ON POLITICAL REALIZATION OF A GIVEN LUXURY GOODS SUPPLY

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To Professor Otakar Borůvka at his Seventieth Birthday

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A certain version of Brouwer fixed point theorem is derived and by means of which one theorem from mathematical politology is presented.

Let S be a d-simplex in E^d.[1]) A map f: bd $S \to$ bd S is said to have an α-property if it holds (for each $L \in \mathscr{F}(S) \dot{-} \{S\}$)

$$(\alpha) \qquad f(L) \cap (-L) = \emptyset.$$

We say f has the property (α) in $Z \in \mathscr{K}(S)$ if (α) holds for all $L \subset Z$. Evidently f has the α-property in $Z_1 \cup Z_2$ if the same holds on Z_1 and Z_2. f_1, f_2 are said to be α-homotopic, if they are homotopic and all f_t, $0 \leqq t \leqq 1$ of the homotopy considered have the α-property. f is called α-deformation if it is α-homotopic with the identity. A map f: $S \to S$ is called to have α-property if $f(\text{bd } S) \subset \text{bd } S$ and $f \mid \text{bd } S$ has α-property.

Lemma: *Let $L \in \mathscr{F}(S) \dot{-} \mathscr{F}_0(S)$, $f : \text{bd } S \to \text{bd } S$ have α-property and $f \mid \text{rlbd } L$ be the identity. Then f is α-homotopic to $g : \text{bd } S \to \text{bd } S$ with $g \mid \mathscr{C}(L) = f \mid \mathscr{C}(L)$ and $g \mid L$ being the identity.*

Proof: For $k = 0, 1, 2, \ldots$ put $Z_k = : (\bigcup K) \cup \mathscr{C}(L)$, where the sum operates on all $K \in \mathscr{F}(S) \dot{-} \{S\}$, $\dim K \leqq \dim L + k$. Evidently $Z_0 = \mathscr{C}(L) \cup L \subset Z_1 \subset \ldots \subset Z_{\bar{k}} = \text{bd } S$ (for some $\bar{k}$). It is $f \mid L : L \to \text{bd } S \dot{-} (-L)$ (because of (α)) and hence in a simple way this α-homotopy f_t can be constructed as $f_t : Z_0 \to \text{bd } S$, $0 \leqq t \leqq 1$ with $f_0 \mid Z_0 = f \mid Z_0, f_t \mid \mathscr{C}(Z) = f \mid \mathscr{C}(L)$ and $f_1 \mid L$ being the identity. Let $Z_0 \neq \text{bd } S$ and choose some $V \in \mathscr{F}(S)$, $\dim V = \dim L + 1$. Put $Z = : \text{rlbd } V$, $T = : Z \cup V$, $U = : \text{bd } S \dot{-} (-V)$, $g_t = f_t \mid Z$ and using the extension theorem construct a homotopy $f_t^* : V \to \text{bd } S \dot{-} (-V)$, $f_t^* \mid Z = g_t$. Define $f_t : Z_0 \cup V \to \text{bd } S$ (on Z_0 by f_t, on V by f_t^*). f_t has on $Z_0 \cup V$ the α-property (because the same holds on Z_0 and V). Step by step in this way we extend f_t first on the whole Z_1, then $Z_2, \ldots, Z_k = \text{bd } S$ and dut $g = f_1$; Q.E.D.

Theorem 1: *A map $f : \text{bd } S \to \text{bd } S$ with the α-property is an α-deformation.*

Proof: Order the set $\mathscr{F}(S) \dot{-} \{S\}$ into a sequence $\{L_i\}_{i=1}^{\bar{d}}$ in such a way that first in the row are all vertices, then edges, then triangles etc. and put $M_i =: \bigcup_{j \leq i} L_j$. In a simple way one constructs an α-homotopy $f_t : \text{bd } S \to \text{bd } S$ with $f_0 = f$ and $f_1 \mid L_1$ being the identity. For $i > 1$, f_{i-1} being α-homotopic to f and $f_{i-1} \mid M_{i-1}$ being the identity construct (according to our Lemma) an α-homotopy $f_t : \text{bd } S \to \text{bd } S$, $i - 1 \leqq \leqq t \leqq i$ with $f_i \mid M_i$ being the identity. Evidently $f_{\bar{d}}$ is the identity, Q.E.D.

Theorem 2: *For a map $f : S \to S$ having the α-property it holds $f(S) = S$.*

Proof: Because of $f(\text{bd } S) = \text{bd } S$ it suffices to consider this case: $x \in \text{int } S$ exists with $x \notin f(S)$. Map linearly the interval $[0, 1]$ on each edge $[v, x]$, $v \in \text{vert } S$ (the corresponding point to t denote by ${}^t v$), ${}^0 v = v$, ${}^1 v = x$, and choose $t_0 \in (0, 1)$ such that $f(\text{bd conv } \{{}^t v\}_{v \in \text{vert} S})$ is sufficiently close to $f(x)$. Project from x on bd S the map $f \mid \text{bd conv } \{{}^t v\}_{v \in \text{vert} S}$ (the projected map denote by f_t). Evidently f_t is bd $S \to$ bd S and f_t, $0 \leqq t \leqq t_0$ is a homotopy with $f_0 = f \mid \text{bd } S$ and $f_{t_0}(\text{bd S}) \neq \text{bd } S$. Hence f_{t_0} is inessential, i.e. $f \mid \text{bd } S$ is inessential—a contradiction to the theorem 1; Q.E.D.

Let n kind of goods be given, n production branches, in each branch (say i) only the good i be produced and for the production of one unit of good i a_{ij} units of good j be destroyed. Put $A =: (a_{ij})$ and let the set $N =: \{1, 2, ..., n\}$ of goods be divided in two nonvoid sets I, II (called production means and consumer goods). Let, for each $i \in N$, it hold $A^i_{\text{I}} \geqslant {}^T o$, $A^i_{\text{I}} \ngeqslant {}^T o$. Denote by $P = \{p \in \mathsf{E}^n \mid p \geqq o,\ {}^T e p = 1\}$ the set (called price simplex) of all so called price vectors p. Denote by $S = = \{s \in \mathsf{E}^n \mid s \geqq o,\ {}^T e s = 1\}$ the set (called power supply simplex) of all so called intensity production vectors s. One says a branch i to be profitable for a given $p \in P$ if $(E - A)^i\, p > 0$ (denote by $\pi(p)$ the set of all profitable i's). Let at least one price vector (say $\bar{p}$) exist with $\pi(\bar{p}) = N$. Evidently $\pi(p)$ is nonvoid for all $p \in P$. One says $p \in P$ (or $s \in S$) is degenerous if it is not $p > o$ ($s > o$). One calls a map $s(p) : P \to S$ a psychology if it holds $s(p)^{\pi(p)} \geqslant o$ for all $p \in P$ and $s(p$, is degenerous if the same holds for p. The pair $(A, s(p))$ with above considered properties is said to be a simple commodity production society (see [3]). Put $Z = \{z \in \mathsf{E}^n \mid {}^T z = {}^T s A,\ s \in S\}$ and such z call a suitable stock. Put $C = \{x \in \mathsf{E}^n \mid x \geqq o,\ {}^T x = {}^T s(E - A),\ s \in S\}$ and call such x a luxury goods supply (the corresponding s's are said to be reproductive). One says $x \in C$ to be economically realizable according to $z \in Z$ if a reproductive $s \in S$ exists with ${}^T z = {}^T s A$ and ${}^T x = {}^T s(E - A)$. Evidently each $x \in C$ is economically realizable according to some $z \in Z$. One says $x \in C$ to be politically realizable according to $z \in Z$ if it is economically realizable according to z and the mentioned s be such that $s = s(p)$ for some $p \in P$.

Theorem 3: *In each simple commodity production society* $(A, s(p))$ *to each* $y \in \mathsf{E}^n$, $y \geqslant o$ *such a number* $\lambda > 0$ *and a suitable stock* z *exist that* λy *is politically realizable (according to* z*) luxury goods supply.*

Proof: Because $\operatorname{conv}({}^T(E - A)^i)_{i \in N}$ is the $(n - 1)$-simplex containing the $(n - 1)$-simplex $\{x \in \mathsf{E}^n \mid x \geqq o\} \bigcap \operatorname{aff}({}^T(E - A))_{i \in N}$ (because of $A^i_{\mathrm{I}} \geqslant {}^To$, $A^i_{\mathrm{II}} \geqslant {}^To$ and the existence of $\bar{p}$), it exists to each $y \geqslant o$ such a number $\lambda > 0$ that λy is economically realizable according to ${}^Ty(E - A)^{-1}A\lambda$. It suffices now to prove $s(P) = S$, but it follows from the theorem 2 because $s(p) : P \to S$ has the α-property if we identify the points from P with those from S having the same coordinates, Q. E. D. Many applications of homotopies in the economy are given in [2].

REFERENCES

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[2] Nikaido, H., Convex structures and economic theory. Academic Press, New York 1968.

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[1]) A Euclidean d-dimensional space denote by E^d. Each point $x \in \mathsf{E}^d$ is considered as to be a column of d reals x^i's, o means the column of zeros, e that with 1's. For $X \subset \mathsf{E}^d$ denote by aff X the smallest space containing X and by dim X the dimension of aff X. For a finite $X \subset \mathsf{E}^d$ the X's convex hull denote by conv X. Denote by ${}^TA$ (or A^{-1}) a transpose (or inverse) to a matrix A, A^V (or A_U), the A's submatrix consisting from the rows (or columns) indexed by elements from V (or U). $A \geqslant B$ means $A \geqq B$ (i.e. $a_{ij} \geqq b_{ij}$) but not $A = B$. AB means the row-by-column matrix multiplication, E the unit matrix. For a d-simplex S (i.e. $\dim S = d$) denote by $\mathscr{F}_k(S)$ the set of all S's k-faces, $\operatorname{vert} S =: \mathscr{F}_0(S)$. $\mathscr{F}(S) =: \bigcup_{k=0}^{d} \mathscr{F}_k(S)$, $\mathscr{K}(S) = \{Z \mid Z = \bigcup_{T \in \mathscr{F}} T,\ \mathscr{F} \subset \mathscr{F}(S)\}$, $\mathscr{C}(L) =: \bigcup_{T \in \mathscr{F}} T$ (where $\mathscr{F} = \{T \in \mathscr{F}(S) \mid T \neq L, L \notin \mathscr{F}(T)\}$) for $L \in \mathscr{F}(S)$ and $-L =: \operatorname{conv}\{\operatorname{vert} S \dot{-} \operatorname{vert} L\}$. The boundary of S denote by bd S, S's interior by int S, the relative boundary of $L \in \mathscr{F}(S)$ by rlbd L. Put $\bar{d} =: \sum_{k=1}^{d} \binom{d+1}{k}$. A continuous transformation $f : X \to Y$ be called a map ($f \mid Z$ is f but on $Z \subset X$ only), f_t, $0 \leqq t \leqq 1$ denote a homotopy, f_0, f_1 are called homotopic. A map $f : \operatorname{bd} S \to \operatorname{bd} S$ is called a deformation if it is homotopic to the identity. A map f is called inessential if it is homotopic to a constant map. Recall, that a deformation is never inessential (see [1], pp. 25—26), that a map $f : \operatorname{bd} S \to \operatorname{bd} S$ with $f(\operatorname{bd} S) \neq \operatorname{bd} S$ is inessential (by a suitable homotopy we contract $f(\operatorname{bd} S)$ into a point), and this extension theorem: if $Z, T \in \mathscr{K}(S)$, $Z \subset T$, $V \in \mathscr{F}(S)$, $U =: \operatorname{bd} S \dot{-} V$, $f_0 : T \to U$, $g_t : Z \to U$, $0 \leqq t \leqq 1$, $g_0 = f_0 \mid Z$, then f_0 admits a homotopy $f_t : T \to U$, $0 \leqq t \leqq 1$ with $f_t \mid Z = g_t$ (see [1], p. 20).