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LATTICE-ORDERED GROUPS WITH MINIMAL PRIME SUBGROUPS SATISFYING A CERTAIN CONDITION

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In this paper one problem of P. Conrad's book [2] is partially solved in connection with one problem of F. Šik. There is proved (Theorem 1) that the set of all cardinal summands of an l-group G is equal to the set of all polars of this group if and only if G is projectable and satisfies a certain property. Further a connection between minimal prime subgroups and cardinal summands and also a connection between minimal prime subgroups and polars is shown here.

Let $G = [G, +, \vee]$ be an l-group. For $x \in G$ we shall denote $|x| = x \vee -x$. If $|a| \wedge |b| = 0$, then elements $a, b \in G$ will be called *disjoint*. If $\emptyset \neq A \subseteq G$, then we denote $A' = \{x \in G: |x| \wedge |a| = 0$ for each $a \in A\}$. Now $A \subseteq G$ is called *polar* if $A'' = A$. (A'' denotes $(A')'$.) Instead of $\{a\}'$, $\{a\}''$ we write a', a'', respectively. It is known that any polar is a convex l-subgroup of G. The set of all polars of G will be denoted by $\Gamma = \Gamma(G)$. If $B \in \Gamma$, then B, B' are called complementary polars.

The following theorem has been proved by F. Šik in [3] (Teorema 1):

Theorem A. (1) *Polars of an l-group form a complete Boolean algebra Γ (ordered by inclusion, an infimum is formed by an intersection).*

(2) *Polars that are l-ideals form a closed subalgebra Γ_1 of Γ.*

(3) *Cardinal summands of G form a subalgebra Γ_2 of Γ_1 (not always complete), where a supremum is formed by a sum of summands.*

It holds that for $B \in \Gamma_2(G)$ it is $G = B \oplus B'$. An l-group G is called an r-group if it is isomorphic to a subdirect product of totally ordered groups. By [4], an l-group is an r-group if and only if each its polar is an l-ideal. A convex l-subgroup P is called *prime* if the following is satisfied:

(i) *If $x \notin P$, then $x' \subseteq P$.*

(i) and the following conditions are equivalent:

(ii) *P contains at least one of polars a'', a' ($a \in G$).*

(iii) *P contains at least one of complementary polars.*

Any prime subgroup contains at least one minimal prime subgroup. In $G \neq \{0\}$, minimal prime subgroups are characterized among convex l-subgroups as: $a \notin P$ iff $a' \subseteq P$. A convex l-subgroup Z is a *z-subgroup* if from $x \in Z$ and $y' = x'$ it follows $y \in Z$. It is known that every polar and every minimal prime subgroup is a z-subgroup. An l-group is called *projectable* if $G = g' \oplus g''$ for each $g \in G$. Clearly any projectable l-group is an r-group.

The following theorem is proved in [1] (Théorème 3.1):

Theorem B. *An l-group G is projectable if and only if any proper prime subgroup contains exactly one prime z-subgroup.*

The problem how to characterize those l-groups for which $\Gamma_2(G) = \Gamma(G)$ has been given by P. Conrad in the book [2, p. 2.8]. Clearly any such l-group will be projectable.

Note. This problem has been solved by F. Šik in [3, p. 8] yet. He has proved that for an l-group the following are equivalent:

(1) *An arbitrary polar is a direct summand.*

(2) *A sum of two arbitrary polars is also a polar.*

(3) *A sum of an arbitrary pair of complementary polars is also a polar.*

(4) *Any pair of complementary polars forms a direct decomposition of this l-group.*

Another characterization is given in [4, Satz 13].

Further denote the following condition:

($*$) *For each minimal prime subgroup A of an l-group G and for each polar K of G it is satisfied: $K \subseteq A$ iff $K' \nsubseteq A$.*

F. Šik has proposed (in a letter) the problem how to characterize l-groups with the property ($*$).

The following theorem shows a certain connection between both problems.

Theorem 1. *For an l-group $G \neq \{0\}$ it holds $\Gamma(G) = \Gamma_2(G)$ if and only if G is projectable and possesses the property* ($*$).

Proof. a) Let $\Gamma(G) = \Gamma_2(G)$ and let A be a minimal prime subgroup of G. Let $K \in \Gamma(G)$, $K, K' \subseteq A$. Since $K \oplus K' = G$, $A = A + A \supseteq K + K' = G$. If $G \neq \{0\}$, then by [5, Folgerung 7.3] $A \neq G$, a contradiction. But since A is a prime subgroup, it contains K or K'. Thus G satisfies ($*$).

b) Let G be projectable and have the property ($*$). Let $K \in \Gamma(G)$ such that $K \oplus K' \neq G$. Let P be a proper prime subgroup of G such that $K \oplus K' \subseteq P$. Let us remind yet that the filet of an element $x \in G$ is $\bar{x} = \{y \in G: y' = x'\}$ and the set of all filets $\mathscr{F}(G)$ form a distributive lattice. Denote thus $\Phi = \{\bar{x} : x \notin P\}$. Evidently Φ is a filter of $\mathscr{F}(G)$. For each $y \in K \cup K'$ it holds $\bar{y} \notin \Phi$. (If, namely, $y \in K$, $\bar{y} \in \Phi$, then $y'' = a''$ for some $a \notin P$ thus $y'' \nsubseteq P$; but $y'' \subseteq K$, and we have a contradiction. Similarly for $z \in K'$.)

Now if $x \in K \cup K'$, then denote a maximal filter of $\mathscr{F}(G)$ that contains Φ and does not $\bar{x}$ by Φ^x. It holds Φ^x is a prime filter. Therefore $Z^x = \{u \in G: \bar{u} \notin \Phi^x\}$ is a prime z-subgroup of G and clearly $Z^x \subseteq P$. Since G is projectable, all prime z-subgroups contained in $P \neq G$ are (by Theorem B) identical, thus for each $x_1, x_2 \in K \cup K'$ $Z^{x_1} = Z^{x_2}$. Further $\Phi^{x_1} = \Phi^{x_2}$ iff $\{u : \bar{u} \notin \Phi^{x_1}\} = \{v : \bar{v} \notin \Phi^{x_2}\}$ and this holds iff $Z^{x_1} = Z^{x_2}$. Thus for each $x_1, x_2 \in K \cup K'$ $\Phi^{x_1} = \Phi^{x_2}$ and therefore $\Psi = \bigcap_{x \in K \cup K'} \Phi^x = \Phi^x$ for each $x \in K \cup K'$. Hence Ψ is a prime filter of $\mathscr{F}(G)$ and $Z = \{w : \bar{w} \notin \Psi\}$ is a prime z-subgroup of G such that $Z \subseteq P$. Consequently, by [1, Proposition 3.1 and its proof], $Z = \bigcup_{a \notin P} a'$.

For each $x \in K \cup K'$ $x \in Z$, therefore $K \subseteq Z$, $K' \subseteq Z$ and this contradicts the assumption that G satisfies ($*$).

Now, it is easy to prove the further

Theorem 2. *For a projectable l-group G the following conditions are equivalent:*

(1) *Any polar of G is a cardinal summand of G. (Thus $\Gamma(G) = \Gamma_2(G)$.)*

(2) *G satisfies the property* ($*$).

(3) *The algebra $\Gamma_2(G)$ is a $\vee$-closed subalgebra of $\Gamma(G)$.*

(4) *The algebra $\Gamma_2(G)$ is a $\wedge$-closed subalgebra of $\Gamma(G)$.*

Proof. (3) $\Rightarrow$ (1): Let $K \in \Gamma(G)$. It holds $K = \bigvee_{\substack{\Gamma \\ a \in K}} a''$ and $a'' \in \Gamma_2(G)$ implies by (3), $K \in \Gamma_2(G)$.

(4) $\Rightarrow$ (1) : If $K \in \Gamma(G)$, then $K = \bigwedge_{\substack{\Gamma \\ b \in K'}} b'$. We have $b' \in \Gamma_2(G)$, thus by (4), $K \in \Gamma_2(G)$.

If H is a prime subgroup of an l-group G, then we say H has the property (**) if it holds:

(**) *If* $K \in \Gamma(G)$ *then* $K \subseteq H$ *iff* $K' \nsubseteq H$.

Further we say $\varnothing \neq A \subseteq G$ is *dense* in G if $A' = \{0\}$.

We get

Theorem 3. *A prime subgroup H of an l-group G is either a polar in G or it is dense in G.*

Proof. Let H not be dense. Then $\{0\} \neq H' \nsubseteq H$. Therefore $H'' \subseteq H$ i.e. H is a polar.

The following theorem is a consequence of Theorems 3 and 1.

Theorem 4. *If a projectable l-group G satisfies* (*) *then each minimal prime subgroup of G is a cardinal summand or it is dense in G.*

Denote now the set of all z-subgroups of an l-group G by $\mathscr{Z}(G)$. It is known (see [1, Proposition 2.3)] $\mathscr{Z}(G)$ forms a complete distributive lattice. It holds $\Gamma(G) \subseteq \mathscr{Z}(G)$ but generally $\Gamma(G)$ need not be a sublattice of $\mathscr{Z}(G)$.

We get

Theorem 5. *Let G be an l-group and $\Gamma(G)$ a closed sublattice of $\mathscr{Z}(G)$. Then a proper prime subgroup H of G has the property* (**) *if and only if H is a polar.*

Proof. If $H \in \mathscr{Z}(G)$, then (by [1, Proposition 2.1)] $H = \bigcup_{a \in H} a'' = \bigvee_{\substack{\mathscr{Z} \\ a \in H}} a''$. By the assumption $\bigvee_{\substack{\Gamma \\ a \in H}} a'' = \bigvee_{\substack{\mathscr{Z} \\ a \in H}} a''$ thus H is a polar. The converse is evident.

Therefore it holds also

Theorem 6. a) *Let an l-group G satisfy* (*) *and let $\Gamma(G)$ be a closed sublattice of $\mathscr{Z}(G)$. Then each minimal prime subgroup of G is a polar in G.*

b) *Let, in addition, G be projectable. Then each minimal prime subgroup is a cardinal summand of G.*

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