued) function on* P_0 *such that*

$$\{| h(x) |; x \in P_0\} \subseteq T.$$

Let f *be the real-valued pay-off function (on* Γ*) such that for an arbitrary* $\mathbf{x} = (x_k; k \in W_l) \in \mathbf{X}$ *there holds*

$$f(\mathbf{x}) = \begin{Bmatrix} \varphi(| h(x_l) |, l) \,.\, \operatorname{sgn} h(x_l) \\ 0 \end{Bmatrix} \quad \textit{if} \quad l \begin{Bmatrix} < \\ = \end{Bmatrix} \infty .$$

Then the conditions (α), (β), (γ) (§ 1.22) *are satisfied (by this* f*).*

Proof. (α) follows immediately from the definition of f, (γ) is to be derived by means of (a.2). Let $x \in Z$, $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Gamma} x$, $\mathbf{x} = (x_k; k \in W_{l_1})$, $\mathbf{y} = (y_k; k \in W_{l_2})$, $|f(\mathbf{x}^{[1]})| \leqq |f(\mathbf{y}^{[1]})|$. If $l_1 = \infty$, then $|f(\mathbf{x})| = 0 \leqq |f(\mathbf{y})|$. If $l_2 = \infty$, then $\mathrm{L}(\mathbf{y}^{[1]}) = \infty$, $|f(\mathbf{x})| \leqq |f(\mathbf{x}^{[1]})| \leqq |f(\mathbf{y}^{[1]})| = 0$ (see (γ), (α)), i.e. $|f(\mathbf{x})| = 0 = |f(\mathbf{y})|$. Let $l_1 < \infty$, $l_2 < \infty$. We put $r_1 = | h(x_{l_1}) |$, $r_2 = | h(y_{l_2}) |$. Thus, $|f(\mathbf{x})| = \varphi(r_1, l_1)$, $|f(\mathbf{y})| = \varphi(r_2, l_2)$, $\varphi(r_1, l_1 - 1) = |f(\mathbf{x})^{[1]}| \leqq |f(\mathbf{y}^{[1]})| = \varphi(r_2, l_2 - 1)$. Hence, $r_1 \leqq r_2$ (see (a.1)). If $r_1 < r_2$, then (cf. above and again (a.1)) $|f(\mathbf{x})| < |f(\mathbf{y})|$. Let $r_1 = r_2 = r$. If $l_1 \geqq l_2$, then $|f(\mathbf{x})| = \varphi(r, l_1) \leqq \varphi(r, l_2) = |f(\mathbf{y})|$ (see (a.2)). Now, let $l_1 < l_2$. Then $l_1 - 1 < l_1 \leqq l_2 - 1$, and $\varphi(r, l_1 - 1) \geqq \varphi(r, l_1) \geqq \varphi(r, l_2 - 1) \geqq \varphi(r, l_1 - 1)$ (cf. above and (a.2)), hence $\varphi(r, l_1 - 1) = \varphi(r, l_1)$, thus (cf. (b)) $|f(\mathbf{x})| = \varphi(r, l_1) = \varphi(r, l_2) = |f(\mathbf{y})|$. We have proved that $|f(\mathbf{x})| \leqq |f(\mathbf{y})|$ in any case. Consequently, there holds even

(β') $$\mathbf{x}_1, \mathbf{x}_2 \in \boldsymbol{\Gamma} x,\ |f(\mathbf{x}_1^{[1]})| \leqq |f(\mathbf{x}_1^{[1]})| \Rightarrow |f(\mathbf{x}_1)| \leqq |f(\mathbf{x}_2)|,$$

which, of course, implies (β).

1.24. Important particular cases of the situations considered in §§ 1.22 – 1.23.

1.24.1. (Cf. § 1.21.) Let Γ be *locally finite*. Let $\mathscr{V} = (V, \leqq)$ be a chain, let

$$F : \{(x, y, v); x \in P, y \in \Gamma x, v \in V\} \to V,$$

let (for any (x, y, v_1), $(x, y, v_2) \in \operatorname{dom} F$)

(*) $$v_1 \leqq v_2 \Rightarrow F(x, y, v_1) \leqq F(x, y, v_2).$$

Let

$$h_0 : P_0 \to V.$$

It is easy to prove (by induction) that there exists exactly one mapping $\boldsymbol{f}: \boldsymbol{X} \to V$ (it, together with $\mathscr{V}$, will be considered as a pay-off function on Γ) such that

$$\boldsymbol{f}((x)) = h_0(x) \quad \text{if} \quad x \in P_0,$$
$$\boldsymbol{f}(x \oplus \boldsymbol{y}) = F(x, y, F(\boldsymbol{y})) \quad \text{if} \quad x \in P, y \in \Gamma x, \boldsymbol{y} \in \boldsymbol{\Gamma} y.$$

Evidently, *f satisfies the condition* D($\bar{1}$).

In particular, if $\circ$ is a binary operation on V such that $(V, \circ, \leqq)$ is a linearly ordered abelian semigroup (hence, if $v_1, v_2, v_3, v_4 \in V$, then $v_1 \leqq v_2 \wedge v_3 \leqq v_4$ implies $v_1 \circ v_3 \leqq v_2 \circ v_4$), and $h : P \to V$ (the so-called *evaluation function*), then, if we choose F and h_0 such that

$$F(x, y, v) = h(x) \circ v \text{ for any } x \in Z, y \in \Gamma x, v \in V,$$
$$h_0 = h \mid P_0,$$

we have a particular case of the above introduced situation, and

$$\boldsymbol{f}(\boldsymbol{x}) = h(x_0) \circ h(x_1) \circ \ldots \circ h(x_l)$$

for any $\boldsymbol{x} = (x_k ; k \in W_l) \in \boldsymbol{X}$. E.g., it is possible to choose the set of real numbers as V, the natural ordering as $\leqq$, and $\circ$ = max or $\circ$ = min (where "max" and "min" are considered as binary operations) or $\circ = +$; clearly, the case $\circ$ = max [$\circ$ = min] gives exactly the "*active*" ["*passive*"] *pay-off functions* (on the locally finite graph Γ) in the sense introduced by Berge (cf., e.g., [2], ch. I. § 2).

It is easy to generalize somewhat the case with ("*vertex-*") *evaluation*, e.g. by introducing some "*mixed*" *evaluation* (given by some *end-vertex evaluation* $h_0 : P_0 \to V$ and an "*edge evaluation*" $h_E : \{(x, y);\, x \in P, y \in \Gamma x\} \to V$).

1.24.2. (Cf. § 1.23.) Let g be a real-valued pay-off function on Γ, let $\boldsymbol{g}(\boldsymbol{x}) = 0$ *if* $\mathrm{L}(\boldsymbol{x}) = \infty$, let $\boldsymbol{g}(\boldsymbol{x})$ *depend only on the terminal position of* $\boldsymbol{x}$ *if* $\mathrm{L}(\boldsymbol{x}) < \infty$. Choosing $T = [0, \infty)$, and (see § 1.23) $\varphi(t, r) = t$ for each $t \in T$, $r = 0, 1, 2, \ldots$, $h(x) = \boldsymbol{g}((x))$ for each $x \in P_0$, we see that $\boldsymbol{g} = \boldsymbol{f}$ where $\boldsymbol{f}$ is given by § 1.23 to the above described T, φ, h.

1.24.3. (Cf. § 1.23.) A real-valued pay-off function $\boldsymbol{f}^*$ is said to be ***qualitative*** iff it satisfies the conditions DD(o) (i), (iii); the interpretation is the same as that (of $\boldsymbol{f}^*$) mentioned in § 1.20. If we wish to express such an interest under which (at a given qualitative pay-off function $\boldsymbol{f}^*$) more rapidly won [lost] plays are considered as better [worse], we can choose, e.g., $T = \{0, 1\}$, $\varphi(t, r) = \dfrac{t}{1 + r}$ $(t \in T, r = 0, 1, 2, \ldots)$ $h(x) = \boldsymbol{f}^*((x))$ (see § 1.23), and then the pay-off function $\boldsymbol{f}$ given by § 1.23 expresses the above mentioned interest.

1.25. $D(\bar{1})$, $D(\bar{4})$, and some stronger conditions. Let f with a chain $\mathscr{V} = (V, \leqq)$ be a pay-off function on Γ.

1.25.1. We introduce condition

$D(\bar{1}^+)$ $y_k \in \Gamma x$, $\mathbf{y}_k \in \Gamma y_k (k = 1, 2,), f(\mathbf{y}_1) \leqq f(\mathbf{y}_2) \Rightarrow f(x \oplus \mathbf{y}_1) \leqq f(x \oplus \mathbf{y}_2)$,

which is, evidently, stronger than $D(\bar{1})$.

E.g., the pay-off functions f in the special case in § 1.24.1 (that in which f is given by h, at $(V, \circ, \leqq)$) and in §§ 1.24.2-3 satisfy even $D(\bar{1}^+)$.

1.25.2. Further, we introduce condition

$D(\bar{4}^+)$ $\qquad f^*(\mathbf{x}) = f^*(\mathbf{y}),\ \ L^*(\mathbf{x}) = \infty^* = L^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) = f(\mathbf{y})$,

which is, evidenty stronger than $D(\bar{4})$.

E.g., the pay-off function f considered in § 1.22 (which, of course, involves the cases considered in §§ 1.23, 1.24.2-3), with L^* and f^* introduced there, satisfies $D(\bar{4}^+)$ (as, even, $L^*(\mathbf{x}) = \infty^*$ implies $f(\mathbf{x}) = 0$, in § 1.22).

1.26. On the linear transformations at real-valued pay-off functions.

Let f be a real-valued pay-off function on Γ, let $\lambda \neq 0$ and c be real numbers, let $g = c + \lambda f (= (c + \lambda f(\mathbf{x}); \mathbf{x} \in \mathbf{X}))$. There is a number of trivial but very useful auxiliary propositions on transfering the properties of f to g; we shall need the following ones:

a) If $\lambda > 0$ [$\lambda < 0$], then f and g express the same preference [antagonistic preferences].

b) If f satisfies (α), (β), (γ) (§ 1.22) and $c = 0$, then g satisfies (α), (β), (γ). If f is given by some φ and h in the sense described in § 1.23, then g is given by φ and $h\,.\,\operatorname{sgn}\lambda$.

c) Let L^* (with some $\mathscr{W}^*$) be a pseudolength (on Γ), let f^* be an L^*-qualitative pay-off function. Let $g^* = f^*\,.\,\operatorname{sgn}\lambda$. Then g^* is an L^*-qualitative pay-off function. If f is an L^*-quasiqualitative pay-off function complying with f^*, and if

either $\lambda > 0$,
or $\lambda < 0$ and f satisfies $D(\bar{4}^+)$,

then g is an L^*-quasiqualitative pay-off function complying with g^*.

Note that linear transformations can be applied also in such a way: real numbers $\lambda_k \neq 0$, $c_k (k = 1, 2)$ are given, and pay-off functions $f_k = c_k + \lambda_k f$ are considered; of course, then $f_{3-k} = \left(c_{3-k} - \dfrac{c_k \lambda_{3-k}}{\lambda_k}\right) + \dfrac{\lambda_{3-k}}{\lambda_k} f_k$ for $k = 1, 2$.

§ 2. GAMES WITH PERFECT INFORMATION

2.1. Partitions. We say that $(P(j);\, j \in J)$ is a partition of a set P iff (J is a set, $P(j)$ are sets, and) $\bigcup_{j \in J} P(j) = P$.

Convention. If a partition $(P(j);\, j \in J)$ of a set P is given, we denote, for each $x \in P$, by $j(x)$ that element of J for which

$$x \in P(j(x))$$

(of course, there exists exactly one such $j(x)$).

2.2. Definition. Under a ***game with perfect information*** (we write only *g. p. i.*, too) we mean a 4-tuple

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (\boldsymbol{f}_j\,; j \in J))$$

such that Γ is an oriented graph, $(P(j);\, j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, $\mathscr{V}_j$ is a chain, and $\boldsymbol{f}_j : \mathbf{X}_\Gamma \to V_j$ (i.e., $\boldsymbol{f}_j$ with $\mathscr{V}_j$ is a pay-off function on Γ) for each $j \in J$. *$\mathscr{G}$ is said to have a property introduced for graphs* (cf. § 1.13) *iff Γ has this property.*

2.3. Definition; remarks on the interpretation. Let

$$\mathscr{G} = (\Gamma, (P(j);\, j \in J), (\mathscr{V}_j = (V_j, \leqq_j);\, j \in J), (\boldsymbol{f}_j\,;\, j \in J))$$

be a g.p.i. Γ is said to be the *graph* of $\mathscr{G}$, $(\Gamma, (P(j); j \in J))$ is called the *game structure* given by $\mathscr{G}$, and

$$(\Gamma, (P(j);\, j \in J);\, (\mathring{\leqq}_{(\boldsymbol{f}_j, \mathscr{V}_j)}\,;\, j \in J))$$

is called the *preference form* of $\mathscr{G}$.

The interpretation of Γ, P, P_0, Z, $\mathbf{X} = \mathbf{X}_\Gamma$ (cf. convention 1.7) *is given by* § 1.8. J is the *set of players* (if $J = 0$, then necessarily $P = \emptyset$). $\boldsymbol{f}_j$ with $\mathscr{V}_j (j \in J)$ is the pay-off function of player j. $P(j)$ is the *set of all the positions at which it is the player j's turn to move* (of course, actually j moves only at the positions from $Z \cap P(j)$). A *play* (starting from some position $x_0 \in P$) is performed in the natural way (cf. § 1.8); if x is a momentary nonfinal position of the play, then the *moving* (i.e. the choice of an element of Γx (as the next position) performing) *player* is $j(x)$.

The preference form of a g.p.i. is not so natural as the formally defined g.p.i. (§ 2.2) itself, but, on the other hand, *all the notions and properties introduced in this paper for games with perfect information could be formulated in terms of the preference forms* [see, in particular, § 2.10; of course, some notions (e.g., plain strategy) can be introduced in terms of game structures or even in terms of graphs (e.g. plays)].

Note that any triple

$$(\Gamma, (P(j); j \in J), (\mathring{\leqq}_j\,; j \in J))$$

where Γ is an oriented graph, $(P(j); j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, and $\overset{\circ}{\leqq}_j (j \in J)$ are preference relations on $\mathbf{X}_\Gamma$ — such an object will be called a *g.p.i. in preference form* — is the preference form of a suitable g.p.i. (e.g., of

$$(\Gamma, (P(j); j \in J), (\mathscr{V}_{\overset{\circ}{\leqq}_j}; j \in J), (f_{\overset{\circ}{\leqq}_j}; j \in J)),$$

cf. § 1.2). Therefore, for the aims of this work *the concept of g.p.i. in preference form could be given as the basic notion*, instead of the notion of g.p.i.

Note that also the notion of g.p.i. in preference form is somewhat redundant from the formal point of view (as the graph Γ can be omitted; namely, if $J = \emptyset$, then $P = \emptyset = \Gamma$, if $J \neq \emptyset$, then any $\overset{\circ}{\leqq}_j$ determines $\mathbf{X}_\Gamma$ and, hence, Γ uniquely).

2.4. Bergean games with perfect information.

2.4.0. Any *Bergean game with perfect information* (that introduced by his general definition in [2], Ch. I, § 2) can be defined formally as

$$(\Gamma, n, (P_1, \ldots, P_n), (\overset{\circ}{\leqq}_1, \ldots, \overset{\circ}{\leqq}_n), N^+, N^-)$$

where Γ is a graph, n is a positive integer, $N^+ \cup N^- = \{1, \ldots, n\}$ $(= N)$, $N^+ \cap N^- = \emptyset$, $(P_j; j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, and $\overset{\circ}{\leqq}_j (j \in J)$ are preference relations *on P*; moreover, Berge supposes that $P(j(x)) \cap \Gamma x = \emptyset$ for each $x \in P$. Here N is the set of *players* (or of the numbers of players), elements of $N^+[N^-]$ are called *active* [*passive*] players. For any $j \in N$, there exists exactly one $\varepsilon \in \{+, -\}$ such that $j \in N^\varepsilon$; we shall denote this ε by $\varepsilon(j)$.

2.4.1. It is easy to see that the adequate description of the interests of the players in a Bergean g.p.i. (in the sense of § 2.4.0) can be given, in terms of preference relations on $\mathbf{X}_\Gamma$, in the following way: if $j \in N$, then the player j's preference relation *on* $\mathbf{X} = \mathbf{X}_\Gamma$ equals $\overset{\circ}{\leqq}_{(g, \overset{\circ}{\mathscr{V}}_j)}$ where $g = (\{x_k;\ k \in W_l\};\ (x_k;\ k \in W_l) \in \mathbf{X})$ $(: \mathbf{X} \to \to (\exp P \setminus \{\emptyset\}))$, and $\overset{\circ}{\mathscr{V}}_j = (\mathbf{X}, \overset{\circ}{\leqq}{}_j^{\varepsilon(j)})$ (§ 1.5). In such a way we have transformed any Bergean g.p.i. to a g.p.i. in preference form, and the latter could be "naturally represented" by a g.p.i. (cf. § 2.3). In this sense and from our point of view (see § 2.3), *Bergean gs. p.i. may be considered as a particular case of gs. p.i. introduced in* § 2.2.

2.4.2. In the most usual case, *the preference relations $\overset{\circ}{\leqq}_j$ on P* of a Bergean g.p.i. (§ 2.4.0) *are given by real-valued "evaluation functions" f_j* on P, i.e. $\overset{\circ}{\leqq}_j = \overset{\circ}{\leqq}_{(f_j, \mathscr{R})}$ (cf. § 1.2) where $\mathscr{R}$ is the chain of real numbers. It is easy to see that in this case the *natural pay-off function* $f_j = f_j^{\varepsilon(j)} \cdot g$ [where $f_j^{\varepsilon(j)}$ is defined by § 1.4 to $\mathscr{V} = \mathscr{R}^* = ([-\infty, +\infty], \leqq)$ ($\leqq$ is the natural ordering), $X = \mathbf{X}$, and $f = f_j$] with $\mathscr{R}^*$ gives exactly the player j's preference relation (on $\mathbf{X}$) described in § 2.4.1, i.e., $\overset{\circ}{\leqq}_{(g, j)} = \overset{\circ}{\leqq}_{(f_j, \mathscr{R}^*)}$.

2.5. Of course, also the usually considered *games* with perfect information, with real-valued pay-off functions, and without chance moves *on trees* are involved in our definition (§ 2.2); the most usual case (the tree is finite, and the pay-offs are determined by the terminal positions) can be considered as a special case of Bergean games with perfect information (each player may be considered as active or as passive, and then his evaluation of nonfinal positions is less than or greater than (respectively) any evaluation of final positions). (Here we consider trees as special oriented graphs, and then the root of any tree is determined by the tree itself; thus, the initial position of a game of the mentioned kind need not be presented in the formal definition of the game.)

Supposition. In the remainder of § 2, let

$$\mathcal{G} = (\Gamma, (P(j);\, j \in J),\, (\mathcal{V}_j = (V_j, \leqq_j);\, j \in J), (f_j\,; j \in J))$$

be a game with perfect information. (Cf. §§ 1.0, 1.7, 2.1.)

2.6. Convention, definition. $\mathfrak{p}$ will have the meaning introduced in § 1.14. We shall write

$$Z(j) = Z \cap P(j),$$
$$\mathring{S}(j) = \mathop{\times}_{z \in Z(j)} \Gamma z, \qquad \mathring{S} = \mathop{\times}_{j \in J} \mathring{S}(j).$$

Elements of $\mathring{S}(j)$ are called the *plain strategies of j* (in $\mathcal{G}$). Elements of $\mathring{S}$ are said to be the *plain strategic situations.*

2.7. Remark. There exists a *natural bijection of $\mathring{S}$ onto* $\mathrm{T_F}(\Gamma)$, namely $(\sigma_j\,; j \in J) \mapsto$ $\mapsto \bigcup_{j \in J} \sigma_j$ (for each $(\sigma_j;\, j \in J) \in \mathring{S}$). (If $J = \emptyset$, then $P = \emptyset$, $\Gamma = \emptyset$, $\mathring{S} = \{\emptyset\} = \mathrm{T_F}(\Gamma)$.)

2.8. In accordance to the well-known fundamental definitions (cf., e.g., [10], Ch. VI. 2), the definition of equilibrium point consisting (only) of plain strategies (or, as we say, of *plain equilibrium point*) *in some* $x^0 \in P$ (this position is considered as an initial position) is to be formulated in such a way: $\sigma = (\sigma_j\,; j \in J) \in \mathring{S}$ is a plain equilibrium point (of $\mathcal{G}$) in x° iff for each $j_0 \in J$ and each $\mathbf{x} \in \boldsymbol{\Gamma} x^\circ$ complying with σ_j for any $j \in J \setminus \{j_0\}$ there holds $f_{j_0}(\mathbf{x}) \leqq_{j_0} f_{j_0}(\mathbf{y})$ where $\mathbf{y}$ is that element of $\boldsymbol{\Gamma} x^0$ which complies with σ_j for each $j \in J$; note that *it may happen that that* $\mathbf{x}$ *does not comply with any plain strategy of the player j.*

Using the 1-1 correspondence among the elements of S and those of $\mathrm{T_F}(\Gamma)$ (§ 2.7), we can give the direct definition of plain absolute equilibrium point (cf. § 0) in terms of full Γ-transformations:

2.9. Definition. Under a ***plain absolute equilibrium point*** of $\mathcal{G}$ we mean $\sigma \in \mathrm{T_F}(\Gamma)$ such that for each $x \in P$, $j \in J$, and for any $\mathbf{x} \in \boldsymbol{\Gamma} x$ complying with $\sigma \mid (Z \setminus Z(j))$ there holds

$$f_j(\mathbf{x}) \leqq_j f_j(\mathfrak{p}(x, \sigma)).$$

2.10. Remark. Of course, in the situation from § 2.9 $f_j(\mathbf{x}) \leqq_j f_j(\mathbf{p}(x, \sigma))$ if and only if $\mathbf{x} \stackrel{\circ}{\leqq}_{(f_j, \mathscr{V}_j)} \mathbf{p}(x, \sigma)$; therefore, *the notion of plain absolute equilibrium point could be formulated in terms of preference forms of gs. p.i.*

2.11. Definition, remark. The game $\mathscr{G}$ is called ***antagonistic*** iff card $J = 2$ and, for $\{j_1, j_2\} = J$, the pay-off functions f_{j_1}, f_{j_2} (with $\mathscr{V}_{j_1}, \mathscr{V}_{j_2}$) express antagonistic preferences.

E.g., if $J = \{j_1, j_2\}$, $j_1 \neq j_2$, and if there exist a real-valued pay-off function f (on Γ) and real numbers λ_k, $c_k (k = 1, 2)$ such that $f_{j_k} = c_k + \lambda_k f (k = 1, 2)$ and $\lambda_1 \lambda_2 < 0$, then $\mathscr{G}$ is antagonistic. (Cf. § 1.2.6.)

If $\mathscr{G}$ is antagonistic, then we also say "... *saddle point*" instead of "... equilibrium point".

§ 3. THE MAIN THEOREM. PARTICULAR CASES

(See the conventions in §§ 1.0, 1.7, 2.1, 2.6.)

3.0. The main theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a game with perfect information.

Let there exist a pseudolength L^* *(on* Γ*) with a chain* $\mathscr{W}^* = (W^*, \leqq^*)$*, and, for each* $j \in J$*, an* L^**-qualitative pay-off function* f_j^* *such that the following statements* (0), (*A*), (*B*) *are satisfied:*

(0) *For each* $j \in J$, f_j *(with* $\mathscr{V}_j$*) is an* L^**-quasiqualitative pay-off function complying with* f_j^*.

(A):

(A.1) $\{\mathrm{L}^*((x)); x \in P_0\}$ *is well-ordered (in* $\mathscr{W}^*$*).*

(A.2) *Let* $\nabla \subseteq Z$, $(y(z); z \in \nabla) \in \mathop{\times}_{z \in \nabla} (\Gamma z \setminus \nabla)$, $(\mathbf{y}(z); z \in \nabla) \in \mathop{\times}_{z \in \nabla} \Gamma y(z)$; *let set* $\{\mathbf{y}(z); z \in \nabla\}$ *be plain, and let each* $\mathbf{y}(z)$ $(z \in \nabla)$ *pass in* $P \setminus \nabla$. *Then* $\{\mathrm{L}^*(z \oplus \mathbf{y}(z)); z \in \nabla\}$ *is well-ordered (in* $\mathscr{W}^*$*).*

(B) *Let* $z \in Z$, $(\mathbf{y}(z); y \in \Gamma z) \in \mathop{\times}_{y \in \Gamma z} \Gamma y$, *and let set* $\{\mathbf{y}(y); y \in \Gamma z\}$ *be plain. Then*

(B/1) $\{f_{j(z)}(z \oplus \mathbf{y}(y)); y \in \Gamma z\}$ *is inversely well-ordered in* $\mathscr{V}_{j(z)}$;

(B/2) *if* $\{f_{j(z)}(\mathbf{y}(y)); y \in \Gamma z\} \left\{\begin{smallmatrix} = \\ \neq \end{smallmatrix}\right\} \{-1\}$, *then set* $\{\mathrm{L}^*(z \oplus \mathbf{y}(y)); y \in \Gamma z\}$ *is* $\left\{\begin{matrix} \textit{inversely well-ordered} \\ \textit{well-ordered} \end{matrix}\right\}$ *(in* $\mathscr{W}^*$*).*

Then the game $\mathscr{G}$ *has a plain absolute equilibrium point.*

3.1. Remark. The proof is presented in § 4. The idea of the construction of a plain absolute equilibrium point of the game $\mathscr{G}$ is based mainly on the condition (0);

each of the conditions DD (1) – (3), (o) – (iii), $(\bar{1})$ – $(\bar{4})$ *is used essentially in the proof.* The realization of this idea necessitates certain auxiliary "technical" conditions; we have chosen the conditions (A) and (B). This choice seems to be the most useful; our aim is not the full utilization of such conditions, we only wish to obtain immediately a number of important particular cases from a common result, and the theorem serves well for this aim, as we show in the following.

3.2. Theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a game with perfect information.

Let L* *be a pseudolength* (*on* Γ) *with a chain* $\mathscr{W}^* = (W^*, \leqq^*)$, *and, for each* $j \in J$, *let* f_j^* *be an* L*-*qualitative pay-off function such that* f_j (*with* $\mathscr{V}_j$) *is an* L*-*quasiqualitative pay-off function complying with* f_j^*.

Let there occur at least one of the following three cases:

(I) P *is finite.*
(II) *im* L* *is well-ordered* (*in* $\mathscr{W}^*$), *and* Γx *is finite for each* $x \in P$.
(III) *im* L* *is finite, im* f_j *is inversely well-ordered for each* $j \in J$.

Then the conditions (*A*) *and* (*B*) *from the main theorem are satisfied, and* (*therefore*) *the game* $\mathscr{G}$ *has a plain absolute equilibrium point.*

(The proof is trivial.)

3.3. Remark. Of course, at applying the theorem, further trivial or simple propositions could be used (**X** is finite $\Rightarrow$ P is finite; $\mathscr{W}^*$ is finite $\Rightarrow$ im L* is finite, etc.; see § 1.1). The case (II) is involved, e.g., in the following *case* (II^+), and the latter *is sufficient* (*in the situation from* § 3.2) *for the satisfaction of* (*A*) *and* (*B*), too:

(II^+) im L* *is well-ordered* (*in* $\mathscr{W}^*$), *and for each* $z \in Z$ *there exists a set* $Y \subseteq \Gamma z$ *such that sets* $\{f_{j(z)}(z \oplus \mathbf{y}); y \in Y, \mathbf{y} \in \Gamma y\}$ *and* $\{\mathrm{L}^*(z \oplus \mathbf{y}); y \in Y, \mathbf{y} \in \Gamma y\}$ *are inversely well-ordered* (*in* $\mathscr{V}_{j(z)}$ *and* $\mathscr{W}^*$, *respectively*), *and* $\Gamma z \setminus Y$ *is finite.*
(Namely, $\{\mathrm{L}^*(z \oplus \mathbf{y}); y \in Y, \mathbf{y} \in \Gamma y\}$ is finite if it is inversely-well ordered, as im L* is well-ordered (cf. § 1.1); the case (II) occurs if the choice $Y = \emptyset$ is admissible for each $z \in Z$.)

3.4. Theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a game with perfect information.

If $\mathscr{G}$ *is locally finite and, for each* $j \in J$, f_j *satisfies the condition* $D(\bar{1})$ *and* im f_j *is inversely well-ordered* (*in* $\mathscr{V}_j$), *then* $\mathscr{G}$ *has a plain absolute equilibrium point.*

(This follows immediately from §§ 1.21 (cf. (C) $\Rightarrow$ (B)) and 3.2 (cf. case (III); namely, $\operatorname{im} L^\circ \subseteq \{\infty\}$).)

3.5. Remark. Important corollaries of § 3.4 can be obtained by means of § 1.24.1, which presents a general construction of pay-off functions (on locally finite graphs) satisfying $D(\bar{1})$. In particular, there holds:

The Berge variant of the Zermelo – von Neumann theorem (cf. §§ 0, 3.6). *Any locally finite Bergean game with perfect information and finite-valued evaluation functions* (§§ 2.4.2, 2.4.0) *has a plain absolute equilibrium point.*

[Namely, in the considered case plain absolute equilibrium points can be defined in terms of the natural pay-off functions (cf. §§ 2.10, 2.4.1, 2.4.2), but, evidently, the latter can be expressed in the way mentioned in § 2.4.1 (and, hence, they satisfy $D(\bar{1})$), and they are finite-valued. Thus, the proposition follows from § 3.4.]

3.6. Remark, definition. Under a ***weak plain absolute equilibrium point*** of a game

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

with perfect information we mean $\sigma \in \mathrm{T_F}(\Gamma)$ such that for any $j \in J$ and $\sigma' \in \mathrm{T_F}(\Gamma)$

$$\sigma' \mid (Z' \setminus Z(j)) = \sigma \mid (Z \setminus Z(j)) \Rightarrow f_j(\mathbf{p}(x, \sigma')) \leqq_j f_j(\mathbf{p}(x, \sigma)).$$

It is clear that if σ is a plain absolute equilibrium point of $\mathscr{G}$, then σ is a weak plain absolute equilibrium point of $\mathscr{G}$. On the other hand, if Γ contains no *cyclic path* (under a cyclic path of Γ we mean a finite sequence $x_0, \ldots, x_m \in P (m \geqq 0)$ such that $x_j \in \Gamma x_{j-1}$ for $j = 1, \ldots, m$, and $x_0 \in \Gamma x_m$), then any weak plain absolute equilibrium point of $\mathscr{G}$ is a plain absolute equilibrium point of $\mathscr{G}$ [namely, if $\sigma \in \mathrm{T_F}(\Gamma)$, $j \in J$, and if $\boldsymbol{x} \in \boldsymbol{X}_\Gamma$ complies with $\sigma \mid (Z \setminus Z(j))$, then there exists $\sigma' \in \mathrm{T_F}(\Gamma)$ such that $\boldsymbol{x} = \mathbf{p}(x, \sigma')$ (etc., cf. § 2.9)]. Clearly, if Γ is locally finite, then it contains no cyclic path.

The notion of weak plain absolute equilibrium point expresses, in terms of this paper (and for the games considered here), the original Berge's notion of "(absolute) equilibrium point" (see [2], Ch. I, §§ 7 and 3). The preceding remarks show that the formulation of the Berge theorem used in § 3.5 is equivalent to the original Berge's formulation (in [2], Ch. I, § 7), although the latter concerns weak plain absolute equilibrium points.

3.7. Theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a two-player game with perfect information, let $J = \{j_1, j_2\}$.

Let there exist real numbers λ_k, $c_k (k = 1, 2)$ and a real-valued pay-off function f (on Γ) such that the conditions (α), (β), (γ) (§ 1.22) *are satisfied, and*

$$f_{j_k} = c_k + \lambda_k f \qquad (k = 1, 2),$$
$$\lambda_1 \, . \, \lambda_2 < 0.$$

Let there occur at least one of the following two cases:

(I') *P is finite.*
(II') $\{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ *is inversely well-ordered, and Γx is finite for each $x \in P$.*

Then $\mathcal{G}$ has a plain absolute saddle point.

Proof. Let $\mathcal{W}^*, f^*$, L* be the same as in § 1.22, let $f_{j_k}^* = f^* . \operatorname{sgn} \lambda_k (k = 1, 2)$. By means of §§ 1.22, 1.25.2, 1.26 (especially, cf. part c)) we conclude that L* with $\mathcal{W}^*$ is a pseudolength (on Γ) and, for each $j \in J$, f_j^* is an L*-qualitative pay-off function such that f_j is an L*-quasiqualitative pay-off function complying with f_j^*. Evidently, if $\{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ is inversely well-ordered, then (cf. § 1.22) im L* is well-ordered. Consequently, if (K') holds for K = I or K = II, then we have the case (K) from § 3.2; therefore (see §§ 1.26, 2.11) $\mathcal{G}$ has a plain absolute saddle point.

3.8. Remark. Of course, it would be possible to derive a certain corollary of the main theorem for the antagonistic case (by means of a suitable re-formulation of the "technical conditions" (A), (B) (§ 3.0) to pay-off functions constructed by means of §§ 1.22 and 1.26, etc.), but we take interest in (immediately applicable) important particular cases (cf. § 3.1); thus, we have based the antagonistic-case result (§ 3.7) directly on § 3.2. (Note that the case (III) in § 3.2 is not useful here, as it would lead to

(III') $\qquad \{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ *and* $\{L(\mathbf{x});\ \mathbf{x} \in \mathbf{X}\}$ *are finite,*

but this gives only a very special case of the situation considered in § 3.4.)

3.9. Remark. Important corollaries of § 3.7 can be obtained by means of § 1.23, which presents a general construction of (some) functions f (§ 1.22), **and by means of the special cases considered in §§ 1.24.2–3.** Especially, the result obtained by means of §§ 3.7 and 1.24.3 [it is easy to see how the direct expression of the notion of plain absolute saddle point is to be formulated (cf. [6], § 3.16, too); note that it is sufficient to take the case

(II") *Γx is finite for each $x \in P$,*
cf. §§ 1.24.3, 1.22] involves the original result of Zermelo (proved for chess in [14]); cf. § 0.

(To be continued)

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SYSTEMS OF EQUATIONS OVER FINITE BOOLEAN ALGEBRAS

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It is possible to construct the theory of systems of Boolean equations on the base of a general theory of systems of equations given in [1] by J. Slominski. The general theory is raised on principle of a homomorphic mapping corresponding to the given system. The investigation of homomorhic mappings can be carried out by means of the so called matrix representaion in finite Boolean algebras (see [6]). The problem of solving the Boolean systems can be easy transformed to problem of extension of a given mapping to the homomorphism. This problem was solved for finite Boolean algebras in [7].

The presented theory solves the problem of existence and number of solutions of Boolean systems and gives a simple algorithm for solving these systems.

This paper is a continuation of papers [6] and [7], the main conceptions and notations are given there.

1.

Definition 1. Let m, n be positive integers and $X = \{x_1, \dots, x_n\}$ $A = \{a_1, \dots, a_m\}$ be sets. The free Boolean algebra freely generated by the set $X \cup A$ is denoted by $\boldsymbol{B}_{a,x}$. Each element of $\boldsymbol{B}_{a,x}$ is called a B-polynom. The elements $x_i \in X$ resp. $a_j \in A$ are called variables resp. coefficients.

The Boolean operation join is denoted by $+$, meet by . and the complement of an element $b \in \boldsymbol{B}_{a,x}$ is denoted by $\bar{b}$.

Every transformation of a given B-polynom by the Boolean operations and identities is called an elementary transformation. On the $\boldsymbol{B}_{a,x}$ there is given the relation of equivalence. The B-polynom Φ is equal to the B-polynom ψ if and only if there exists a finite sequence of elementary transformations which performs Φ onto ψ. This relation of equivalence is denoted by $=$.

The algebra $\boldsymbol{B}_{a,x}$ has just 2^{m+n} elements because it has just 2^{m+n} atoms, i.e. elementary conjuctive forms

$$\tilde{x}_1 . \tilde{x}_2 . \dots . \tilde{x}_n . \tilde{a}_1 . \dots . \tilde{a}_m,$$

where $\tilde{x}_i = x_i$ or $\bar{x}_i$, $\tilde{a}_j = a_j$ or $\bar{a}_j$.

Definition 2. Let E_0 be the Cartesian product $\boldsymbol{B}_{a,x} \times \boldsymbol{B}_{a,x}$. Each subset $E \subseteq E_0$ is called the system of Boolean equations (or simply Boolean system or B-system) with parameters $a_1, \ldots, a_m$. The elements of X are called unknowns of the B-system E. Each pair $\langle \Phi, \psi \rangle \in E$ will be called a B-equation.

A subalgebra of $\boldsymbol{B}_{a,x}$ generated by the set A is denoted by $\boldsymbol{B}_a$. The Boolean algebra $\boldsymbol{B}_a$ has 2^{2^m} elements.

Definition 3. The B-system $E_1 \subseteq \boldsymbol{B}_a \times \boldsymbol{B}_a$ is called a compatible system if and only if the relation $\langle \Phi, \psi \rangle \in E_1$ implies the equivalence $\Phi = \psi$.

Definition 4. A mapping h is said to be a characteristic mapping of a system E provided that:

(a) h is a homomorphic mapping of $\boldsymbol{B}_{a,x}$ into $\boldsymbol{B}_a$.

(b) $\langle \Phi, \psi \rangle \in E$ implies $h(\Phi) = h(\psi)$.

(c) $h \mid \boldsymbol{B}_a$ is a homomorphic mapping of $\boldsymbol{B}_a$ onto $h(\boldsymbol{B}_{a,x})$. A characteristic mapping h of a system E is called proper if instead of (c) the following condition holds:

(d) $h \mid \boldsymbol{B}_a$ is an identical isomorphic mapping of $\boldsymbol{B}_a$ onto $\boldsymbol{B}_a$.

It is obvious that (d) implies (c).

Definition 5. The congruence relation $\sim_h$ induced by the characteristic mapping h of E (obviously holding $\Phi \sim_h \psi$ for each $\langle \Phi, \psi \rangle \in E$) is called the regularizer of the system E.

Definition 6. Each set $\{F_1, \ldots, F_n\}$ of B-polynoms $F_i \in \boldsymbol{B}_a$ is called a solution of the B-system E if the substitution of F_i instead of x_i in all places in Φ, ψ implies

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_h \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$, where $\sim_h$ is a regularizer of the B-system E. If $\sim_h$ is equal to $=$, the solution is called proper.

Accordingly, the solution is proper iff the characteristic mapping corresponding to the regularizer $\sim_h$ is proper. It is a solution by a classical definition.

Definition 7. Let $\sim_1, \sim_2$ be two congruences on $\boldsymbol{B}_{a,x}$. We define the ordering: $\sim_1 \leqq \sim_2$ iff for arbitrary elements $a, b \in \boldsymbol{B}_{a,x}$ the implication $a \sim_1 b \Rightarrow a \sim_2 b$ holds.

A regularizer of E which is minimal with respect to the ordering $\leqq$ on the set of all regularizers of E is called a minimal regularizer of E (see [1]).

Definition 8. The B-systems $E \subseteq E_0$, $E' \subseteq E_0$ are equivalent iff they have identical set of solutions.

Theorem 1. Let $E = \{\langle \Phi_1, \psi_1 \rangle, \ldots, \langle \Phi_k, \psi_k \rangle\}$ be a B-system of $\boldsymbol{B}_{a,x}$; π be a permutation of the set $\{1, \ldots, k\}$, f be a B-polynom of k variables and E^* be a compatible B-system of $\boldsymbol{B}_a$. Then the system E is equivalent with systems $E' =$

$= \{\langle \Phi_{\pi(1)}, \psi_{\pi(1)}\rangle, \ldots \langle \Phi_{\pi(k)}, \psi_{\pi(k)}\rangle\}$, $E'' = E \cup E^*$, $E''' = E \cup \{\langle f(\Phi_1, \ldots, \Phi_k)$ $f(\psi_1, \ldots, \psi_k\}$.

Proof. The equivalence of E, E' is obvious. Equivalence of E, E''' follows from the fact that the regularizer is a congruence. Equivalence of E, E'' follows from the relation $= \leqq \sim_h$.

Theorem 2. Each characteristic mapping h of B-system E induces a solution of the B-system E.

Proof. Let h be a characteristic mapping of E and $\sim_h$ be the regularizer induced by h. Denote $h(x_i) = C_i \in \boldsymbol{B}_a$. By the definition 4 (c) or (d) each element of $h(\boldsymbol{B}_{a,x})$ has a preimage in $\boldsymbol{B}_a$. Let F_i be an element from $\boldsymbol{B}_a$ fulfilling $h(F_i) = G_i$, thus $x_i \sim_h$ $\sim_h F_i \in \boldsymbol{B}_a$. From $h(\Phi) = h(\psi)$ and $x_i \sim_h F_i$ it follows

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_n \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$. Thus $\{F_1, \ldots, F_n\}$ is a solution of E.

Theorem 3. Let h_1 be a characteristic mapping of the B-system E and $\sim_1$ be a corresponding regularizer. Let h_2 be a homomorphic mapping of $\boldsymbol{B}_{a,x}$ into $\boldsymbol{B}_a$ and corresponding congruence relation $\sim_2$ fulfil $\sim_1 \leqq \sim_2$. Then h_2 is a characteristic mapping of E and the set R_1 of all solutions of E induced by h_1 is a subset of the set R_2 of all solutions of E induced by h_2.

Proof. Relation $\sim_1 \leqq \sim_2$ implies the condition (b) of definition 4, validity of (a), (c) is evident, thus h_2 is a characteristic mapping of E. Relation $\{F_1, \ldots, F_n\} \in$ $\in R_1$ holds iff for each $\langle \Phi, \psi \rangle \in E$ there holds $\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_1 \psi(F_1, \ldots,$ $\ldots, F_n, a_1, \ldots, a_m)$. This relation and $\sim_1 \leqq \sim_2$ imply

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_2 \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$. Accordingly $\{F_1, \ldots, F_n\} \in R_2$, i.e. $R_1 \subseteq R_2$.

Theorems 2 and 3 give a method for the finding of solutions of the B-system E by means of characteristic mappings of this system. The B-system E is solved if we find all characteristic mappings h_i of E whose regularizers are minimal. Then a set of solutions of E is the set of all $\{F_1, \ldots, F_n\}$, where $F_j \in h^{-1}h(x_j)$, $F_j \in \boldsymbol{B}_a$ and h is arbitrary homomorphism whose congruence $\sim_h$ fulfils $\sim_h \geqq \sim_{h_i}$ (we can write $h \geqq h_i$ iff $\sim_h \geqq \sim_{h_i}$). If $\sim_{h_i}$ is equal to $=$, the solution of E induced by h_i is proper.

2.

Investigations of solutions of B-systems we shall deal by means of an isomorphic representation of $\boldsymbol{B}_{a,x}$. Each Boolean algebra having 2^m elements is isomorphic with the direct power $\{0, 1\}^m$ of two-elements Boolean algebra $\{0, 1\}$ by Birkhoff's theorem. Let us denote $\{0, 1\}2^{m+n}$ by $\mathfrak{M}_{a,x}$ and $\{0, 1\}2^m$ by $\mathfrak{M}_a$. Elements of $\mathfrak{M}_{a,x}$ (resp. $\mathfrak{M}_a$) are called 2^{m+n}-dimensional (resp. 2^m-dimensional) B-vectors*). The Boolean algebra $\boldsymbol{B}_{a,x}$ is isomorphic with $\mathfrak{M}_{a,x}$, $\boldsymbol{B}_a$ with $\mathfrak{M}_a$. Let us fix the isomorphism i of $\boldsymbol{B}_{a,x}$ onto $\mathfrak{M}_{a,x}$ such that it holds:

$$i(x_j) = (11\ldots100\ldots011\ldots100\ldots0 \;\; . \;\; . \;\; . \;\; 11\ldots100\ldots0)$$

for $j = 1, \ldots, n$, where each group of 1 or 0 has 2^{j-1} elements and

$$i(a_k) = (11\ldots100\ldots011\ldots100\ldots0 \;\; . \;\; . \;\; . \;\; 11\ldots100\ldots0)$$

for $k = 1, \ldots, m$, where each group of 1 or 0 has 2^{n+k-1} elements.

For $\boldsymbol{B}_a$, $\mathfrak{M}_a$ we fix the isomorphism j of $\boldsymbol{B}_a$ onto $\mathfrak{M}_a$ for which $j(a_k) = (11\ldots100\ldots011\ldots100\ldots0 \;\; . \;\; . \;\; . \;\; 11\ldots100\ldots0)$ for $k = 1, \ldots m$, where each group of 1 or 0 has 2^{k-1} elements.

Obviously $i(0) = (00\ldots0)$, $j(0) = (00\ldots0)$. They are called the zero-vectors of $\mathfrak{M}_{a,x}$ or $\mathfrak{M}_a$ respectively; $i(J) = (11\ldots1)$, $j(J) = (11\ldots1)$, they are called the unit-vectors of $\mathfrak{M}_{a,x}$ or $\mathfrak{M}_a$ respectively. The element of $\mathfrak{M}_{a,x}$ isomorphic to B-polynom $\Phi \in \boldsymbol{B}_{a,x}$(by i) will be denoted by Φ again.

We shall consider now a finite B-system E of $\boldsymbol{B}_{a,x}$ given by relations:

$$(E) \qquad \begin{gathered} \Phi_i(x_1, \ldots, x_n, a_1, \ldots, a_m) = \psi_i(x_1, \ldots, x_n, a_1, \ldots a_m) \\ i = 1, \ldots, k \end{gathered}$$

We shall determine a characteristic mapping of (E) in "B-vectors representation", i.e. a homomorphic mapping h of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$ fulfilling $h(\Phi_i) = h(\psi_i)$ for $i = 1, \ldots, k$ and $h_y(\mathfrak{M}_{a,x}) = h(\mathfrak{N})$, where $\mathfrak{N}$ is a subalgebra of $\mathfrak{M}_{a,x}$ generated by B-vectors $\{a_1, \ldots, a_m\}$ of $\mathfrak{M}_{a,x}$. This mapping corresponding to the characteristic mapping of (E) in the given isomorphic representation.

We shall not differ between the characteristic mapping of (E) and this mapping h of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$ corresponding to the characteristic mapping in given representation. By the theorem 2 in [6] there exists just one B-matrix of the type $2^{m+n}/2^m$ representing the characteristic mapping.

Theorem 4. Let C be a B-matrix representing a characteristic mapping of B-system E. Let $f_j^{(i)}$ (resp. $g_j^{(i)}$) be the j-th coordinate of B-vector Φ_i (resp. ψ_i). If there exists an index $i \in \{1, \ldots, k\}$ such that $f_j^{(i)} \neq g_j^{(i)}$, then all elements in the j-th row of C are equal to zero (so called "zero row").

*) See [6]; $\mathfrak{M}_{a,x}$ is $\mathfrak{M}_{m+n}$, $\mathfrak{M}_a$ is $\mathfrak{M}_m$ from this paper and conception of B-vector is identical.

Proof. Let for example $f_j^{(i)} = 1$, $g_j^{(i)} = 0$ and $c_{js} = 1$ for an index $s \in \{1, \ldots, 2^m\}$, where c_{js} is the element of B-matrix C in the j-th row and the s-th column. Then:

$$h(\Phi_i) = (t_1, \ldots, t_{s-1}, 1, t_{s+1}, \ldots, t_{2^m}),$$

$h(\psi_i) = (v_1, \ldots, s_{s-1}, 0, v_{s+1}, \ldots, v_{2^m})$ because C has at most one unity in each column by the theorem 1 in [6]. Accordingly $h(\Phi_i) \neq h(\psi_i)$ which is a contradiction.

Let us denote by the r-th section a sequence

$$\langle f^{(i)}_{r.2^n+1}, f^{(i)}_{r.2^n+2}, \ldots, f^{(i)}_{r.2^n+2^n} \rangle$$

of coordinates of B-vector Φ_i or a sequence

$$\langle c_{r.2^n+1}, c_{r.2^n+2}, \ldots, c_{r.2^n+2^n} \rangle$$

of rows of B-matrix C, where $r = 0, 1, \ldots, 2^m - 1$.

Theorem 5. Let C be a B-matrix representing a characteristic mapping of B-system E. There are not two different non-zero rows in an arbitrary section of C.

Proof. Let there be a unity in the r-th section of C in the t-th row and v-th column and in the s-th row and w-th column, $t \neq s$. By the theorem 1 in [6] we have $v \neq w$ because h is a homomorphic mapping. Then an image of B-vector $(\underbrace{00\ldots0}_{t\text{-}1 \text{ zero-elements}}10\ldots0) \in \mathfrak{M}_{a,x}$ is equal to $b = (b_1, \ldots, b_{v-1}, 1, b_{v+1}, \ldots, b_{w-1}, 0, b_{w+1}, \ldots, b_{2^m})$. But an image of each B-vector of $\mathfrak{N}$ which has unity in the r-th section has the v-th and w-th coordinates equal to 1 and an image of each B-vector of $\mathfrak{N}$ which has not unity in the r-th section has the v-th and w-th coordinates equal to 0. Accordingly, there does not exist a B-vector of $\mathfrak{N}$ whose image is equal to b, i.e. $h(\mathfrak{M}_{a,x}) \neq h(\mathfrak{N})$ which is a contradiction with the definition 4 (c).

Theorem 6. A B-matrix C of the type $2^{m+n}/2^m$, having at most one unity in each column, represents a characteristic mapping of B-system E if and only if it holds:

(a) if there exists an index $i \in \{1, \ldots, k\}$ such that $f_j^{(i)} \neq g_j^{(i)}$, then C has in the j-th row only 0.

(b) C has not two different non-zero rows in an arbitrary section of C.

Proof. Necessity follows from the theorems 4 and 5. Sufficiency: Let C fulfil assumptions of the theorem 6. Then C represents a homomorphic mapping of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$. (a) implies $h(\Phi_i) = h(\psi_i)$ for $i = 1, 2, \ldots, k$, (b) implies $h(\mathfrak{M}_{a,x}) = h(\mathfrak{N})$. q.e.d.

Let us consider the case when h is a proper characteristic mapping of E. Then $h(a_i) = a_i$. The B-matrix C representing a proper characteristic maping of E is quasidiagonal (see Fig. 1), i.e. all elements out of frames are equal to 0 and in frame there is a section of column.

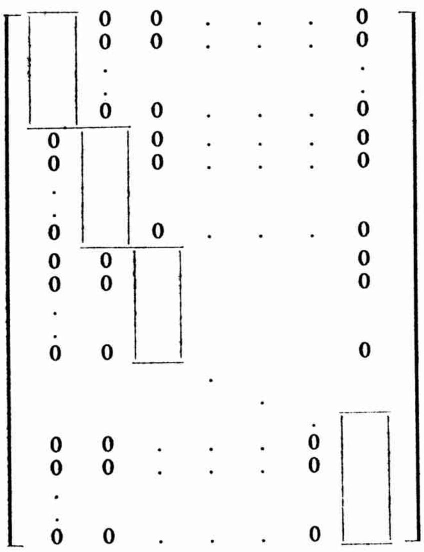

Fig. 1

In this case we must fill in each column just one "diagonal" element. These elements must fulfil the assumptions of theorem 6. We can fill 0 in these sections by theorem 4 comparing coordinates in all pairs $\langle \Phi_i, \psi_i \rangle$ of B-vectors corresponding to equations of E.

If h is not a proper characteristic mapping, we can not assume $h(a_i) = a_i$. We can only fill in B-matrix C zero rows by theorem 4. Other elements are equal to 1 or 0 but they must fulfill assumption of the theorem 6.

Let us fill in B-matrix C zero row if $f_j^{(i)} \neq g_j^{(i)}$ for at least one index i and unit row if $f_j^{(i)} = g_j^{(i)}$ for all $i = 1, \ldots, k$. The matrix constructed by this way is called the matrix of solutions of B-system E.

Definition 9. By a section decomposition of matrix of solutions C we understand the set $\{C_1, \ldots, C_s\}$ of all B-matrices of the type identical with the type of C such that:

(a) each C_i has in each section at most one unit row

(b) if C has in the p-th row only zero elements, each C_i has in the p-th row only zero elements

(c) each C_i has only zero rows and unit rows

(d) $C_1 + C_2 + \ldots + C_s = C$ (the sum of B-matrices is defined in [6]).

It is easy to show that all B-matrices representing all characteristic mappings of E which regularizers are minimal are included in decompositions (defined in [6]) of B-matrices $C_1, \ldots, C_s$ forming a section decomposition of matrix of solutions of E.

Moreover, each B-matrix from decompositions of a section decomposition of matrix of solutions represents a characteristic mapping of E. All other characteristic mappings of E are represented by matrices which are obtained from matrices of decompositions of section decomposition of matrix of solutions by substitution 0 instead 1 respectively in all unit elements.

Theorem 7. The B-system E has a proper solution if and only if the matrix of solutions of E has at least one unit row in each section.

Proof. If E has a proper solution, then there exists a quasidiagonal matrix C' with non-zero columns, i.e. decompositions of section decomposition of matrix of solutions have in each section non-zero elements and the statement of the theorem holds. Conversely, if matrix of solutions fulfils assumption of the theorem, decompositions of a section decomposition of this matrix contain a quasidiagonal matrix of desirable property.

It is easy to show the following.

Theorem 8. Let the matrix of solutions of given B-system E has in the j-th section just k_j unit rows. Then the B-system E has just $s = k_1 . k_2 \dots k_2$ different proper solutions.

Theorem 9. Let the matrix C of solutions of B-system E have in the j-th section just k_j unit rows, $p_j = \max(k_j, 1)$, $r_j = \min(k_j, 1)$ $q = 2^m . \min(1, \sum_{i=1}^{2^m} r_i)$. Then the B-system E has just

$$s = p_1 \cdot p_2 \cdot \dots \cdot p_{2^m} \cdot (\sum_{i=1}^{2^m} r_i)^{2^m} . (2^q - 1) + 1 \qquad \text{solutions.}$$

Proof. The section decomposition of matrix of solutions C contains only matrices $C_1, \dots, C_s$, where $s = p_1 \cdot p_2 \dots p_{2^m}$ (by theorem 8). Each C_i, $i = 1, \dots, s$, contains just $\sum_{i=1}^{2^m} r_i$ unities in each column (it has 2^m columns), thus (see to [6]) their decompositions contain $p_1 \dots p_{2^m} \cdot (\sum_{i=1}^{2^m} r_i)^{2^m}$ matrices. Each matrix of decompositions of section decomposition of C contains q unities, i.e. we receive 2^q B-matrices replacing 1 by 0. Disregarding the zero matrix, we receive $2^q - 1$ matrices.

Accordingly, we receive at all $p_1 \dots p_{2^m} (\sum_{i=1}^{2^m} r_i)^{2^m} . (2^q - 1) + 1$ B-matrices representing all characteristic mappings of E by the theorem 6.

Theorem 10. Each B-system E has at least one solution.

Proof. The zero matrix (it contains only zero elements) fulfils assumptions of the theorem 6, thus the zero-homomorphism h_0 with $h_0(\mathfrak{M}_{a,x}) = \{o\}$ is a characteristic mapping of E (o is the zero-vector). Then the corresponding regularizer is induced by the ideal $I = \mathfrak{M}_{a,x}$ (or $I = \boldsymbol{B}_{a,x}$) and the set $\{F_1, \ldots, F_n\}$, where F_i is an arbitrary B-polynom of $\boldsymbol{B}_a$, is a solution of E.

These theorems form a complete theory of solution of Boolean equations over finite Boolean algebras. We can state when the given B-system has proper solutions and enumerate the number of them, we can enumerate number of all solutions and by a simple algorithm (constructing a matrix of solutions, section decomposition, decompositions and substitution 1 by 0) construct matrices representing all characteristic mappings. If h is a characteristic mapping of E, we can determine a solution from relations $h(x_i) = h(F_i)$, $F_i \in \boldsymbol{B}_a$. Corresponding regularizer, i.e. congruence relation replacing the equivalence, is induced by an ideal I (i.e. the set of all B-vectors of $\mathfrak{M}_{a,x}$ fulfilling $h(I) = o$). Finally, each B-system has a solution.

The solving of a given B-system can demonstrated by an example.

Example. Consider the B-system

$$\begin{aligned} (x_1 + x_2)\,\bar{a}_1 a_2 + \bar{x}_1 x_2 a_1 a_2 &= \bar{x}_1 x_2 \\ x_1 x_2 \bar{a}_1 &= (x_1 + \bar{x}_2)\,\bar{a}_1\bar{a}_2 \\ a_1(\bar{x}_1 x_2 + x_1\bar{x}_2) &= a_1 \end{aligned} \qquad (E')$$

We can write vertically the B-vectors $\Phi_1, \psi_1, \Phi_2, \psi_2, \Phi_3, \psi_3$, corresponding to equations of E' and by theorems 4 and 6 construct the matrix of solutions $\boldsymbol{C}$ of E'.

$$i(x_1) = (1010101010101010)$$
$$i(x_2) = (1100110011001100)$$
$$i(a_1) = (1111000011110000)$$
$$i(a_2) = (1111111100000000)$$
$$j(a_1) = (1010) \qquad j(a_2) = (1100)$$

Φ_1	Ψ_1	Φ_2	Ψ_2	Φ_3	Ψ_3	
0	0	0	0	0	1	
1	1	0	0	1	1	*1*-st section
0	0	0	0	1	1	
0	0	0	0	0	1	
1	0	1	0	0	0	
1	1	0	0	0	0	2-nd section
1	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	1	
0	1	0	0	1	1	3-rd section
0	0	0	0	1	1	
0	0	0	0	0	1	
0	0	1	1	0	0	
0	1	0	0	0	0	4-th section
0	0	0	1	0	0	
0	0	0	1	0	0	

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this $k_1 = 2,\ k_2 = 2,\ k_3 = 1,\ k_4 = 1$

$$p_1 = 2,\ p_2 = 2,\ p_3 = 1,\ p_4 = 1$$

$$r_1 = 1,\ r_2 = 1,\ r_3 = 1,\ r_4 = 1$$

From theorem 8 it follows that the B-system E' has just 4 proper solutions. Corresponding B-matrices C_1', C_2', C_3', C_4' form a decomposition of the quasidiagonal matrix C' (derived from C):

$$C' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}, \text{ i.e. } C_1' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix} \quad C_2' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}$$

$$C_3' = \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix} \quad C_4' = \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}$$

We can determine solutions. From C_1':

$$x_1 \to (0\ 0\ 1\ 1),\ \bar{a}_2 \to (0\ 0\ 1\ 1) \qquad \text{from this } x_1 = \bar{a}_2$$

$$x_2 \to (1\ 1\ 0\ 1),\ \bar{a}_1 + a_2 \to (1\ 1\ 0\ 1) \qquad x_2 = \bar{a}_1 + a_2$$

is the 1-st solution. Other proper solutions are obtained from C_2', C_3', C_4' analogously:

$$x_1 = \bar{a}_1 \qquad x_1 = a_1 + \bar{a}_2 \qquad x_1 = \bar{a}_2$$

$$x_2 = a_1 s_2 + \bar{a}_1\bar{a}_2 \qquad x_2 = \bar{a}_1 \qquad x_2 = \bar{a}_1\bar{a}_2$$

Further we can construct a section decomposition of C (4 matrices form it). Let us choose from this section decomposition for example this matrix:

$$C_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

The decomposition of C_1 is formed by $(\sum_{i=1}^{2^2} r_i)^{2^2} = 4^4 = 256$ B-matrices. Let us choose from them for example the following one:

$$C_{11} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

This matrix C_{11} represents a characteristic mapping h of E' with

$$\left.\begin{array}{l} h(x_1) = (1\ 1\ 1\ 1)\ \ h(J) = (1\ 1\ 1\ 1) \\ h(x_2) = (0\ 0\ 1\ 0)\ \ h(\bar{a}_1\bar{a}_2) = (0\ 0\ 1\ 1) \end{array}\right\} \quad x_1 \sim_h J,\ x_2 \sim_h \bar{a}_1\bar{a}_2$$

We can determine the ideal corresponding to congruence $\sim_h$. But for the investigation of solutions it is only intersection of this ideal with $\boldsymbol{B}_a$ necessary. The intersection is equal to $\{0, \bar{a}_1 a_2\}$. We can say that $x_1 = J$, $x_2 = \bar{a}_1\bar{a}_2$ is a „conditional" solution of E′ iff the condition $\bar{a}_1 a_2 = 0$ holds.

If we replace for example the unity in the first column by 0, we obtain the solution $x_1 \sim_{h'} (\bar{a}_1 + \bar{a}_2)$, $x_2 \sim_{h'} \bar{a}_1\bar{a}_2$ and $\sim_{h'}$ on $\boldsymbol{B}_a$ is given by the ideal $\{0, \bar{a}_1 a_2\}$.

All other solutions of E' can be determined analogously.

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DIE KOMBINATIONEN GEGEBENEN PROFILS I

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Das Hauptziel dieser Arbeit ist es einige neue graphische Prinzipe kombinatorischer Betrachtungen einzuführen.

Man löst hier folgende Aufgabe: Es ist 1) die Gesamtheit, 2) die Anzahl aller Kombinationen festzustellen, die aus n Elementen

$$x_1, x_2, \dots, x_n, \tag{1}$$

mit event. Wiederholung so zusammengesetzt werden, dass die hintereinanderfolgenden Indexe in jeder Kombination $(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ die Bedingungen erfüllen:

$$i_1 \mathrm{Z}\, i_2 \mathrm{Z} \dots \mathrm{Z}\, i_k, \tag{2}$$

wo wir für jedes Symbol Z je nach unserer Wahl immer eines der Zeichen einsetzen:

$$\leqq, <, \geqq, >. \tag{3}$$

1. Vereinbarung. Die durch unsere Wahl realisierte Beziehung (2) bezeichnen wir mit dem Termin „*Profil*". Weiter heben wir zwei Haupttypen von Profilen hervor: 1) *Die monotonen Profile* – bestehend nur aus den gleichgerichteten Zeichen, z. B. $\leqq\ <\ <$, oder $>\ >\ >$, u. s. w., 2) *Die nichtmonotonen Profile*, die in ihren Zusammenstellungen wenigstens ein entgegengerichtetes gegenüber den übrigen Zeichen haben, z. B. $\leqq\ \leqq\ \geqq$, oder $\geqq\ \leqq\ >\ >$, u. s. w.

2. Bemerkung. Die Gruppen vom gegebenen Profil wäre es möglich durch Einführung einer grösseren Anzahl von Zeichen zu erweitern. Die Zeichen müssten wir dann durch eine andere Symbolik ausdrücken. So z. B. $i_1(+2)\, i_2$ wäre ein Profil für zweistellige Kombinationen, die aus den Elementen (1) so zusammengestellt würden, dass bei der Besetzung der ersten Stelle mit dem Element x_j die zweite Stelle nur mit einem von den Elementen $x_{j+2}, x_{j+3}, \dots, x_n$ besetzt werden kann. Alle weiteren Betrachtungen würden analog wie diejenigen in dieser Arbeit geführt werden.

Alle Erwägungen gehen hier von gewissem Graphen aus, der eigentlich die möglichst knappe Hinschreibung aller entsprechenden Fälle ist. Diesem Graphen werden wir dann seine Matrix und dieser wieder ihr Vektorfeld zuordnen.

3. Definition. *Der Entwicklungsgraph.* Seine Bestandteile sind *die Elemente* (1) und *die Zweige*, die hier als Verbindungen gewisser Elemente fungieren. Die Elemente (mit event. Wiederholung) werden in dem Graph in den Spalten eingerichtet. Die Elemente, die sich in der r-ten Spalte befinden, erfassen wir als „*die r-te Folge des Graphen*", die von oben nach unten hinzielt; $r = 1, 2, 3, \ldots$

Der Graph entwickelt sich stufenweise, d. h. immer die $(r + 1)$-e Folge entwickelt sich aus der r-ten. Diese Entwicklung bezeichnen wir als „*die Verzweigung*". Es gelten hier folgende Regeln:

1) In der ersten Spalte liegt immer „*die Grundfolge des Graphen*", d. h. die Folge $\{x_1, x_2, x_3, \ldots, x_n\}$.

2) Es sei x_j ein beliebiges Element der r-ten Spalte, und es gelte a) $i_r \leqq i_{r+1}$, *b*) $i_r < i_{r+1}$, c) $i_r \gtrless i_{r+1}$, d) $i_r > i_{r+1}$. Dann entwickelt sich die entsprechende Folge $P(x_j)$ in der $(r + 1)$-en Spalte folgendermassen: a) $P(x_j) = \{x_j, x_{j+1}, \ldots, x_n\}$, b) $P(x_j) = \{x_{j+1}, x_{j+2}, \ldots, x_n\}$, c) $P(x_j) = \{x_1, x_2, \ldots, x_j\}$, d) $P(x_j) = \{x_1, x_2, \ldots, x_{j-1}\}$. Die Folge $P(x_j)$ nennen wir „*die Verzweigung des Elementes* x_j" und wir führen von dem Urheber x_j zu jedem ihren Element eine Verbindung, insofern wenigstens ein Element dieser Folge existiert. Jede solche Verbindung bezeichnen wir als „*elementaren Zweig*" und im Sinne der Graphenentwicklung erfassen wir auch seine Orientierung.

3) Es seien x_j, x_k zwei Nachbarelemente der r-ten Spalte derart, dass das Element x_j über dem Element x_k liegt. Dann sind die Folgen $P(x_j)$, $P(x_k)$ zwei Nachbarfolgen der $(r + 1)$-en Spalte derart, dass die Elemente der Folge $P(x_j)$ über den Elementen der Folge $P(x_k)$ liegen.

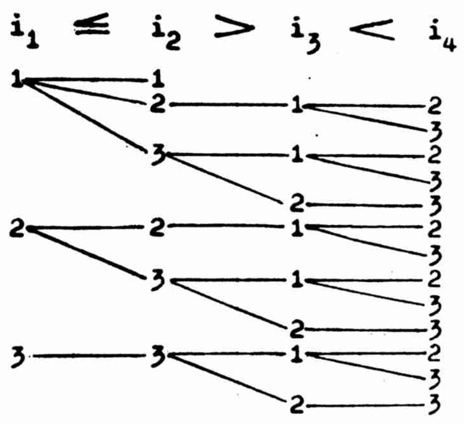
Aus den elementaren, hintereinander angeknüpften Zweigen entstehen einzelne einfachen (orientierten) mehrgliedrigen Zweige. Jeden solchen Zweig, der von der esten bis zur letzten Graphenspalte führt, bezeichnen wir als den „*Totalzweig*".

4. Bemerkung. Einen ähnlichen Entwicklungsgraph können wir leicht auch für die Variationen (mit oder ohne Wiederholung) konstruieren; besonders gut kann man hiemit das Wesen der Fakultät ausklären.

5. Satz. Die Anzahl aller Kombinationen k-ter Klasse, aus n Elementen, die einem gegebenen Profil entsprechen, ist gleich der Anzahl aller Elemente der k-ten Folge des zugehörigen Entwicklungsgraphen.

Beweis. Aus dem Graphen kann man gerade alle möglichen Totalzweige ausnehmen; jeder von diesen gehört immer einer anderen, und immer entsprechender Kombination, über deren Elemente läuft er durch. Dabei endet jeder Totalzweig bei seinem eigenen Element.

6. Beispiel. Es ist die Gesamtheit aller Kombinationen vierter Klasse, aus den Elementen $\{x_1, x_2, x_3\}$, festzustellen, die dem Profil $(i_1 \leqq i_2 > i_3 < i_4)$ entsprechen.



Lösung: Stellen wir den entsprechenden Graph zusammen. (Anstatt des ganzen Elementes legen wir immer nur seinen Index). Nach dem Satz 5, bekommen wir folgenden, gut geordneten Inbegriff:

$$(1, 2, 1, 2), (1, 2, 1, 3), (1, 3, 1, 2), (1, 3, 1, 3), (1, 3, 2, 3), (2, 2, 1, 2), (2, 2, 1, 3),$$
$$(2, 3, 1, 2), (2, 3, 1, 3), (2, 3, 2, 3), (3, 3, 1, 2), (3, 3, 1, 3), (3, 3, 2, 3).$$

7. Definition. Die Matrix eines Entwicklungsgraphen, die wir mit dem Symbol M bezeichnen, besteht aus den Elementen a_{jr}, die in n Zeilen und k Spalten angeordnet sind, wobei n bzw. k die Anzahl aller gegebenen Elemente, bzw. aller Spalten des Graphen bedeutet.

Die Zahl a_{jr} gibt an, wieviel Elemente x_j sich in der r-ten Spalte des Entwicklungsgraphen befindet.

8. Satz. Die Anzahl aller Kombinationen k-ter Klasse, aus n Elementen, die einem gegebenen Profil entsprechen, ist gleich der Summe aller Elemente von der k-ten Spalte der zugehörigen Matrix M.

Beweis. Der Satz ist eine ersichtliche Folgerung des Satzes 5, und der Def. 7.

9. Satz.

Es sei a_{jr} die Grösse aus der j-ten Zeile und der r-ten Spalte einer n-zeilen und k-spalten Matrix M, und es gelte nach dem Profil:

a) $i_r \leqq i_{r+1}$ b) $i_r < i_{r+1}$ c) $i_r \geqq i_{r+1}$ d) $i_r > i_{r+1}$.

Dann gelten die Rekurrenzen:

$$\text{(4)} \quad \begin{array}{ll} \text{a)} \ a_{j,r+1} = a_{j-1,r+1} + a_{j,r}, & \text{b)} \ a_{j,r+1} = a_{j-1,r+1} + a_{j-1,r}, \\ \text{c)} \ a_{j,r+1} = a_{j+1,r+1} + a_{j,r}, & \text{d)} \ a_{j,r+1} = a_{j+1,r+1} + a_{j+1,r}, \end{array}$$

und die Randbedingungen:

(5) $\quad a_{r,1} = 1;\; r = 1, 2, \ldots, n. \qquad$ b) $a_{rj} = 0$ für $r > n, r < 1$.

Beweis. Aus den gegenseitig übereinstimmenden Elementen x_j der r-ten Graphenspalte, entwickelt sich immer dieselbe Folge $P(x_j)$, d. h. bei jedem Zufall dieses Elementes. Deshalb wird der Zahlenwert des Matrixelementes a_{jr} in die $(r + 1)$-e Spalte übertragen, und zwar in die Zeilen vom Index: a) $j, j + 1, \ldots, n$. b) $j + 1$, $j + 2, \ldots, n$. c) $1, 2, \ldots, j$. d) $1, 2, \ldots, j - 1$. Dies gilt für $j = 1, 2, \ldots, n$, so dass gilt

$$(6)\qquad \begin{aligned} &\text{a)}\quad a_{j,r+1} = \sum_{i=1}^{j} a_{i,r} \qquad &&\text{b)}\quad a_{j,r+1} = \sum_{i=1}^{j-1} a_{i,r} \\ &\text{c)}\quad a_{j,r+1} = \sum_{i=j}^{n} a_{i,r} \qquad &&\text{d)}\quad a_{j,r+1} = \sum_{i=1+1}^{n} a_{i,r} \end{aligned}$$

Weiter nehmen wir in Betracht, dass es nach (6) gleichzeitig gilt

$$(7)\qquad \begin{aligned} &\text{a)}\quad a_{j-1,r+1} = \sum_{i=1}^{j-1} a_{i,r} \qquad &&\text{b)}\quad a_{j-1,r+1} = \sum_{i=1}^{j-2} a_{i,r} \\ &\text{c)}\quad a_{j+1,r+1} = \sum_{i=j+1}^{n} a_{i,r} \qquad &&\text{d)}\quad a_{j+1,r+1} = \sum_{i=j+2}^{n} a_{i,r} \end{aligned}$$

Als die Differenz der entsprechenden Gleichungspaare von (6), (7), bekommen wir dann die Relationen (4). Die Bedingungen (5) sind ersichtliche Folgerungen der Def. 3. Siehe auch die Figur 1.

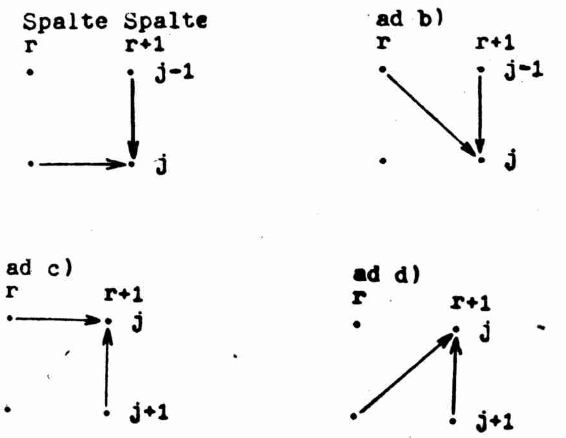


Fig. 1

10. Definition. Es ist eine n-zeilen und k-spalten Matrix M gegeben. Dieser ordnen wir das Vektorfeld F und seine Operationen folgendermassen zu:

Das Gitter des Vektorfeldes ist das System aller n Zeilen und k Spalten der

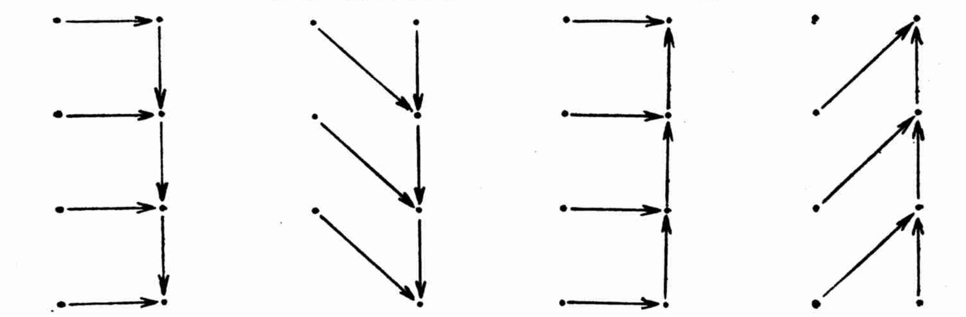
Matrix M, die hier als einzelne gezeichnete Strecken gedacht sind. Der Gitterpunkt (i, j) ist der Durschnittspunkt der i-ten Zeile mit der j-ten Spalte. Der elementare Vektor ist ein gebundener Vektor; er geht immer aus einem Gitterpunkt aus und zielt in einen anderen hin.

In den Gitterpunkten der ersten Spalte liegen immer die Einser. Diese Einser werden wir „*die Anfangswerte des Vektorfeldes*" nennen. Jeder Vektor teilt den Wert seines Anfangspunktes seinem Zielpunkt mit. Der seinem Zielpunkt mitgeteilte Wert wird zum Wert dieses Punktes. Wenn in den Punkt (i, j) mehrere Vektoren hinzielen, werden die von ihnen mitgeteilten Werte summiert und diese Summe wird zum Wert des Punktes (i, j). Zielt kein Vektor in einen (Rand)-punkt hin, so ist der Wert eines solchen Punktes Null. Ausserhalb des Definitionsbereiches werden keine Werte mitgeteilt und umgekehrt.

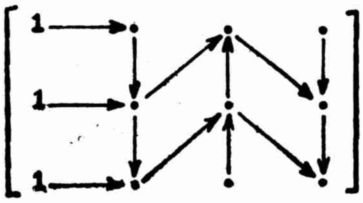
11. Satz. Jedem von den Zeichen (3) entspricht (immer zwischen den zugehörigen Spalten) sein eigener *Vektorfeldsstreifen*, in dem die gebundenen Vektoren, was es ihrer Richtung und Orientierung betrifft, sich mit jeder Zeile periodisch wiederholen.

Beweis. Der Satz ist eine ersichtliche Folgerung der Formeln (4), und der Def. 10. Siehe auch die Figur 2:

a) $i_r \leq i_{r+1}$ b) $i_r < i_{r+1}$ c) $i_r \geq i_{r+1}$ d) $i_r > i_{r+1}$



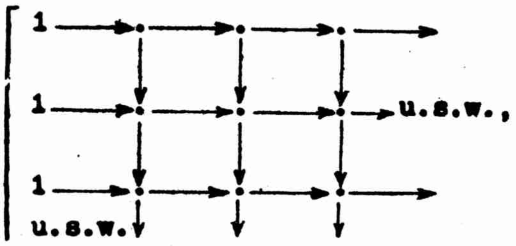
12. Beispiel — ist die Fortsetzung der Aufgabe aus dem Beispiel 6. Es ist jetzt die Anzahl aller entsprechenden Kombinationen unvermittelbar festzustellen — mit kurzer theoretischen Begleitung:



$$\begin{bmatrix} 1 & 1 & 5 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 0 & 8 \end{bmatrix}$$

Die Entwicklung der Matrix M als die Operation ihres Vektorfeldes: die Abbildungen zeigen das Vektorfeld am Anfang und am Ende. Also nach dem Satz 8 gilt: $0 + 5 + 8 = 13$.

13. Bemerkung. Die Definition 10 gilt auch für das Rechtecksystem der binomischen Koeffizienten, siehe (8). Dies ist aus unserer Hinsicht der Zufall eines speziellen Profils, in dem sich das Zeichen $\leqq$, bzw. $\geqq$, stets wiederholt:



$$\left[\begin{matrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \binom{3}{3} \\ \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & \binom{4}{3} \\ \binom{2}{0} & \binom{3}{1} & \binom{4}{2} & \binom{5}{3} \\ \text{u.s.w.} & & & \end{matrix}\right.$$

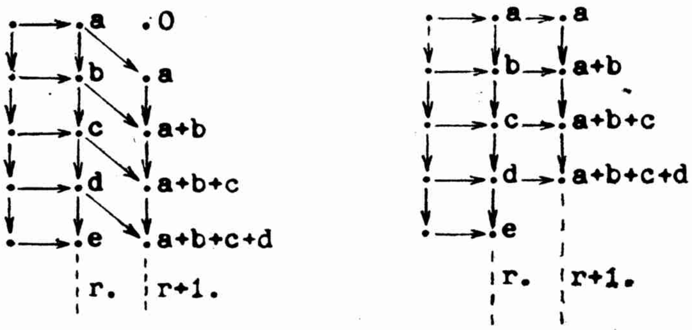
14. Vereinbarung. Die Kombinationen, die einem monotonen Profil entsprechen, bezeichnen wir im allgemeinen „*die Kombinationen mit örtlicher Wiederholung*". Die Anzahl aller Kombinationen aus n Elementen, k-ter Klasse und mit örtlicher Wiederholung, wobei das Zeichen $\leqq$ (bzw. $\geqq$) r-mal und das Zeichen $<$ (bzw. $>$) s-mal zugelassen wird, $(r + s = k - 1)$, bezeichnen wir mit dem Symbol $C_r^k(n)$.

In weiterem werden wir nur die Zeichen $\leqq$, $<$ betrachten; analog würden wir die Betrachtung für die Zeichen $\geqq$, $>$ durchführen.

15. Lemma. Es seien zwei monotone Profile gegeben, die ausser folgender Ausnahme überall übereinstimmen: 1. $i_r < i_{r+1}$, 2. $i_r \leqq i_{r+1}$.

Dem ersten, bzw. dem zweiten Profil gehöre die n-zeilen und k-spalten Matrix M, bzw. M^+. Dann, falls wir in der Matrix M^+, mi der $(r + 1)$-en Spalte beginnend, den nachfolgenden Teil der n-ten Zeile nicht registrieren, bekommen wir bei diesen zwei Matrizen dieselbe Folge der Spaltensummen.

Fig. 3



Beweis. Aus der Folge der r-ten Spalte einer Matrix M: $(a_{r1}, a_{r2}, \ldots, a_{rn})$, bekommen wir mit dem Gebrauch der Formel (4 – b), bzw. (4 – a), in beiden diesen Fällen zwar dieselbe, doch aber in dem zweiten Fall um eine Zeile höher anfangende Folge der $(r + 1)$-en Spalte.

Siehe auch die Figur 3.

16. Satz. Es gelte: $n \geqq 1$, $k - 1 \geqq r \geqq 0$. Dann gilt die Formel:

$$C_r^k(n) = \binom{n + r}{k}. \tag{9}$$

Beweis. Ersetzen wir in dem gegebenen Profil jedes Zeichen $<$ mit dem Zeichen $\leqq$. Dann, nach dem Satz 8 und dem Lemma 15, müssen wir in der k-ten Spalte der Matrix des neu geformten Profils nur die ersten $n - (k - 1 - r)$ Glieder summieren, so dass wir nach (8) folgende Reihe bekommen: $\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+r-1}{k-1}$, die nach [1], S. 248, Formel 11, mit der rechten Seite aus (9) gleich ist.

Damit sind wir zu einer Universalformel für die Kombinationen mit örtlicher Wiederholung angelangt. Es ist ersichtlich, dass die „*Kombinationen ohne Wiederholung*" $(r = 0)$ und die „*Kombinationen mit Wiederholung*" $(r = k - 1)$ die Randfälle sind. Es ist gleichzeitig ersichtlich, dass die Anzahl der entsprechenden Kombinationen nicht davon abhängt, zwischen welchen Indexen die Wiederholung zugelassen wurde.

Aus den obigen Betrachtungen kommt auch folgende Formierung unserer Resultäte hervor:

17 Korolar. Der Funktionsausdruck $\binom{n}{1}$, der der ersten Spaltenfolge in dem Graphen entspricht, entwickelt sich bei den monotonen Profilen folgendermassen:

a) *Mit jeder Verzweigung mit erlaubter Wiederholung wächst sowohl der obere als auch der untere Zeiger um eins.*

b) *Mit jeder Werzweigung ohne erlaubte Wiederholung wächst nur der untere Zeiger um eins.*

Nun wollen wir prüfen, ob es auch bei einem willkürlichen, nicht monotonen Profil eine ähnliche, dem Graphen entsprechende, Entwicklung von $\binom{n}{1}$ existiert.

18. Definition. Es seien n Elemente (1) gegeben und x_j sei ein willkürliches Element in der r-ten Spalte des Graphen.

Die vollständige Verzweigung des Elementes x_j bedeutet, dass diesem Element entsprechende Folge $P(x_j)$, die in der $(r + 1)$-en Spalte liegt, mit der Grundfolge übereinstimmt, siehe Def. 3.

Die vollständige Verzweigung der r-ten Folge des Graphen bedeutet, dass jedem Element dieser Folge seine vollständige Verzweigung entspricht, wobei für die

Entwicklung der elementaren Zweige die Bedingungen aus der Def. 3 gelten. Dies noten wir in einem Profil mit dem Zeichen $\lesseqgtr$, d. h. $i_r \lesseqgtr i_{r+1}$.

Die einem von den Zeichen (3) entsprechende Verzweigung des Elementes x_j wird durch die Verbindung mit ihrer Ergänzungsverzweigung desselben Elementes x_j zur vollständigen Verzweigung dieses Elementes gebracht.

19. Satz. Es sei x_j ein beliebiges Element aus der r-ten Spalte des Graphen. Setzen wir voraus, dass seine Verzweigung dem i-ten Zeichen aus der Folge $\{\leqq, <, \overline{\geqq}, >\}$ entspricht. Dann entspricht seine Ergänzungsverzweigung dem $(5 - i)$-ten Zeichen aus derselben Folge.

Die Gültigkeit des Satzes ist ersichtlich aus den Def. 3 und 18.

20. Vereinbarung. Die Zerlegung gewisser Folge in zwei Teilfolgen bezeichnen wir mit den Zeichen $+$, $-$. Zum Beispiel: $P(x_j) + q(x_j) = R(x_j)$, oder $R(x_j) - P(x_j) = = q(x_j)$. Dasselbe in Worten.

21. Satz. Es sei x_j ein beliebiges Element aus der r-ten Spalte des Graphen und es sei $H_1(x_j), H_2H_1(x_j), H_3H_2H_1(x_j), \ldots$ seine Verzweigung in der $(r + 1)$-en, $(r + 2)$-en, $(r + 3)$-en ... Spalte. Dabei anstatt H_i bezeichnen wir P_i, p_i, Q_i, q_i, R_i, falls die Entwicklung nach dem (entsprechenden) Zeichen $\leqq, <, \overline{\geqq}, >, \lesseqgtr$ läuft. Dann gillt:

$$P_2Q_1(x_j) = P_2[R_1(x_j) - p_1(x_{j_2}] = P_2R_1(x_j) - P_2p_1(x_j), \tag{10}$$

und weiter:

$$P_3P_2Q_1(x_j) = P_3P_2[R_1(x_j) - p_1(x_j)] = P_3P_2R_1(x_j) - P_3P_2p_1(x_j), \tag{11}$$

und so ähnlich in den übrigen Fällen.

Beweis. Das angezeigte Prinzip „*Die Verzweigung der Differenz ist gleich der Differenz der Verzweigung*“ gilt hier aus dem Grund, dass die Entwicklungsbedingungen für ein beliebiges Element der r-ten Spalte des Graphen unabhängig daran sind, in welche ihre Teilfolge dieses Element zugeteilt wird.

22. Satz. Es seien die Elemente (1) gegeben; A sei die Anzahl aller Elemente der r-ten Folge des Graphen und es gelte $i_r \lesseqgtr i_{r+1}$. Dann kann man die $(r + 1)$-e Folge dieses Graphen als die A-fache der Grundfolge erfassen.

Beweis. Nach der Def. 18 gilt es hier: $a_{r+1\ 1} = a_{r+1,2} = \ldots = a_{r+1,n} = A$. Also nach den Formeln (4) kann man die Konstante A aus jeder weiteren Spalte dieser Matrix ausklammern.

Die Zusammenfassung der Regeln für die Verzweigung bei den nicht monotonen Profilen können wir folgendermassen ausdrücken:

23. Korolar

a) ***Die Verzweigung der r-ten Folge des Graphen nach einem entgegengerichteten Zeichen erfassen wir als die Differenz zwischen der vollständigen Verzweigung und der Ergänzungsverzweigung von derselben r-ten Folge des Graphen.***

b) *Die Ergänzungsverzwigung läuft dann nach den Regeln für die monotonen Profile.*

c) *Mit der vollständigen Verzweigung endet die bisherige Entwicklung des zugehrigen Ausdruckes-er ist eine Konstante geworden, und zugleich ein neuer Ausdruck fängt seine Entwicklung wieder von* $\binom{n}{1}$ *an.*

d) *Für die weitere Entwicklung gilt stets das Prinzip: „Die Verzweigung der Differenz wird gleich der Differenz der Verzweigung".*

24. Beispiel. Es sind die Elemente (1) gegeben. Es ist die Anzahl aller 9-stelligen Kombinationen auszudrücken, die dem Profil $i_1 \leqq i_2 < i_3 \leqq i_4 > i_5 < i_6 \gtreqless i_7 \leqq \leqq i_8 < i_9$ entsprechen. (Der Strich über dem Ausdruck wird hier bedeuten, dass sich derselbe nicht mehr entwickelt). S_r sei hier die Anzahl aller Elemente aus der r-ten Spalte des Graphen:

$$S_1 = \binom{n}{1}, \quad S_2 = \binom{n+1}{2}, \quad S_3 = \binom{n+1}{3}, \quad S_4 = \binom{n+2}{4},$$

$$S_5 = \overline{\binom{n+2}{4}}\binom{n}{1} - \binom{n+3}{5}, \quad S_6 = \overline{\binom{n+2}{4}}\binom{n}{2} - \binom{n+3}{6},$$

$$S_7 = \overline{\left[\binom{n+2}{4}\binom{n}{2} - \binom{n+3}{6}\right]}\cdot\binom{n}{1} - \overline{\binom{n+2}{4}}\binom{n}{3} + \binom{n+3}{7},$$

$$S_8 = \overline{\left[\binom{n+2}{4}\binom{n}{2} - \binom{n+3}{6}\right]}\cdot\binom{n+1}{2} - \overline{\binom{n+2}{4}}\binom{n+1}{4} + \binom{n+4}{8}.$$

$$S_9 = \overline{\left[\binom{n+2}{4}\binom{n}{2} - \binom{n+3}{6}\right]}\cdot\binom{n+1}{3} - \overline{\binom{n+2}{4}}\binom{n+1}{5} + \binom{n+4}{9}.$$

Es ist ersichtlich, dass die am Anfang dieser Arbeit vorgelegte Aufgabe bewältigt worden ist. Man sieht aber auch, dass es zwecklos wäre, für unregelmässige Profile besondere Formeln zu suchen. Bei speziellen Profilen führt es jedoch zum Auffinden von neuen Beziehungen.

Als ersten Zufall eines speziellen Profils führen wir an

DIE ZWEIPHASIGEN KOMBINATIONEN 1. ART

Diese Kombinationen entsprechen folgendem Profil:

$$i_1 \leqq i_2 \leqq \ldots \leqq i_k \geqq i_{k+1} \geqq \ldots \geqq i_{k+r}. \tag{12}$$

25. Satz. Es sei mit dem Symbol $D_r^k(n)$ die Anzahl aller zweiphasigen Kombinationen bezeichnet, die aus n Elementen mit Wiederholung und $(k+r)$-ter Klasse sind, und dabei dem Profil (12) entsprechen.

Dann gelten folgende zwei Formeln:

$$D_r^k(n) = \sum_{i=0}^{n-1} \binom{k-1+i}{k-1}\binom{r+i}{r}, \tag{13}$$

$$D_r^k(n) = \sum_{i=1}^{k} (-1)^{i+1} \binom{n+k-1-i}{k-i}\binom{n+r}{i+r}, \tag{14}$$

wo für beide diese Formeln gilt: $n \geqq 1$, $k \geqq 1$, $r \geqq 0$.

Beweis der Formel (13)

Betrachten wir die Elemente a_{is} aus der dem Profil (12) entsprechenden Matrix: Die Elemente aus den ersten k Spalten übereinstimmen offenbar mit den, was es ihrer Lage betrifft, entsprechenden Elementen aus dem System (8). Nun wollen wir beweisen, dass es für die Matrixelemente der weiteren Spalten gilt:

$$a_{1+i,k+s} = \binom{s-1}{s-1}\binom{k-1+i}{k-1} + \binom{s}{s-1}\binom{k+i}{k-1} + \dots + \\ + \binom{n+s-i-2}{s-1}\binom{n+k-2}{k-1}, \tag{15}$$

wo $i = 0, 1, \dots, n-1$; $s = 1, 2, \dots, r$.

Den Beweis führen wir mittels der strengen Induktion durch:

Da es hier nach (6-c) gilt:

$$a_{1+i,k+s+1} = \sum_{j=1}^{n-i} a_{i+j,k+s}, \qquad i = 0, 1, \dots, n-1;\ s = 0, 1, \dots, r-1, \tag{16}$$

muss es hier, falls (15) richtig ist, folgende Gleichung gelten:

$$\begin{aligned} a_{i+1,k+s+1} = {} & \binom{s-1}{s-1}\binom{k-1+i}{k-1} + \binom{s}{s-1}\binom{k+i}{k-1} + \dots + \binom{n+s-i-2}{s-1}\binom{n+k-2}{k-1} \\ & + \binom{s-1}{s-1}\binom{k+i}{k-1} + \binom{s}{s-1}\binom{k+i+1}{k-1} + \dots + \binom{n+s-i-3}{s-1}\binom{n+k-2}{k-1} \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \binom{s-1}{s-1}\binom{n+k-3}{k-1} + \binom{s}{s-1}\binom{n+k-2}{k-1} \\ & + \binom{s-1}{s-1}\binom{n+k-2}{k-1} \end{aligned}$$

Addieren wir diese Glieder in Diagonalen, so bekommen wir:

$$\begin{aligned} a_{i+1,k+s+1} = {} & \binom{s-1}{s-1}\binom{k-1+i}{k-1} + \left[\binom{s-1}{s-1} + \binom{s}{s-1}\right]\cdot\binom{k+i}{k-1} + \dots + \\ & + \left[\binom{s-1}{s-1} + \binom{s}{s-1} + \dots + \binom{n+s-i-2}{s-1}\right]\cdot\binom{n+k-2}{k-1}, \end{aligned}$$

und nach der Umformung, zu der wir die Formel 11 aus [1], Seite 248, benützen:

$$a_{i+1,k+s+1} = \binom{s}{s}\binom{k-1+i}{k-1} + \binom{s+1}{s}\binom{k+i}{k-1} + \dots +$$
$$+ \binom{n+s-i-1}{s}\binom{n+k-2}{k-1}.$$

Wir haben also bewiesen: wenn (15) für $k+s$ gilt, dann gilt sie auch für $k+s+1$. Es gilt weiter, mit Hinsicht auf (8) und (6-c):

$$a_{i+1,k+1} = \binom{k+i-1}{k-1} + \binom{k+i}{k-1} + \dots + \binom{n+k-2}{k-1},$$

oder:

$$a_{i+1,k+1} = \binom{0}{0}\binom{k+i-1}{k-1} + \binom{1}{0}\binom{k+i}{k-1} + \dots + \binom{n-i-1}{0}\binom{n+k-2}{k-1},$$

was die Form (15) für $s = 1$ ist.

Nun nach (15) und nach dem Satz 8 bekommen wir $D_r^k(n)$ durch die Summation der rechten Seiten folgender Gleichungen:

$$a_{1,k+r} = \binom{r-1}{r-1}\binom{k-1}{k-1} + \binom{r}{r-1}\binom{k}{k-1} + \dots + \binom{n+r-2}{r-1}\binom{n+k-2}{k-1},$$
$$a_{2,k+r} = \binom{r-1}{r-1}\binom{k}{k-1} + \binom{r}{r-1}\binom{k+1}{k-1} + \dots + \binom{n+r-3}{r-1}\binom{n+k-2}{k-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n,k+r} = \binom{r-1}{r-1}\binom{n+k-2}{k-1}$$

Addieren wir diese Glieder in Diagonalen, so bekommen wir:

$$D_r^k(n) = \binom{r-1}{r-1}\binom{k-1}{k-1} + \left[\binom{r-1}{r-1} + \binom{r}{r-1}\right]\cdot\binom{k}{k-1} + \dots +$$
$$+ \left[\binom{r-1}{r-1} + \dots + \binom{n+r-3}{r-1} + \binom{n+r-2}{r-1}\right]\cdot\binom{n+k-2}{k-1},$$

und daraus, mit Hilfe der oben erwähnten Formel, bekommen wir sofort (13).

Beweis der Formel (14).

Es sei E_s die Anzahl aller Elemente aus der s-ten Spalte des zugehörigen Entwicklungsgraphen; $s = 1, 2, \dots, k$. Es gelte hier das Zeichen $\geqq$, bzw. $\leqq$ für die Grund- bzw. Gegenrichtung. Dann gilt folgende Relation:

(17) $$E_s = \sum_{i=1}^{s} (-1)^{i+1} \cdot E_{s-i} \cdot \binom{n}{i}, \qquad E_0 = 1, \quad s = 1, 2, \dots, k.$$

Beweis. Es gelte (17) für $s < k$. Weil zwischen den Indexen i_s, i_{s+1}, ein inverses Zeichen liegt, so gilt es – nach Korolar 23:

$$E_{s+1} = E_s \cdot \binom{n}{1} - \sum_{i=1}^{s} (-1)^{i+1} \cdot E_{s-i} \cdot \binom{n}{i+1}.$$

Nach der Substitution der Grenzen:

$$i + 1 = j, \text{ woraus } 1 \leqq i \leqq s \Rightarrow 2 \leqq i + 1 \leqq s + 1 \Rightarrow 2 \leqq j \leqq s + 1,$$

gilt:

$$E_{s+1} = E_s \cdot \binom{n}{1} + \sum_{j=2}^{s+1} (-1)^{j+1} \cdot E_{s+1-j} \cdot \binom{n}{j} = \sum_{j=1}^{s+1} (-1)^{j+1} \cdot E_{s+1-j} \cdot \binom{n}{j},$$

was die Relation (17) für den Index $s + 1$ ist.

Für $s = 1$ lautet (17) so: $E_1 = E_0 \cdot \binom{n}{1} = \binom{n}{1}$, was der Wirklichkeit entspricht.

Es gilt allerdings zugleich: $E_s = \binom{n+s-1}{s}$, $s = 1, 2, \ldots, k$, da dem betrachteten Teile des Profils $i_1 \leqq i_2 \leqq \ldots \leqq i_k$ genügen ersichtlich die Kombinationen mit (dauernder) Wiederholung.

Es gilt also

$$E_k = \sum_{i=1}^{k} (-1)^{i+1} \cdot \overline{\binom{n+k-i-1}{k-i}} \binom{n}{i}, \tag{18}$$

in dem der Strich über die Kombinationszahl die schon bekannte Bedeutung hat. Die weitere Verzweigung des Graphen verläuft weiter stets nach dem Zeichen $\geqq$, das hier das Grundzeichen ist. Nach den r Verzweigungen wachsen daher die beiden Zeiger des nicht konstanten Ausdruckes im (18) um r Einheiten. Dadurch bekommen wir Formel (14).

26. Folgerung. Durch die Verbindung der Formeln (13), (14), bekommen wir unmittelbar eine neue kombinatorische Identität.

Illustrationen.

Fig. 1. Die Konfigurationen für die Formeln (4)—a, b, c, d.

Fig. 2. Die Vektorfeldsstreifen für die Zeichen (3).

Fig. 3. Die Darstellung für das Lemma 15.

In folgenden Absätzen dieser Arbeit werden wir zwei weitere speziellen Arten des Profils prüfen:

DIE OSZILLIERENDEN KOMBINATIONEN 1. UND 2. ART

27. Definition. Die oscillierenden Kombinationen erster bzw. zweiter Art sind die Kombinationen aus den Elementen (1), mit Wiederholung, k-ter Klasse, deren Profil aus zwei fortwährend wechselnden Zeichenarten besteht: bei der ersten bzw. zweiten Art sind es die Zeichen $\leqq \geqq$, bzw. $< >$. Es ist unwichtig, mit welchem von den zwei zusammenwirkenden Zeichen das Profil beginnt.

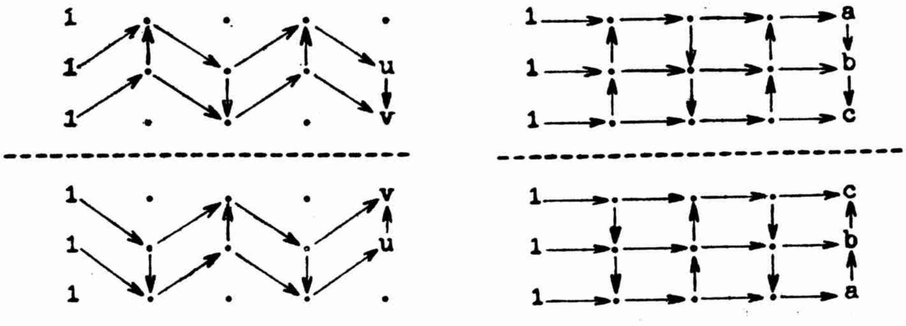
Die oscillierenden Kombinationen werden wir mit der Abkürzung *OK* bezeichnen.

Die Anzahl aller *OK* der 1-en, bzw. der 2-ten Art, k-ter Klasse, aus n Elementen, bezeichnen wir A_n^k, bzw. B_n^k. Dazu definieren wir:

$$A_n^0 = B_n^0 = 1; \qquad n = 1, 2, 3, \ldots \tag{19}$$

28. Satz. Die Anzahl aller *OK* erster bzw. Zweiter Art, aus n Elementen, k-ter Klasse, ist unabhängig von der Art des ersten Profilzeichens.

Beweis. Zeichnen wir zwei, aus n Zeilen und k Spalten bestehende Vektorfelder, so dass ihre (disjunkte) Symetrieachse mit ihren Zeilen parallel sei. Die ersten Profilzeichen dieser Felder seien gegeneinander entgegengesetzt: $\leqq$, $\geqq$, bzw. $<$, $>$. Dann aus den Gleichungen (4)-a, -c, bzw. (4)-b, -d, geht es hervor, dass zwischen den entsprechenden zwei Feldern die Spiegelsymmetrie existiert. Dann aber, je nach der Def. 10, gilt dasselbe für die Grössen a_{ik} der zugehörigen Matrizen, und daraus, nach dem Satz 8, geht unser Satz hervor. Siehe auch die Figur 4:



29. Satz. Es sei $n \geqq 1$. Dann gilt:

$$\text{a) } A_n^k = B_{n+1}^k \quad \text{für} \quad k \geqq 2, \quad \text{oder} \quad k = 0. \qquad \text{b) } A_n^1 = B_{n+1}^1 - 1. \tag{20}$$

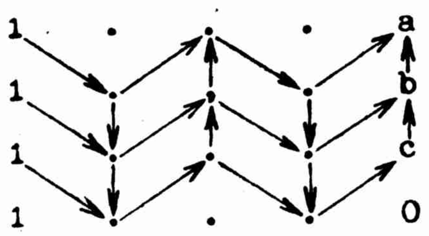
Beweis. Es sei ein n-zeilen und k-spalten Vektorfeld F für die *OK* der ersten Art gegeben. Das zugehörige System aller gebundenen Vektoren dieses Feldes sei in der Deckung mit einem zusammenhängenden Netz, dessen elastische, bzw. nicht elastische Elemente in der Richtung der Zeilen, bzw. der Spalten des Vektorfeldes sind.

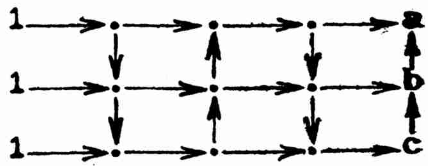
Nun, wenn wir jede gerade (ungerade) Spalte dieses Netzes um eine Zeile nach unten aufspannen und dagegen jede ungerade (gerade) Spalte in der ursprünglichen Lage behalten, dann wird (nach 11. und 27.) dieses Netz mit jenem System gebundener Vektoren in die Deckung übergehen, der dem $(n + 1)$-zeilen und k-spalten Vektorfeld F^+ für die OK der zweiten Art gehört.

Betrachten wir nun die Matrizen M, M^+, die den Vektorfeldern F, F^+ entsprechen: Nach 10] liegt in der s-ten Spalte der beiden Matrizen dieselbe Folge der Nichtnullelemente; $s = 2, 3, \ldots, k$. Die einzelne Ausnahme entsteht nur in der ersten Spalte, weil die Matrix M, bzw. M^+, aus n, bzw. $n + 1$ Zeilen besteht.

Daraus, nach dem Satz 8, folgt sofort der zu beweisende Satz.

Siehe auch die Figur 5:





Die Matrix M

30. Definition. M_1 bzw. M_2 bedeutet eine endliche Menge von n verschiedenen Elementen:

$$M_1 = \{x_1, x_2, \ldots, x_n\}, \qquad M_2 = \{y_1, y_2, \ldots, y_n\}, \tag{21}$$

die die Ungleichung erfüllen:

$$x_i > y_j, \qquad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n. \tag{22}$$

Mit der Benennung „*die komplexen oszillierenden Kombinationen*" und der Abkürzung KOK bezeichnen wir eine k-stellige, aus den Elementen der Menge $(M_1 \cup M_2)$ nach der Vorschrift

$$(x_{i_1} x_{i_2} \ldots x_{i_r} y_{i_{r+1}} y_{i_{r+2}} \ldots y_{i_k}), \qquad 0 \leqq r \leqq k, \tag{23}$$

gebildete Gruppe, wo $(x_{i_1} x_{i_2} \ldots x_{i_r})$, bzw. $(y_{i_{r+1}} y_{i_{r+2}} \ldots y_{i_k})$ eine OK erster, bzw. zweiter Art ist, siehe Def. 27.

Das Profil von KOK (23) darf nicht zwei gleichgerichtete Zeichen hintereinander enthalten. Dabei statt des Zeichens zwischen i_r, i_{r+1} registrieren wir dasjenige zwischen den Elementen $x_{i_r}, y_{i_{r+1}}$, d. h. nach (22). Also die Fälle a) $\ldots i_{r-1} \geqq \geqq i_r > i_{r+1} \ldots$, b) $\ldots i_r > i_{r+1} > i_{r+2} \ldots$ dürfen nicht entstehen. Es muss also gelten:

$$\begin{aligned} &i_1 \leqq i_2 \quad \text{für gerade } r, \qquad i_1 \geqq i_2 \quad \text{für ungerade } r, \\ &i_{r+1} < i_{r+2} \quad \text{für alle } r. \end{aligned} \tag{24}$$

31. Vereinbarung. Benützen wir die Strukturalbezeichnung

$$(r, k - r); \quad 0 \leqq r \leqq k, \tag{25}$$

für jede *KOK*, die in ihrer Zusammenstellung r bzw. $k - r$ Elemente aus der Menge M_1 bzw. M_2 hat. Dabei bezeichnen wir eine *KOK* als gerade, bzw. ungerade, wenn die Zahl r in (25) eine gerade bzw. ungerade ist.

32. Satz. Es sei k eine beliebige natürliche Zahl. Die Menge aller *KOK* k-ter Klasse, die aus den Elementen $\in (M_1 \cup M_2)$ nach der Def. 30 zusammengesetzt sind, besitzt eine gleiche Anzahl von geraden und ungeraden *KOK*.

Beweis. Betrachten wir eine gewisse Abbildung der Menge aller, durch die Def. 30 gegebener *KOK* auf sich: diese Abbildung wollen wir „*Transformation durch umwerfen des Randelelementes*" nennen.

Das Prinzip der Transformation: Bei jeder *KOK* der betrachteten Menge wird immer eines der Randelemente auf die entgegengelegene Seite umgesetzt, d. h. es wird bei dieser Kombination zu dem Randelement von der entgegengesetzten Seite, und es wird dabei das umzusetzende Element auf gewisse Art transformiert. Alle anderen Elemente bleiben dabei in *KOK* ohne Änderung und in ursprünglicher Zusammenstellung.

Die Transformation des umzuwerfenden Elementes:

1) Es sei die Zahl k (die Klasse *KOK* angebend) eine gerade: dann wird das Element x_i zu dem y_{n+1-i} oder y_j wird zum x_{n+1-j}.

2) Es sei k eine ungerade Zahl: dann wird das Element x_i zu dem y_i, oder das Element y_j wird zu dem x_j.

Es ist allerdings Bedingung der Transformation, dass wieder eine *KOK* entstehen muss, die die Bedingungen (23) und (24) erfüllt.

Zwecks des Beweises unseres Satzes nehmen wir zuerst in Betracht, dass durch die erwähnte Transformation immer aus der geraden, bzw. ungeraden *KOK* eine ungerade, bzw. gerade *KOK* entstehen muss.

Wir beweisen weiter, dass es bei jeder *KOK* möglich ist immer nur eines ihrer Randelemente auf die angeführte Art umzustellen und zu transformieren, wenn wieder eine die Bedingungen (23) und (24) erfüllende *KOK* entstehen soll:

A) Betrachten wir alle *KOK*, für die $r = 0$, oder $r = k$ gilt. Diese Kombinationen haben je nach (23) ersichtlich die einzige Transformationsmöglichkeit.

B) Betrachten wir alle *KOK*, für die gilt: $0 < r < k$, wo k eine gerade Zahl ist. Dann beweisen wir, dass einer *KOK* (23) nicht beide Varianten ihrer Transformation, d. i.

a) $$(x_{i_2} \ldots x_{i_r} y_{i_r+1} \ldots y_{i_k-1} y_{i_k} y_{n+1-i_1}),$$

b) $$(x_{n+1-i_k} x_{i_1} x_{i_2} \ldots x_{i_r} y_{i_r+1} \ldots y_{i_k-1}),$$

entsprechen können. Betrachten wir in Bezug auf (24) beide Möglichkeikeiten : 1) Wenn r eine gerade Zahl ist, dann muss in der ursprünglichen *KOK* $i_1 \leqq i_2$ und gleichzeitig $i_{k-1} < i_k$ gelten. Damit aus dieser wieder eine *KOK* entstehe, muss es also gelten: im Falle a) $i_k > n + 1 - i_1$, im Falle b) $n + 1 - i_k \geqq i$. Aber von diesen zwei Möglichkeiten kann nur eine gelten. 2) Wenn r eine ungerade Zahl ist, dann muss in der ursprünglichen *KOK* $i_1 \geqq i_2$, (oder $i_1 > i_2$, falls schon das zweite Element $\in M_2$), und gleichzeitig $i_{k-1} > i_k$ gelten. Damit aus dieser wieder eine *KOK* entstehe, muss es also gelten: im Falle a) $i_k < n + 1 - i_1$, im Falle b) $n + 1 - i_k \leqq i_1$, was wieder nicht gleichzeitig gelten kann.

C) Betrachten wir alle *KOK*, für die gilt $0 < r < k$, wo k eine *ungerade* Zahl ist. Wir beweisen wieder, dass einer *KOK* (23) nicht beide Varianten ihrer Transformation, d. i.

$$\text{a)} \qquad (x_{i_2} \ldots x_{i_r} y_{i_r+1} \ldots y_{i_{k-1}} y_{i_k} y_{i_1}),$$

$$\text{b)} \qquad (x_{i_k} x_{i_1} x_{i_2} \ldots x_{i_r} y_{i_r+1} \ldots y_{i_{k-1}}),$$

entsprechen können. Betrachten wir in Bezug auf (24) beide Möglichkeiten: 1) Wenn r eine gerade Zahl ist, dann muss in der ursprünglichen *KOK* $i_1 \leqq i_2$ und gleichzeitig $i_{k-1} > i_k$ gelten. Damit aus dieser wieder eine *KOK* entstehe, muss es also gelten: im Falle a) $i_k < i_1$, im Falle b) $i_k \geqq i_1$, was nicht gleichzeitig gelten kann. 2) Wenn r eine ungerade Zahl ist, dann muss in der ursprünglicher *KOK* $i_1 \geqq i_2$ (oder $i_1 > i_2$) und gleichzeitig $i_{k-1} < i_k$ gelten. Damit aus dieser wieder eine *KOK* entstehe, muss es also gelten: im Falle a) $i_k > i_1$, im Falle b) $i_k \leqq i_1$, was nicht gleichzeitig gelten kann.

Es ist also ersichtlich, dass durch die angeführte Transformation auf der Menge aller gegebenen *KOK* eine (1,1)-deutige Abbildung zwischen der Untermenge aller geraden und der Untermenge aller ungeraden *KOK* definiert wird. Dadurch ist aber der Beweis des Satzes erbracht.

33. Lemma. Die Anzahl aller *KOK*, die nach der Def. 30 zusammengestellt werden und die eine gemeinsame Struktur $(r, k - r)$ haben, ist $A_n^r \cdot B_n^{k-r}$.

Beweis. Betrachten wir 1) die *OK* 1-er Art, r-ter Klasse, aus M_1, 2) die *OK* 2-ter, Art, $(k - r)$-ter Klasse, aus M_2. Mit Rücksicht auf (24) verlangen wir, dass bei jeder *OK* and 1) das erste Zeichen des Profils $\leqq$ bzw. $\geqq$ sei, wenn die Zahl r gerade, bzw. ungerade ist. Bei jeder *OK* ad 2) verlangen wir, dass das erste Profilzeichen $<$ sei. Diese Forderungen haben nach dem Satz 5 keinen Einfluss auf die Bestimmung der Formel für die Anzahl aller entsprechenden *OK*. Wählen wir nun eine beliebige von den angeführten *OK* 1-er Art und fügen sukzessiv alle der angeführten *OK* der 2-ten Art dazu. Damit erhalten wir B_n^{k-r} *KOK* k-ter Klasse, aus $(M_1 \cup M_2)$ von gemeinsamer Struktur $(r, k - r)$. Weil wir von den *OK* 1-er Art im Ganzen A_n^r wählen können, so ist es ersichtlich, dass die Anzahl aller *KOK* der $(r, k - r)$ Struktur $A_n^r B_n^{k-r}$ ist.

34. Satz. Für $n > 1$, $k \geqq 1$, gilt:

$$\sum_{r=0}^{k} (-1)^{k-r} \, . \, A_n^r \, . \, B_n^{k-r} = 0, \tag{26}$$

Beweis. Nach dem Satz 32 und Lemma 33 handelt es sich um die Differenz zwischen der Anzahl aller geraden und ungeraden, nach der Def. 30 gegebenen *KOK*.

35. Definition. Für die Zahlen A_n^k, $n \geqq 1$, $k \geqq 0$, und für die Zahlen B_n^k, $n \geqq 2$, $k \geqq 0$, siehe die Def. 27, bestimmen wir und folgendermassen bezeichnen die erzeugende Funktion:

$$\varphi_n(x) = A_n^0 + A_n^1 \, . \, x + A_n^2 \, . \, x^2 + \ldots + A_n^k \, . \, x^k + \ldots \tag{27}$$

$$\psi_n(x) = B_n^0 + B_n^1 \, . \, x + B_n^2 \, . \, x^2 + \ldots + B_n^k \, . \, x^k + \ldots \tag{28}$$

36. Lemma. Die Potenzreihe (27), bzw. (28) konvergiert zum weingsten im Interval $|x| < \dfrac{1}{n}$.

Beweis. Sind n verschiedene Elemente und das Profil der dauernden vollständigen Verzweigung gegeben (siehe Def. 18): $\lessgtr, \lessgtr, \ldots, \lessgtr$, dann ist die Anzahl der in der k-ten Spalte des betreffenden Graphen liegenden Elemente ersichtlich n^k, $k = 1, 2, 3, \ldots$ Daraus folgt weiter, dass die Potenzreihe $R_n(x) = 1 + nx + (nx)^2 + \ldots + (nx)^k + + \ldots$ für $x > 0$ eine majorante Reihe zu den Reihen (27), (28) ist. Dabei $R_n(x)$ konvergiert im Interval $|x| < \dfrac{1}{n}$.

37. Satz. Für $n \geqq 2$ und $|x| < \dfrac{1}{n}$ gilt:

$$\varphi_n(x) \, . \, \psi_n(-x) = 1. \tag{29}$$

Beweis. Wir führen das Cauchy-sche Produkt der Potenzreihen $\varphi_n(x)$ $\psi_n(-x)$ durch. Die das Produkt dieser zwei Potenzreihen darstellende Potenzreihe sei bezeichnet: $C_0 + C_1 \, . \, x + C_2 \, . \, x^2 + \ldots$ Dann, wie bekannt, gilt: $C_k = \sum_{r=0}^{k} (-1)^{k-r} \times \times A_n^r \, . \, B_n^{k-r}$. Hier aber, nach dem Satz 34 gilt: $C_k = 0$ für $k = 1, 2, 3, \ldots$ Ausserdem, nach der Def. 27 gilt: $C_0 = A_n^0 \, . \, B_n^0 = 1$.

38. Satz. Für die fortschreitende Entwicklung der ausschaffenden Funktionen $\varphi_n(x)$ gilt folgende Rekurrenz:

$$\varphi_n(x) \, . \, (-x + \varphi_{n-1}(-x)) = 1; \qquad n = 2, 3, 4, \ldots \tag{30}$$

$$\varphi_1(x) = \frac{1}{-x + 1}. \tag{31}$$

Beweis ad (30) – ist eine direkte Folgerung der Sätze 29 und 37.

Beweis ad (31) – Weil $A_1^k = 1$ für $k \geqq 0$, so gilt nach (27):

$$\varphi_1(x) = 1 + x + x^2 + \ldots + x^k + \ldots = \frac{1}{-x+1} \qquad \text{fur} \qquad |x| < \frac{1}{n}.$$

Auf Grund dieser Rekurrenzen versuchen wir in weiterem die direkten Formeln zu finden.

39. Definition. Es sei $[m]$ die Abrundung der Zahl m auf ihr nächstes kleineres Ganze. Dann definieren wir die Zahl

(32) $$D_n^k = \binom{\left[\frac{n+k}{2}\right]}{k},$$

für den Indexenbereich

(33) $$k = 0, 1, \ldots, n; \qquad n = 0, 1, 2, \ldots$$

Die dem Indexenbereich entsprechenden Zahlen D_n^k ordnen wir in ein Dreiecksystem an, das wir *„das verdoppelte Pascal'sche Dreieck"* benennen. Es ist ein System von Zeilen und Spalten, siehe die Nebendarstellung, worin sich die Zahl D_n^k in der n-te Zeile von oben und k-ten Spalte von links befindet.

$$\begin{array}{llll} D_0^0 & & & \\ D_0^1 & D_1^1 & & \\ D_2^0 & D_2^1 & D_2^2 & \\ D_3^0 & D_3^1 & D_3^2 & D_3^3 \end{array}$$

40. Bemerkung. Die Zahlen D_n^k, die sich ausserhalb des Dreiecksystems befinden, sind im Hinblick auf die Elementardefinition der Zahlen $\binom{n}{k}$ gleich null.

Die Zahl D_n^k bedeutet auch die Anzahl aller Kombinationen $(x_{i_1} x_{i_2} \ldots x_{i_k})$, wo die Elemente x_i die natürlichen Zahlen der Menge $\{1, 2, \ldots, n\}$ sind und gleichzeitig die folgenden 2 Bedingungen erfüllen: 1) $x_{i_1} < x_{i_2} < \ldots < x_{i_k}$, 2) $x_{i_{2r-1}}$ resp. $x_{i_{2r}}$ ist eine ungerade, bzw. gerade Zahl; $r = 1, 2, \ldots$. Siehe [1], Paragraph 49.

41. Satz. Für die Zahlen D_n^k gilt die Rekursionformel:

(34) $$D_n^k = D_{n-2}^k + D_{n-1}^{k-1},$$

Beweis. Es genügt zu überlegen, dass es gilt: $\left[\frac{n+k-2}{2}\right] = \left[\frac{n+k}{2} - 1\right] = \left[\frac{n+k}{2}\right] - 1$. Wenn man dann $\left[\frac{n+k}{2}\right] = m$ bezeichnet, lautet die Gleichung

(34) folgendermassen: $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$, was eine Elementarbekannte Relation ist.

42. Definition. Wir legen fest und bezeichnen mit einem Symbol das folgende Polynom:

$$f_n(x) = \sum_{i=0}^{n} (-1)^{\binom{i+1}{2}} . D_n^i . x^i; \qquad n = 0, 1, 2, \dots \tag{35}$$

43. Bemerkung. Wie ersichtlich, stimmt die Koeffizientenfolge des Polynoms $f_n(x)$ – bis auf die Vorzeichen – mit der Folge der n-ten Zeile des Zahlensystems D_n^k überein. Wir gebrauchen daher auch im weiteren Text die Bezeichung für das Polynom nullter Stufe: $f_0(x) = D_0^0 = 1$.

44. Satz. Für Polynome $f_n(x)$, $n \geqq 2$, gilt die Rekursionsgleichung:

$$f_n(x) = f_{n-2}(x) - x . f_{n-1}(-x); \qquad f_0(x) = 1, \qquad f_1(x) = 1 - x. \tag{36}$$

Beweis. $f_{n-2}(x) - x . f_{n-1}(-x) = \sum_{i=0}^{n-2} (-1)^{\binom{i+1}{2}} . D_{n-2}^i . x^i - x . \sum_{i=0}^{n-1} (-1)^{\binom{i+1}{2}} \times$ $\times D_{n-1}^i . (-x)^i = D_{n-2}^0 + \sum_{i=0}^{n-3} (-1)^{\binom{i+1}{2}} . D_{n-2}^{i+1} . x^{i+1} + \sum_{i=0}^{n-1} (-1)^{\binom{i+1}{2} + i + 1} \times$ $\times D_{n-1}^i . x^{i+1} = D_n^0 + \sum_{i=0}^{n-1} (-1)^{\binom{i+2}{2}} . (D_{n-2}^{i+1} + D_{n-1}^i) . x^{i+1} = D_n^0 + \sum_{i=0}^{n-1} (-1)^{\binom{i+2}{2}} \times$ $\times D_n^{i+1} . x^{i+1} = D_n^0 . x^0 + \sum_{i=1}^{n} (-1)^{\binom{i+1}{2}} . D_n^i . x^i = \sum_{i=0}^{n} (-1)^{\binom{i+1}{2}} . D_n^i . x^i = f_n(x)$,
w. z. b. w.

45. Definition. Wir setzen fest und bezeichnen die rationale gebrochene Funktion

$$\Phi_n(x) = \frac{f_{n-1}(-x)}{f_n(x)}; \qquad n = 1, 2, 3, \dots \tag{37}$$

46. Satz. Für die Funktion $\Phi_n(x)$, $n > 1$, gilt die Rekursionsbeziehung:

$$\Phi_n(x) . (\Phi_{n-1}(-x) - x) = 1; \qquad \Phi_1(x) = \frac{1}{1-x}. \tag{38}$$

Beweis. Es gilt nach (36) und (37):

$$\Phi_n(x) = f_{n-1}(-x) \,/\, f_n(x) = f_{n-1}(-x) \,/\, (f_{n-2}(x) - x . f_{n-1}(-x)) =$$

$$= 1 \Big/ \left(-x + \frac{f_{n-2}(x)}{f_{n-1}(-x)}\right) = \frac{1}{-x + \Phi_{n-1}(-x)}, \qquad \text{w. z. b. w.}$$

47. Korolar.

Die ausschaffende Funktion (27) für die oszillierenden Kombinationen der ersten Art, in ihrer beendeten Form, lautet:

$$\varphi_n(x) = \frac{f_{n-1}(-x)}{f_n(x)}\,; \qquad n \geqq 1. \tag{39}$$

Beweis. Es genügt ersichtlich die Identität der Funktionen $\varphi_n(x)$, $\Phi_n(x)$ zu beweisen, siehe (27), (37). Das ist aber leicht: weil ihre Rekursionsformeln (30), (38) formal übereinstimmen, und weil auch $\varphi_1(x) = \Phi_1(x)$, siehe (31) und (35) für $n = 1$, wird damit die Identität bewiesen.

48. Satz. Die Summe der r-ten Zeile im Dreiecksystem der Zahlen D_n^k gibt die Anzahl aller oszillierenden Kombinationen erster Art, r-ten Klasse, aus zwei Elementen an.

Beweis. Nach dem Korolar 47 und nach den Definitionen 35 und 42, gilt:

$\dfrac{1 + x}{1 - x - x^2} = A_2^0 + A_2^1 \,.\, x + A_2^2 \,.\, x^2 + \ldots$ Es gilt doch zugleich:

$\dfrac{1 + x}{1 - x - x^2} = G_0 + G_1 \,.\, x + G_2 \,.\, x^2 + \ldots$, wo gilt: $G_r = D_r^0 + D_r^1 + \ldots + D_r^r$; $r = 0, 1, 2, \ldots$, siehe [1], Paragraph 49.

In dem nächsten Abschnitt der Arbeit werden wir die Erörterung der Wurzeleigenschaften des Polynoms $f_n(x)$ in Angriff nehmen.

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ON THE INFINITESIMAL GEOMETRIC OBJECTS OF SUBMANIFOLDS

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As a further development of the invariant method of investigation for submanifolds of homogeneous spaces by É. Cartan, some authors (in particular S. P. Finikov and his disciples) have applied a natural modification of this method for finding of infinitesimal invariants and infinitesimal relative invariants of submanifolds. Though they have investigated only concrete situations, we can observe that their evaluations have the following general form. Besides the "variations with respet to the secondary parameters" of some functions, one also finds the corresponding variations of a basis of the principal forms and then proceeds in the same way as in the case of the usual (or "finite") invariants. Since we feel such a "classical" approach to be unsatisfactory from theoretical point of view, we present an intrinsic explanation of this algorithm. Moreover, the present state of the theory of geometric objects of submanifolds, [10], [3], [7], enables us to treat arbitrary infinitesimal geometric objects of submanifolds. We conclude with a detailed discussion of a classical example from our point of view. Since we use a specialization of frames in the example, we also add a general remark to an aspect of this procedure.

Our considerations are in the category C^∞. The standard terminology of the theory of jets is used throughout the paper, cf. [3], [6].

1. Consider the fibre bundle $K^1_m(M)$ of all regular contact m^1-elements on a differentiable manifold M, $m < n = \dim M$. Taking into account that every $\xi \in K^1_{m,x}(M)$ is naturally identified with an m-dimensional subspace $\tau(\xi)$ of $T_x(M)$, we define

$$VK^1_m(M) = \{(v, \xi) \in T(M) \oplus K^1_m(M);\ v \in \tau(\xi)\}, \tag{1}$$

where $\oplus$ means the fibre product over M. The space $VK^1_m(M)$ can be considered either as a vector bundle over $K^1_m(M)$ of fibre dimension m or as an associated fibre bundle of the symbol $(M, \gamma^n_m, L^1_n, H^1(M))$, where $\gamma^n_m = VK^1_{m,0}(\mathbf{R}^n)$ (i.e. the fibre of $VK^1_m(\mathbf{R}^n)$ over $0 \in \mathbf{R}^n$), cf. [1]. Further, let $K^r_m(M)$ be the fibre bundle of all contact m^r-elements on M and let $\rho^s_r : K^r_m(M) \to K^s_m(M)$, $s \leqq r$, be the jet projection. Then we set

$$VK^r_m(M) = \{(v, \xi) \in T(M) \oplus K^r_m(M);\ v \in \tau(\rho^1_r(\xi))\}. \tag{2}$$

Even this space can be considered either as a vector bundle over $K^r_m(M)$ of fibre dimension m or as an associated fibre bundle of the symbol $(M, VK^r_{m,0}(\mathbf{R}^n), L^r_n, H^r(M))$. If S is an m-dimensional submanifold of M, then we have a natural injection $T(S) \subset \subset VK^r_m(M)$, $v \to (v, k^r_{\eta(v)}S)$, where $\eta : T(M) \to M$ is the bundle projection and $k^r_x S$ means the contact m^r-element determined by S at a point $x \in S$.

Remark 1. Quite analogously, one can treat the tangent vectors of higher order. Let $T^r(M)$ be the bundle of all tangent vectors of order r on M. (In particular, we recall the exact sequence

$$0 \to T^{r-1}(M) \to T^r(M) \to S^r T(M) \to 0 \tag{3}$$

established by Pohl, [12].) Let $\xi = XL^r_m \in K^r_{m,x}(M)$, where X is an m^r-velocity on M at x. Since X is an r-jet of $\mathbf{R}^m$ into M with source 0 and target x, it determines a linear mapping $X_* : T^r_0(\mathbf{R}^m) \to T^r_x(M)$. One sees directly that the subspace $X_*(T^r_0(\mathbf{R}^m)) \subset \subset T^r_x(M)$ is well determined by ξ, i.e. it does not depend on the choice of a representative X of ξ; we shall denote this subspace by $\tau_r(\xi)$. Then we define

$$V^s K^r_m(M) = \{(v, \xi) \in T^s(M) \oplus K^r_m(M);\ v \in \tau_s(\rho^s_r(\xi))\}.$$

Obviously, it is $VK^r_m(M) = V^1 K^r_m(M)$. In the differential geometry of submanifolds, the higher order tangent vectors are sometimes investigated in such a way, that the vectors of $T^r(M)$ are transformed by (3) into the elements of $S^r T(M)$. In this special case, the following method of investigation can be applied. However, we do not see any "natural" invariant algorithm for the higher order tangent vectors in the general case.

2. To clear some fundamental ideas up, we shall start with an auxiliary consideration. The tangent bundle $T(M)$ of M is an associated fibre bundle of the symbol $(M, \mathbf{R}^n, L^1_n, H^1(M))$, so that one has a relative image map $\mu : H^1(M) \oplus T(M) \to \mathbf{R}^n$, see [5]. This mapping is in the following close relation to the canonical form Θ of $H^1(M)$, [2], [4]. Let $\pi : H^1(M) \to M$ be the bundle projection. Then its differential $\pi_* : T(H^1(M)) \to T(M)$ together with the bundle projection $\nu : T(H^1(M)) \to H^1(M)$ determine a mapping $\lambda : T(H^1(M)) \to H^1(M) \oplus T(M)$, $\lambda(v) = (\nu(v), \pi_*(v))$.

Lemma 1. *The following diagram commutes*

$$\begin{tikzcd} \mathbf{R}^n & T(H^1(M)) \arrow[l, "\Theta"'] \arrow[d, "\lambda"] \\ & H^1(M) \oplus T(M) \arrow[ul, "\mu"] \end{tikzcd} \tag{4}$$

Proof. If $u \in H^1(M)$ and $v \in T_u(H^1(M))$, then $\Theta(v) = u^{-1}(\pi_*(v))$ by definition of Θ. On the other hand, if $w \in T_x(M)$, then $\mu(u, w) = u^{-1}(w)$ by definition of μ. Hence $\mu(\lambda(v)) = \mu(u, \pi_*(v)) = u^{-1}(\pi_*(v)) = \Theta(v)$, QED.

Let $K^1_{n,m} = K^1_{m,0}(\mathbf{R}^n)$ and let $\hat{K}^1_{n,m} \subset K^1_{n,m}$ be the subspace of all elements transversal with respect to the canonical projection $p : \mathbf{R}^n \to \mathbf{R}^m$. On $\hat{K}^1_{n,m}$, there are natural coordinates y^J_p, see [6]. Further, let $\hat{\gamma}^n_m \subset \gamma^n_m$ be the subspace of all pairs (v, ξ), such that $\xi \in \hat{K}^1_{n,m}$. On $\hat{\gamma}^n_m$, we introduce the coordinates y^J_p, y^p by $y^J_p(v, \xi) = y^J_p(\xi)$ and by

$$(5) \qquad y^p(v, \xi) = x^p(p(v)) = y^p(v), \qquad \begin{array}{l} p, q, \ldots = 1, \ldots, m, \\ J, K, \ldots = m + 1, \ldots, n, \end{array}$$

where x^p are the canonical coordinates on $\mathbf{R}^m$. Obviously, v is completely determined by $p(v) \in \mathbf{R}^m$ and by ξ (the coordinates of $v \in \mathbf{R}^n$ are $y^p(v)$, $y^J_p(\xi)\, y^p(v)$).

Consider an m-dimensional submanifold $S \subset M$. According to [6], we denote by $Q^1(S)$ the restriction of $H^1(M)$ over S and we set

$$\hat{Q}^1(S) = \{u \in Q^1(S);\, u^{-1}(k^1_x S) \in \hat{K}^1_{n,m},\, x = \pi(u)\}.$$

Every $u \in Q^1_x(S)$ carries $T_x(S)$ into an m-dimensional subspace $u^{-1}(T_x(S))$ of $\mathbf{R}^n$, so that it plays a role of a frame for $T_x(S)$. If $u \in \hat{Q}^1_x(S)$, then $u^{-1}(T_x(S))$ is transversal with respect to $p : \mathbf{R}^n \to \mathbf{R}^m$. Consequently, we obtain the natural coordinate functions A^p on $\hat{Q}^1(S) \oplus T(S)$ defined by $A^p(u, v) = x^p(p(u^{-1}(v)))$. Let

$$\tilde{\Theta} = (\tilde{\Theta}_i), \qquad i, j \ldots = 1, \ldots, n$$

be the restriction of the canonical form of $H^1(M)$ to $Q^1(S)$. By (4) and (5), we deduce the commutativity of the diagram

$$(6) \qquad \begin{tikzcd} \mathbf{R} & T(\hat{Q}^1(S)) \arrow[l, "\tilde{\Theta}^p"'] \arrow[d, "\lambda_S"] \\ & \hat{Q}^1(S) \oplus T(S) \arrow[ul, "A^p"] \end{tikzcd}$$

where $\lambda_S = \lambda \mid T(\hat{Q}^1(S))$. Hence the differential forms $\tilde{\Theta}^p : T(\hat{Q}^1(S)) \to \mathbf{R}$ can be considered as a special kind of some "coordinate functions" for $T(S)$ in the following sense. If $v \in T_x(S)$, $u \in \hat{Q}^1_x(S)$ and $w \in T_u(\hat{Q}^1(S))$ is a vector such that $\pi_*(w) = v$, then it is $\hat{\Theta}^p(w) = A^p(u, v)$. In addition, if $a^J_p : \hat{Q}^1(S) \to \mathbf{R}$ are the coordinate functions of the fundamental field of the first order of S, then the coordinates of the vector $u^{-1}(v) \in \mathbf{R}^n$ are $\tilde{\Theta}^p(w)$, $a^J_p(u)\, \tilde{\Theta}^p(w)$.

We shall now deduce the equations of the fundamental distribution on $L^1_n \times \gamma^n_m$ with respect to the coordinates y^J_p, y^p. Though it is not complicated to use direct evaluations, we shall proceed in an indirect way. For the sake of simplicity, we shall assume that S is homotopically trivial. Take a vector field $\zeta : S \to T(S)$ and denote by $a^p : \hat{Q}^1(S) \to \mathbf{R}$ its coordinate functions, i.e. $a^p(u) = A^p(u, \zeta(\pi(u)))$. Consider a fundamental vector field Y on $\hat{Q}^1(S) \subset Q^1(S)$ and choose a vector field Z on $\hat{Q}^1(S)$ such that Z is π-related with ζ and satisfies $[Y, Z] = 0$. (Such a vector field can be constructed easily be means of a trivialization of principal fibre bundle $Q^1(S)$.) According to (6), it holds $a^p = \tilde{\Theta}^p(Z)$. To evaluate $da^p(Y) = Y\tilde{\Theta}^p(Z)$, we use, on the one hand, the well known formula for exterior derivative

$$\text{(7)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = \frac{1}{2}\{Y\tilde{\Theta}^p(Z) - Z\tilde{\Theta}^p(Y) - \tilde{\Theta}^p([Y, Z])\}$$

and, on the other hand, the structure equations of Θ

$$\text{(8)} \qquad \mathrm{d}\Theta^i = \Theta^j \wedge \Theta'^i_j,$$

where (Θ^i, Θ'^i_j) is an admissible extension of Θ, [4]. Using the relation $\tilde{\Theta}^J = a^J_p \tilde{\Theta}^p$, [6], we obtain

$$\text{(9)} \qquad \mathrm{d}\tilde{\Theta}^p = \tilde{\Theta}^q \wedge (\tilde{\Theta}'^p_q + a^J_q \tilde{\Theta}'^p_J).$$

As Y is a vertical vector field, it is $\tilde{\Theta}^p(Y) = 0$. Hence (7) is simplified to

$$\text{(10)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = \frac{1}{2}\, Y\tilde{\Theta}^p(Z)$$

and (9) implies

$$\text{(11)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = -\frac{1}{2}\tilde{\Theta}^q(Z)\,[\tilde{\Theta}'^p_q(Y) + a^J_q \tilde{\Theta}'^p_J(Y)].$$

Comparing (10) and (11), we obtain finally

$$\text{(12)} \qquad da^p(Y) + a^q \Theta'^p_q(Y) + a^q a^J_q \tilde{\Theta}'^p_J(Y) = 0.$$

In [6], we have deduced the equations of the fundamental distribution on $L^1_n \subseteq K^1_{n,m}$ in the form

$$\text{(13)} \qquad \mathrm{d}y^J_p - y^J_q \pi^q_p - y^J_q y^K_p \pi^q_K + y^K_p \pi^J_K + \pi^J_p = 0,$$

where π^j_i is the natural basis of the Maurer–Cartan forms of L^1_n. Applying Lemma 2 of [5] to (12), we prove

Proposition 1. *The equations of the fundamental distribution on* $L^1_n \subseteq \gamma^n_m$ *are* (13) *and*

$$\text{(14)} \qquad \mathrm{d}y^p + y^q \pi^p_q + y^q y^J_q \pi^p_J = 0.$$

Remark 2. Our previous consideration gives a precise explanation of the "classical" manipulation with two "exchangeable symbols of differentiation d, δ, where δ means the differentiation with respect to the secondary parameters" according to É. Cartan, see e.g. [11].

Assume now that M is a homogeneous space with fundamental group G. Fix a point $c \in M$ and denote by H its stability group. Then G has a natural structure of a principal fibre bundle over M with structure group H and $VK^1_m(M)$ can be considered as an associated fibre bundle of the symbol $(M, VK^1_{m,c}(M), H, G)$. Fix a local coordinate system $\varkappa$ on M at c, so that $VK^1_{m,c}(M)$ is identified with γ^n_m. Let

$$
\begin{aligned}
(15)\qquad & d\omega^i = \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k + c^i_{j\lambda}\omega^j \wedge \omega^\lambda, && \lambda, \mu, \ldots = n+1, \ldots, \dim G, \\
& d\omega^\lambda = \frac{1}{2} c^\lambda_{\alpha\beta}\omega^\alpha \wedge \omega^\beta, && \alpha, \beta, \ldots = 1, \ldots, \dim G,
\end{aligned}
$$

be the corresponding structure equations. By means of the homomorphism of H into L^1_n investigated in [6], we find directly the equations of the fundamental distribution on $H \times \gamma^n_m$ in the form

$$
(16)\qquad dy^J_p - (y^J_q c^q_{p\lambda} + y^J_q y^K_p c^q_{K\lambda} - y^K_p c^J_{K\lambda} - c^J_{p\lambda})\pi^\lambda = 0,
$$

$$
(17)\qquad dy^p + y^q c^p_{q\lambda}\pi^\lambda + y^q y^J_q c^p_{J\lambda}\pi^\lambda = 0,
$$

where π^λ is the restriction of ω^λ to H. From the practical point of view it is remarkable that these equations can be deduced directly from (15) as follows.

Consider an m-dimensional submanifold $S \subset M$, denote by $Q(S)$ the restrition of principal fibre bundle $G(M, H)$ over S and introduce $\hat{Q}(S) \subset Q(S)$ in the same way as in [6]. Then every $u \in Q_x(S)$ plays a role of a frame for $T_x(S)$, since it caries $T_x(S)$ into an m-dimensional subspace $u^{-1}(T_x(S))$ of $T_c(M)$. If $u \in \hat{Q}(S)$, then this subspace is transversal with respect to $p : \mathbf{R}^n \to \mathbf{R}^m$, provided the local identification of M and $\mathbf{R}^n$ determined by $\varkappa$ is applied. Hence we can define some coordinate functions A^p on $\hat{Q}(S) \oplus T(S)$ by $A^p(u, v) = x^p(p(u^{-1}(v)))$. We also have a mapping $\lambda : T(G) \to G \oplus T(M)$, $v \to (\nu(v), \pi_*(v))$, where $\nu : T(G) \to G$ and $\pi : G \to M$ are the bundle projections. Let $\lambda_S = \lambda \mid T(\hat{Q}(S))$ and let $\tilde{\omega}^\alpha$ be the restriction of ω^α to $\hat{Q}(S)$. According to (6) and [6], we obtain a commutative diagram

$$
(18)\qquad
\begin{tikzcd}
\mathbf{R} & T(\hat{Q}(S)) \arrow[l, "\tilde{\omega}^p"'] \arrow[d, "\lambda_S"] \\
 & \hat{Q}(S) \oplus T(S) \arrow[ul, "A^p"]
\end{tikzcd}
$$

Taking into account the relations $\tilde{\omega}^J = a^J_p \tilde{\omega}^p$, where $a^J_p : \hat{Q}(S) \to \mathbf{R}$ are the coordinate functions of the fundamental field of the first order of S, we can deduce (17) analogously to (7) and (11), provided the structure equations (15) are used instead of (8).

If we now consider the fibre bundle $VK^r_m(M)$ for an arbitrary differentiable manifold M, we obtain the equations of the fundamental distribution on $L^r_n \times VK^r_{m,0}(\mathbf{R}^n)$ immediately as the equations (21) of [6] together with (14). Analogously, if M is a homogeneous space, then $VK^r_m(M)$ should be considered as an associated fibre bundle of the symbol $(M, VK^r_{m,c}(M), H, G)$ and the equations of the fundamental distribution on $H \times VK^r_{m,c}(M)$ consist of the equations of the fundamental distribution on $H \times K^r_{m,c}(M)$, [6], and of (17).

3. We shall present the definition of the infinitesimal geometric objects of submanifolds in the most important case, i.e. for submanifolds of a homogeneous space (the case of a space with a fundamental Lie pseudogroup can be treated quite similarly to [3], [6]).

Definition 1. *An infinitesimal geometric m^r-object ψ on a homogeneous space M is an equivariant mapping of H-space $VK^r_{m,c}(M)$ into another H-space W, $\psi : VK^r_{m,c}(M) \to \to W$. Let ψ_2 be the induced mapping $\psi_2 : VK^r_m(M) \to (M, W, H, G)$, see* [3]. *If $S \subset M$ is an m-dimensional submanifold, then the restriction of ψ_2 to $T(S) \subset VK^r_m(M)$ will be called the value of ψ on S. Moreover, taking into account* (18), *we also introduce the auxiliary form of the value of ψ on S as the mapping $\bar{\psi}_S : T(Q(S)) \to W$ defined by $\bar{\psi}_S(v) = \psi(u^{-1}(\pi_*(v)), u^{-1}(k^r_{\pi(u)}S))$, $u = v(v)$.*

Let $y^J_p, \ldots, y^J_{p_1 \ldots p_r}, y^p$ be the local coordinates on $VK^r_{m,c}(M)$ determined by $\varkappa$, let z^a be some local coordinates on W and let

$$z^a = f^a(y^J_p, \ldots, y^J_{p_1, \ldots, p_r}, y^p), \tag{19}$$

be the coordinate expression of ψ. (Since we have the equations of the fundamental distribution on $H \times VK^r_{m,c}(M)$, the method of G. F. Laptěv, see [10], p. 301 and Appendix to [6], can be used for local analytic constructions of such equivariant mappings.)

Proposition 2. *Let S be an m-dimesional submanifold of M, let $a^J_p, \ldots, a^J_{p_1 \ldots p_r} : : \hat{Q}(S) \to \mathbf{R}$ be the coordinate functions of the fundamental field of order r of S, let $\tilde{\omega}^p$ be the restriction of ω^p to $T(\hat{Q}(S))$ and let* (19) *be the coordinate expression of an infinitesimal geometric m^r-object ψ on M. Then the coordinate expression of the auxiliary form $\bar{\psi}_S$ of the value of ψ on S is*

$$z^a = f^a(a^J_p, \ldots, a^J_{p_1, \ldots, p_r}, \tilde{\omega}^p). \tag{20}$$

Proof. This follows directly from (18) and from our results of [6].

Remark 3. Considering the space $\bar{K}^r_m(M)$ of all semi-holonomic contact m^r-elements on M, [7], we introduce $V\bar{K}^r_m(M)$ quite similarly to (2) and we define a semi-holonomic infinitesimal geometric m^r-object on M as an equivariant mapping ψ of H-space $V\bar{K}^r_{m,c}(M)$ into another H-space W. The values of ψ can be considered on m-dimensional manifolds with connection of type M, [7], or on non-holonomic m-dimensional distributions on M, [9].

Remark 4. As mentioned in the introduction, in the „classical" differential geometry of submanifolds one meets infinitesimal invariants and relative invariants. he property "to be an invar iant" or "to be a relative invariant" is the following general property of equivariant mappings. Let G be a Lie group and let f be an equivariant mapping of a G-space F into $\mathbf{R}$. If G acts on $\mathbf{R}$ by the identity representation, then f is said to be an (absolute) invariant. If G acts on $\mathbf{R}$ by a homomorphism into the group $\mathbf{R} \setminus \{0\}$ of all homothetic transformations of $\mathbf{R}$, then f will be called a relative invariant. Finally, if G acts on $\mathbf{R}$ by a homomorphism into the subgroup $\mathbf{R}^+ \subset \mathbf{R} \setminus \{0\}$, then f will be said to be an oriented relative invariant. Let y^A be some local coordinates on F, let

$$\mathrm{d}y^A + \eta^A_\alpha(y^B)\,\omega^\alpha = 0 \tag{21}$$

be the equations of the fundamental distribution on $G \times F$ and let $f(y^A)$ be the coordinate formula for $f : F \to \mathbf{R}$. Taking account of Appendix of [6], we shall denote by $\mathrm{d}f$ the expression

$$\mathrm{d}f = -(\partial f/\partial y^A)\,\eta^A_\alpha(y^B)\,\omega^\alpha. \tag{22}$$

One sees directly that f is an invariant if and only if

$$\mathrm{d}f = 0, \tag{23}$$

while f is a relative invariant or an oriented relative invariant if and only if

$$\mathrm{d}f + f\omega = 0, \tag{24}$$

where ω is a linear combination of ω^α with constant coefficients. Combining this general remark with the results of § 2 and with Proposition 2, one can treat the infinitesimal invariants and relative invariants of submanifolds.

Remark 5. A practical inconvenience by the standard use of the so-called Cartan's methods consists in the fact that one cannot decide by (24) whether the relative invariant f is oriented or not. An essential difference between both cases is that an oriented relative invariant f determines three invariant subspaces $f^{-1}(\mathbf{R}^+)$, $f^{-1}(\mathbf{R}^-)$, $f^{-1}(0)$ of F, while in the non-oriented case only two invariant subspaces $f^{-1}(0)$ and $f^{-1}(\mathbf{R} \setminus \{0\})$ are determined by f.

Example 1. To illustrate Definition 1 and Proposition 2, we shall discuss a classical example. To simplify our evaluations, we shall apply a specialization of frames, see also Remark 6. Consider a surface S of the 3-dimensional affine space A_3 and fix an affine coordinate system on A_3. Let H be the stability group of the point $c = (0, 0, 0)$ and let

$$\omega^i, \omega^i_j, \qquad i, j, \ldots = 1, 2, 3,$$

be the natural basis of the Maurer–Cartan forms of the fundamental group of A_3. Let k^1S be the fundamental field of the first order of S, let c_1 be the subspace $x^3 = 0$ of $T_c(A_3)$ considered as an element of $K^1_{2,c}(A_3)$ and let $Q_1(S)$ be the reduction of

$Q(S)$ determined by the pair (k^1S, c_1), [8]. In particular, the differential equations of the structure group $H_1 \subset H$ of $Q_1(S)$ are

(25) $$\omega^i, = 0, \omega_p^3 = 0, \qquad p, q, \ldots = 1, 2.$$

Let $\tilde{\omega}^i$, $\tilde{\omega}_j^i$ be the restrictions of ω^i, ω_j^i to $Q_1(S)$; then it holds

(26) $$\tilde{\omega}^3 = 0.$$

Introduce $\rho : VK^2_{2,c}(A_3) \to K^1_{2,c}(A_3)$, $(v, \xi) \to \rho_2(\xi)$, and set $N = \rho^{-1}(c_1)$. On $VK^2_{2,c}(A_3)$ we have the coordinates $y_p^3 = y_p$, $y_{pq}^3 = y_{pq}$, y^p and N is characterized by $y_p = 0$. Applying the previous method, we find the equations of the fundamental distribution on $H_1 \times N$ in the form

(27) $$\begin{aligned} &\mathrm{d}y_{11} - y_{11}(2\pi_1^1 - \pi_3^3) - 2y_{12}\pi_1^2 = 0. \\ &\mathrm{d}y_{12} - y_{12}(\pi_1^1 + \pi_2^2 - \pi_3^3) - y_{11}\pi_2^1 - y_{22}\pi_1^2 = 0, \\ &\mathrm{d}y_{22} - y_{22}(2\pi_2^2 - \pi_3^3) - 2y_{12}\pi_2^1 = 0, \end{aligned}$$

(28) $$\begin{aligned} &\mathrm{d}y^1 + y^1\pi_1^1 + y^2\pi_2^1 = 0, \\ &\mathrm{d}y^2 + y^1\pi_1^2 + y^2\pi_2^2 = 0, \end{aligned}$$

where the π's are the restrictions of the corresponding ω's to H_1. (In practice, one can deduce (28) by the following evaluation, in which the "classical" notation mentioned in Remark 2 is used as a kind of shorthand. On $Q_1(S)$, the structure equations of A_3 give

$$\begin{aligned} \mathrm{d}\tilde{\omega}^1 &= \tilde{\omega}^1 \wedge \tilde{\omega}_1^1 + \tilde{\omega}^2 \wedge \tilde{\omega}_2^1, \\ \mathrm{d}\tilde{\omega}^2 &= \tilde{\omega}^1 \wedge \tilde{\omega}_1^2 + \tilde{\omega}^2 \wedge \tilde{\omega}_2^2. \end{aligned}$$

Analogously to (7) and (11) one obtains,

(29) $$\begin{aligned} \delta(\omega^1(\mathrm{d})) &= -\tilde{\omega}^1(\mathrm{d})\,\pi_1^1 - \tilde{\omega}^2(\mathrm{d})\,\pi_2^1, \\ \delta(\omega^2(\mathrm{d})) &= -\tilde{\omega}^1(\mathrm{d})\,\pi_1^2 - \tilde{\omega}^2(\mathrm{d})\,\pi_2^2. \end{aligned}$$

By Lemma 2 of [5], (29) implies (28).) Using (27) and (28), we deduce that the mapping

$$f = y_{11}(y^1)^2 + 2y_{12}y^1y^2 + y_{22}(y^2)^2$$

satisfies $\mathrm{d}f + f\pi_3^3 = 0$. Hence f is a relative invariant. Further, let $a_{pq} : Q_1(S) \to \mathbf{R}$ be the coordinate functions of the fundamental field of the second order of S. By Proposition 2, the restriction to $Q_1(S)$ of the auxiliary form of the value of f on S is

$$a_{11}(\tilde{\omega}^1)^2 + 2a_{12}\tilde{\omega}^1\tilde{\omega}^2 + a_{22}(\tilde{\omega}^2)^2,$$

which is the well known asymptotic form of S.

Remark 6. In Example 1, we have constructed an equivariant mapping f of the H_1-space N, though Definition 1 requires an equivariant mapping of the H-space

$VK^2_{2,c}(A_3)$. But f can be naturally extended to such a mapping as follows. In general, consider a homogeneous space A with fundamental group H and denote by H_1 the stability group of a point $p \in A$. Let A_1 be another H-space and let $\eta : A_1 \to A$ be an equivariant surjection. Set $A_0 = \eta^{-1}(p)$, which is an H_1-space. Consider another H_1-space $\bar{A}_0$ and an H_1-equivariant mapping $\varphi : A_0 \to \bar{A}_0$. Since H has a natural structure of a principal fibre bundle $H(A, H_1)$ over A with structure group H_1, we can construct an associated fibre bundle $\bar{A}_1 = \bar{A}_1(A, \bar{A}_0, H_1, H)$. Every element of $\bar{A}_1$ being an equivalence class $\{(h, y)\}$ with respect to the equivalence relation $(h, y) \sim (hh_1^{-1}, h_1 y)$, $h_1 \in H_1$, we first introduce a left action of H on A_1 by $\bar{h}\{(h, y)\} = \{(\bar{h}h, y)\}$, $\bar{h} \in H$. This definition is correct, since $\bar{h}\{(hh_1, h_1^{-1}y)\} = \{(\bar{h}hh_1, h_1^{-1}y)\} = \{(\bar{h}h, y)\}$. Then we define an H-equivariant mapping $\tilde{\varphi} : A_1 \to \bar{A}_1$ by $\tilde{\varphi}(hy) = h\varphi(y)$, $h \in H$, $y \in A_0$. Even this is a correct definition, since $hy = \bar{h}\bar{y}$, $y, \bar{y} \in A_0$, implies $h^{-1}\bar{h} \in H_1$, so that $\tilde{\varphi}(\bar{h}\bar{y}) = \bar{h}\varphi(\bar{y}) = hh^{-1}\bar{h}\varphi(\bar{y}) = h\varphi(y)$. The H-equivariant mapping $\tilde{\varphi} : A_1 \to \bar{A}_1$ is the above-mentioned natural extension of an H_1-equivariant mapping $\varphi : A_0 \to \bar{A}_0$.

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