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The asymptotics for the number of eigenvalue branches for the magnetic Schrödinger operator $H - \lambda W$ in a gap of H

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1. Introduction

1.1. Let $H = H(A, V) = (i\nabla + A(x))^*(i\nabla + A(x)) + V(x)$ be a Schrödinger operator in $\mathbb{R}^n$, with a real bounded potential V and a magnetic potential A, such that the spectrum $\sigma(H)$ has gaps (for instance, this is the case for many uniform A and periodic V), let W be a potential decaying at infinity, and consider eigenvalue branches of families of Schrödinger operators $H \mp \lambda W$, crossing an energy level E in a gap of $\sigma(H)$.

Problems involving eigenvalues in a spectral gap of a Schrödinger operator arise e.g. in the investigation of impurity levels in insulators and semiconductors (see the bibliography in [DH], [H1], [GHKSV]).

There are a number of papers devoted to the study of counting functions

$$N_{\pm}(\lambda, H - E, W) = \sum_{0<\mu<\lambda} \dim \operatorname{Ker}(H - E \mp \mu W) ,$$

as $\lambda \to +\infty$ (for $A = 0$ see [DH], [GHKSV], [H1], [H2], [ADH], [B1], [B2], for $A \neq 0$ – [BR], [R]; see also the bibliography in [MR]), but even some cases of polynomially decreasing W remain to be investigated.

For instance, the asymptotics of $N_-(\lambda, H-E, W)$ was studied only if $A = 0$ or $V = 0$ ([ADH] and [R] respectively), and the asymptotics of $N_+(\lambda, H-E, W)$ for slowly decreasing W was computed only in the case $V = 0$ [R].

1.2. In the paper, we suppose that A is uniform, i.e. the magnetic tensor

$$B = [b_{jk}]_{j,k=1}^n, \qquad b_{jk} = \partial_j A_k - \partial_k A_j ,$$

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is constant, $V \in L_\infty(\mathbb{R}^n)$, and $\rho(\cdot, H)$, the integrated density of states for H, exists:

$$\rho(\lambda, H) = \lim_{\operatorname{vol} Q_R \to +\infty} \frac{1}{\operatorname{vol} Q_R} N(\lambda, H(Q_R)), \tag{1.1}$$

where Q_R are expanding cubes, and $H(Q_R)$ is the operator of the Dirichlet problem for H in Q_R.

We suppose that a non-negative $W \in L_\infty$ decays at infinity as $|x|^{-m}$ with some $m > 0$:

$$|W(x)| \leqq C(1 + |x|)^{-m}, \tag{1.2}$$

and stabilizes to a positively homogeneous continuous function $\tilde{W} : \mathbb{R}^n \backslash 0 \to \mathbb{R}$ of order $-m$;

$$(W(tx) - \tilde{W}(tx))t^m \to 0 \quad \text{as } t \to +\infty, \tag{1.3}$$

uniformly in $x \in S_{n-1}$.

1.3 Main theorems

Theorem 1.1. *Let* (1.2) *and* (1.3) *hold with* $m \in (0, 2)$, *and let a limit* (1.1) *exist for all* λ.

Then

$$\lim_{\lambda \to +\infty} N_+(\lambda, H - E, W)\lambda^{-n/m} = c^+(\tilde{W}), \tag{1.4}$$

where

$$c^+(\tilde{W}) = \int_{\mathbb{R}^n} dx \int_{E < t < E + \tilde{W}(x)} d\rho(t, H). \tag{1.5}$$

Theorem 1.2. *Let* (1.2) *and* (1.3) *hold with* $m > 0$, *and let a limit* (1.1) *exist for all* λ. *Then*

$$\lim_{\lambda \to +\infty} N_-(\lambda, H - E, W)\lambda^{-n/m} = c^-(\tilde{W}), \tag{1.6}$$

where

$$c^-(\tilde{W}) = \int_{\mathbb{R}^n} dx \int_{E - \tilde{W}(x) < t < E} d\rho(t, H). \tag{1.7}$$

Theorem 1.3. *Let* (1.2) *and* (1.3) *hold. with* $m = 2$. *Then*

$$\lim_{\lambda \to +\infty} N_+(\lambda, H - E, W)\lambda^{-n/2}(\ln \lambda)^{-1} = c_{\text{int}}(\tilde{W}), \tag{1.8}$$

where

$$c_{\text{int}}(\tilde{W}) = (2\pi)^{-n}\frac{1}{2}|v_n| \int_{S_{n-1}} \tilde{W}(x)^{n/2} dS(x), \tag{1.9}$$

and $|v_n|$ *is the volume of the unit ball.*

Remark 1.1. The integral in (1.5) converges if and only if $m \in (0, 2)$.

1.4. The plan of the paper is as follows. In Sect. 2 we gather some auxiliary results, and in Sects. 3–5 we prove Theorems 1.1–1.3, respectively.

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2. Auxiliary results

2.1. We shall need the following definition and lemmas.

Let $\mathfrak{A}$ be a quadratic form in a Hilbert space $\mathfrak{H}$ with the domain $D(\mathfrak{A})$, and let $V \subset D(\mathfrak{A})$ be a subspace. We assume that $\mathfrak{A}$ is closable, and we set

$$\mathcal{N}(\lambda, \mathfrak{A}, V) = \sup\{\dim L \mid L \subset V,\ \mathfrak{A}[u] < \lambda \|u\|_{\mathfrak{H}}^2\ \forall (0 \neq) u \in L\} .$$

For A an operator in $\mathfrak{H}$ and $\mathfrak{A}[u] = \langle Au, u\rangle_{\mathfrak{H}}, D(\mathfrak{A}) = D(A)$, write $\mathcal{N}(\lambda, A, V)$ instead of $\mathcal{N}(\lambda, \mathfrak{A}, V)$.

We also set $N((a,b);A) = \dim P_{(a,b)}(A)$, where $P_{(a,b)}(A)$ is a spectral projection of A, and we set $N(\lambda;A) = N((-\infty,\lambda),A)$.

The following Lemmas are well-known and widely used; for proofs, see e.g. Appendix in [L] and [RS].

Lemma 2.1. *Let A be a semibounded self-adjoint operator with the domain $D(A)$.*

Then $N(\lambda;A) = \mathcal{N}(\lambda;A;D(A))$.

Lemma 2.2. *$\mathcal{N}(\lambda;\mathfrak{A};V)$ is independent of V provided V is a core of the form $\mathfrak{A}$.*

Lemma 2.3. *If $\mathfrak{A}[u] \leqq \mathfrak{A}_1[u]\ \forall u \in V$, then*

$$\mathcal{N}(\lambda;\mathfrak{A}_1;V) \leqq \mathcal{N}(\lambda;\mathfrak{A};V) .$$

Lemma 2.4. *If $V \subset V_1$, then*

$$\mathcal{N}(\lambda;\mathfrak{A};V) \leqq \mathcal{N}(\lambda;\mathfrak{A};V_1) .$$

Lemma 2.5. *Let H_s be a Hilbert space, $\mathfrak{A}_s$ a quadratic form in $H_s, V_s \subset D(\mathfrak{A}_s)$ a subspace $(s = 0,1,\dots,r)$, let*

$$l : \bigoplus_{1 \leqq s \leqq r} V_s \to V_0$$

be an isomorphism, and

$$\mathfrak{A}_0[l(u_1,\dots,u_r)] = \sum_{s=1}^{r} \mathfrak{A}_s[u_s] .$$

Then

$$\mathcal{N}(\lambda;\mathfrak{A}_0;V_0) = \sum_{s=1}^{r} \mathcal{N}(\lambda;\mathfrak{A}_s;V_s) .$$

2.2. Weighted Sobolev spaces associated with a magnetic Schrödinger operator

Let A be a uniform magnetic field, and $s \in \mathbb{Z}_+$. Set

$$H_A^s(\mathbb{R}^n) = \left\{ u \in L_2(\mathbb{R}^n) \mid \|u\|_{A,s} = \left(\sum_{|\alpha| \leqq s} \|(D + A(x))^\alpha u\|_{L_2}^2 \right)^{1/2} < \infty \right\} . \quad (2.1)$$

This is a Hilbert space, and $C_0^\infty(\mathbb{R}^n)$ is dense in it (see e.g. [S]).

Lemma 2.6. *The following norm defines a topology in* $H_A^s(\mathbb{R}^n)$:

$$\|u\|'_{A,s} = (\langle H(A,0)^s u, u\rangle_{L_2} + \|u\|_{L_2}^2)^{1/2} . \quad (2.2)$$

Proof. For the case $s = 1$, see [S]; the proof in general case is similar. □

Lemma 2.7. *For* $s \in \mathbb{Z}_+$, $H(A,0)^s : \mathbb{C}_0^\infty(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ *is essentially self-adjoint, and we denote its closure by* $H(A,0)^s$.

Proof. For the case $s = 1$, see e.g. [ADH],[S]; the proof in general case is similar. □

2.3. Some bounds for counting functions

Denote by B_l an open ball of the radius l, centered at the origin, and let $\Omega(R,R_1)$ stand for any of sets $Q_R, B_R, \bar{Q}_R, \bar{B}_R, \bar{B}_R \backslash B_{R_1}, Q_R \backslash \bar{Q}_{R_1}$, and $\Omega_0(R,R_1)$ – for any of sets $B_R, Q_R, B_R \backslash \bar{B}_{R_1}, Q_R \backslash \bar{Q}_{R_1}$.

The following lemma (even in a more general form) is well-known – see e.g. [RSS].

Lemma 2.8. *Let* $N \geqq 1$, $n \geqq 1$ *be integers.*

Then there exists $C = C_{n,N}$ *such that for any* $\lambda > 0$ *and any* $R/2 > R_1 > 1$

$$\mathcal{N}(\lambda; (-\Delta)^N; C_0^\infty(\Omega_0(R,R_1))) \leqq C\lambda^{nN/2} \operatorname{meas} \Omega_0(R,R_1) , \quad (2.3)$$

$$\mathcal{N}(\lambda; \| \cdot \|_{0,N}^2; C_0^\infty(\Omega(R,R_1))) \leqq C(1 + \lambda^{nN/2}) \operatorname{meas} \Omega(R,R_1) . \quad (2.4)$$

Lemma 2.9. *Let* $N \geqq 1$, $n \geqq 1$ *be integers, let* A *be a uniform magnetic potential, and let* $V \in L_\infty(\mathbb{R}^n)$.

Then there exists $C = C_{n,N}$ *such that for any* $\lambda > 0$, *any* $R/2 > R_1 > 1$, *and any* $x_0 \in \mathbb{R}^n$,

$$\mathcal{N}(\lambda; H(A,V)^N; C_0^\infty(x_0 + \Omega_0(R,R_1))) \leqq C(1 + \lambda^{nN/2}) \operatorname{meas} \Omega_0(R,R_1) , \quad (2.5)$$

$$\mathcal{N}(\lambda; \|H(A,V)^N \cdot \|^2; C_0^\infty(x_0 + \Omega(R,R_1))) \leqq C(1 + \lambda^{nN/4}) \operatorname{meas} \Omega(R,R_1) . \quad (2.6)$$

Proof. It is easy to see that there exist $C_j = C_j(N, V, R)$, $j = 1, 2$, such that

$$\langle(-\Delta)^N u, u\rangle_{L_2} \leqq C_1\langle H(A,V)^N u, u\rangle_{L_2} + C_2\|u\|^2_{L_2}, \ \forall u \in \mathbf{C}_0^\infty(\Omega_0(R,R_1)),$$

and

$$\|u\|^2_{0,N} \leqq C_1\langle H(A,V)^N u, u\rangle_{L_2} + C_2\|u\|^2_{L_2}, \ \forall u \in \mathbf{C}_0^\infty(\Omega(R,R_1)).$$

By applying Lemma 2.3, we see that (2.5) and (2.6) follow from (2.3) and (2.4), respectively, with the constant depending on x_0, R, R_1, though.

To reduce the proof in general case to the case of a fixed R and $x_0 = 0$, construct a partition of unity

$$\sum_{k\in\mathbb{Z}^n} \chi^2_{r,k} = 1, \quad \operatorname{supp}\chi_{r,k} \subset r\cdot k + (-r^{1/2}/4, r + r^{1/2}/4)^n, \tag{2.7}$$

$$|D^\alpha \chi_{r,k}| \leqq C_\alpha r^{-|\alpha|/2}, \quad \forall\alpha, \tag{2.8}$$

with C_α independent of $r \geqq 1/4$. It follows from (2.8) and (2.2) that

$$|\langle[H(A,V)^N, \chi_{r,k}]\chi_{r,k}u, u\rangle| \leqq C_1 r^{-1/2}(\langle H(A,V)^N u, u\rangle + C_2\|u\|^2_{L_2}), \tag{2.9}$$

with C_1, C_2 independent of N and r. Due to (2.7) and the locality of $H(A,V)$, (2.9) holds for $|\sum_k [H(A,V)^N, \chi_{r,k}]\chi_{r,k}|$ as well, and we conclude that there exist C_1, C_2 such that uniformly in $r \geqq 1$.

$$\langle H(A,V)^N u, u\rangle \geqq (1 - C_1 r^{-1/2})\sum_k \langle H(A,V)^N \chi_{r,k}u, \chi_{r,k}u\rangle - C_2 r^{-1/2}\|u\|^2_{L_2}. \tag{2.10}$$

Now, (2.10) and Lemmas 2.5 and 2.3 give for $r = 2C_2^2$:

$$\begin{aligned}&\mathcal{N}(\lambda; H(A,V)^N; \mathbf{C}_0^\infty(x_0 + \Omega_0(R,R_1)))\\ &\quad\leqq \sum_{k\in I(R,R_1,r)} \mathcal{N}(C_3(1+\lambda); H(A,V)^N; \mathbf{C}_0^\infty(U^+_{r,k})),\end{aligned} \tag{2.11}$$

where $U^+_{r,k} = r\cdot k + (-r^{1/2}/4, r + r^{1/2}/4)^n$, and

$$I(R,R_1,r) = \{k \mid U^+_{r,k} \cap (x_0 + \Omega_0(R,R_1)) \neq \emptyset\};$$

C_3 is independent of k, of course.

Set $V_k(\cdot) = V(rk + \cdot)$ and notice that $H(A,V)$ is unitarily equivalent to $H(A,V_k)$. Hence,

$$\mathcal{N}(C_3(1+\lambda); H(A,V)^N; \mathbf{C}_0^\infty(U^+_{r,k})) = \mathcal{N}(C_3(1+\lambda); H(A,V_k)^N; \mathbf{C}_0^\infty(U^+_{r,0})),$$

and since V_k is uniformly bounded with respect to k, we deduce from (2.3) an estimate

$$\mathcal{N}(C_3(1+\lambda); H(A,V)^N; \mathbf{C}_0^\infty(U^+_{r,k})) \leqq C_4(1+\lambda^{n/2N})\operatorname{meas} U^+_{r,k}, \tag{2.12}$$

with C_4 independent of k. By substituting (2.12) into (2.11), we obtain (2.5).

(2.6) is proved similarly, (2.4) being used. □

3. Proof of Theorem 1.1

3.1. Without loss of generality we may assume that $H \geqq 1$.

Due to (1.2), W is non-negative and relatively compact with respect to $H - E$, therefore the Birman–Schwinger principle implies that

$$N_{\pm}(\lambda; H - E; W) = N(0, \lambda^{-1} \mp W^{1/2}(H - E)^{-1} W^{1/2}) . \tag{3.1}$$

We apply the Glazman lemma 2.1 to (3.1), and then use Lemma 2.5. We fix $\delta > 0$ and C_0, set $R = C_0 \lambda^{1/m}$ and $r = \delta \lambda^{1/m}$, and consider

$$\langle [(H - E)^{-1}, \chi_{r,k}] \chi_{r,k} u, u \rangle = \langle A_k H^{-1/2} u, H^{-1/2} u \rangle , \tag{3.2}$$

where

$$A_k = H^{1/2}(H - E)^{-1}[\chi_{r,k}, H - E](H - E)^{-1} \chi_{r,k} H^{1/2} ,$$

and $\chi_{r,k}$ are the same as in (2.7) and (2.8).

We have

$$\begin{aligned} A_l A_k^* = & H^{1/2}(H - E)^{-1}[\chi_{r,l}, H - E](H - E)^{-1} \chi_{r,l} H \chi_{r,k} (H - E)^{-1} \\ & \times [H - E, \chi_{r,k}](H - E)^{-1} H^{1/2} . \end{aligned}$$

H is local, and there exists $C > 0$ such that $\operatorname{supp} \chi_{r,k} \cap \operatorname{supp} \chi_{r,l} = \emptyset$ provided $|k - l| > C$, therefore (2.9) gives

$$\sum_k \| A_l A_k^* \|^{1/2}_{L_2 \to L_2} \leqq C_1 r^{-1/2} , \tag{3.3}$$

uniformly in l. Using (2.9) and the exponential decay of the kernel of $(H–E)^{-1}$ off the diagonal (see [S]), we obtain

$$\| A_l^* A_k \|^{1/2}_{L_2 \to L_2} \leqq C_2 r^{-1/2} \exp(-\delta \,|\, l - k \,|\, r) ,$$

with some $\delta > 0$, and hence

$$\sum_k \| A_l^* A_k \|^{1/2}_{L_2 \to L_2} \leqq C_3 r^{-1/2} . \tag{3.4}$$

It follows from (3.3), (3.4) and the Cotlar lemma that $\sum A_k$ strongly converges to a bounded A with $\|A\| \leqq C r^{-1/2}$, hence if C_1 is sufficiently large and $\varepsilon = C_1 r^{-1/2}$, then (3.2) yields an estimate

$$\begin{aligned} & \langle (\lambda^{-1} - W^{1/2}(H - E)^{-1} W^{1/2}) u, u \rangle \\ & = \langle (\lambda^{-1} - W^{1/2}((H - E)^{-1} + \varepsilon H^{-1}) W^{1/2}) \sum_k \chi^2_{r,k} u, u \rangle + \varepsilon \langle W^{1/2} H^{-1} W^{1/2} u, u \rangle \\ & \geqq \sum_k \langle (\lambda^{-1} - W^{1/2}((H - E)^{-1} + \varepsilon H^{-1}) W^{1/2}) \chi_{r,k} u, \chi_{r,k} u \rangle , \end{aligned}$$

and Lemmas 2.5 and 2.3 give

$$N_+(\lambda, H-E, W) \leqq \sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}^+)) . \tag{3.5}$$

Using an equality $\chi_{r,k} u = u \ \forall u \in \mathbb{C}_0^\infty(U_{r,k}^-)$, where

$$U_{r,k}^- = \{x \in U_{r,k}^+ \mid \operatorname{dist}(x, \partial U_{r,k}^+) > r^{-1/2}/2\} ,$$

we similarly obtain for

$$u = (u_j) \in \bigoplus_k \mathbb{C}_0^\infty(U_{r,k}^-) \subset \mathbb{C}_0^\infty(\mathbb{R}^n)$$

an estimate

$$\begin{aligned}
&\langle(\lambda^{-1} - W^{1/2}(H-E)^{-1}W^{1/2})u, u\rangle_{L_2} \\
&\quad = \left\langle (\lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2}) \sum_k \chi_{r,k}^2 u, u \right\rangle_{L_2} \\
&\qquad - \varepsilon \langle W^{1/2} H^{-1} W^{1/2} u, u\rangle_{L_2} \\
&\quad \leqq \sum_k \langle(\lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2})\chi_{r,k} u, \chi_{r,k} u\rangle_{L_2} \\
&\quad = \sum_k \langle(\lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2} u_k, u_k\rangle_{L_2} .
\end{aligned}$$

Using Lemmas 2.4, 2.5 and 2.3 we derive from this estimate

$$N_+(\lambda, H-E, W) \geqq \sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}^-)) . \tag{3.6}$$

Similarly to (3.5) we have

$$\begin{aligned}
&\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}^+)) \\
&\quad \leqq \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty((\partial U_{r,k})^+)) \\
&\qquad + \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}))
\end{aligned}$$

– here and below ε_j stands for a function which is $o(1)$ as $r \to +\infty$, and

$$U_{r,k} = r \cdot k + (0, r)^n, (\partial U)_{r,k}^+ = \{x \mid \operatorname{dist}(x, \partial U_{r,k}) < r^{1/2}/2\} ,$$

therefore we can rewrite (3.5) as

$$\begin{aligned}
&N_+(\lambda, H-E, W) \\
&\quad \leqq \sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty((\partial U)_{r,k}^+)) \\
&\qquad + \sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k})) .
\end{aligned} \tag{3.7}$$

Similarly, for given $\varepsilon \to 0$ we can find $\varepsilon_1 \to 0$ such that

$$\begin{aligned}&\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}))\\ &\leqq \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}^-))\\ &\quad + \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty((\partial U)_{r,k}^+)),\end{aligned}$$

and we deduce from (3.6) an estimate

$$\begin{aligned}&N_+(\lambda, H-E, W)\\ &\geqq -\sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty((\partial U)_{R,k}^+))\\ &\quad + \sum_k \mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k})). \quad (3.8)\end{aligned}$$

If $C_0 > 0$ is sufficiently large then

$$\|W^{1/2}((H-E)^{-1} \pm \varepsilon_1 H^{-1})W^{1/2}\| < \lambda^{-1}/2 \text{ on } U_{r,k}^+ \not\subset B_R,$$

therefore we may assume that k in (3.7) and (3.8) satisfies $U_{r,k}^+ \subset B_R$.

Fix $\varepsilon_0 \to 0$ as $\lambda \to +\infty$, and set $I_{00} = \{k \mid |k| \leqq 1\}$,

$$I_{++} = \{k \notin I_{00} \mid U_{r,k} \subset B_R,\ W(x) > \varepsilon_0 |x|^{-m},\ \forall x \in U_{r,k}\},$$

$$I_{--} = \{k \notin I_{00} \cap I_{++} \mid U_{r,k} \subset B_R\}.$$

To treat terms with $k \in I_{++}$, set

$$w_{R,k}^+ = \sup_{U_{R,k}} W(x), \qquad w_{R,k}^- = \inf_{U_{R,k}} W(x),$$

and notice that $w_{r,k}^- > 0$ for $k \in I_{++}$, and due to (1.3),

$$|W(x) - w_{r,k}^\pm| \leqq \nu w_{r,k}^\pm \quad \forall x \in U_{r,k},$$

where $\nu \to 0$ uniformly in k provided $\varepsilon_0 \to 0$ sufficiently slowly. It follows from these observations, (3.7), (3.8) and Lemma 2.3 that for $k \in I_{++}$,

$$\begin{aligned}&\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} + \varepsilon_1 H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}))\\ &\leqq \mathcal{N}(0, (w_{R,k}^+\lambda)^{-1} - (1+\varepsilon_1)(H-E)^{-1} - \varepsilon_2 (H-E)^{-2}, \mathbb{C}_0^\infty(U_{r,k})), \quad (3.9)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{r,k}))\\ &\geqq \mathcal{N}(0, (w_{R,k}^-\lambda)^{-1} - (1-\varepsilon_1)(H-E)^{-1} + \varepsilon_2 (H-E)^{-2}, \mathbb{C}_0^\infty(U_{r,k})). \quad (3.10)\end{aligned}$$

The RHS of (3.9) and (3.10) can be estimated by means of the following lemma.

Lemma 3.1. *Let $\omega \to 0$ as $R \to +\infty$.*

Then there exist $C > 0$ and $\varepsilon \to 0$ as $R \to +\infty$ such that for all $\lambda > 0$

$$\begin{aligned} &N(E+\lambda-\lambda\varepsilon, H(Q_R)) - N(E+\lambda\varepsilon, H(Q_R)) - CR^{n-1/2}(1+\lambda^{n/2}) \\ &\quad \leqq \mathcal{N}(0, 1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}), L_2(Q_R)) \\ &\quad \leqq N(E+\lambda+\lambda\varepsilon, H(Q_R)) - N(E-\lambda\varepsilon, H(Q_R)) + CR^{n-1/2}(1+\lambda^{n/2}), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &N(E-\lambda\varepsilon, H(Q_R)) - N(E-\lambda+\lambda\varepsilon, H(Q_R)) - CR^{n-1/2}(1+\lambda^{n/2}) \\ &\quad \leqq \mathcal{N}(0, 1+\lambda((H-E)^{-1}-\omega(H-E)^{-2}), L_2(Q_R)) \\ &\quad \leqq N(E+\lambda\varepsilon, H(Q_R)) - N(E-\lambda-\lambda\varepsilon, H(Q_R)) + CR^{n-1/2}(1+\lambda^{n/2}). \end{aligned} \tag{3.12}$$

Proof. We prove (3.11); (3.12) can be proved similarly. For $v \in \mathbb{C}_0^\infty(Q_R)$, we have $(H-E)v|_{Q_R} \in L_2(Q_R)$, therefore

$$\begin{aligned} &\mathcal{N}(0, 1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}), L_2(Q_R)) \\ &\quad \geqq \mathcal{N}(0, (H-E)^2 - \lambda((H-E)+\omega), \mathbb{C}_0^\infty(Q_R)). \end{aligned} \tag{3.13}$$

To obtain a similar estimate from above, we construct a function χ_R such that

$$\chi_R|_{Q_R} = 1, \quad \operatorname{supp}\chi_R \subset Q_{R+R^{1/2}}, \quad |D^\alpha \chi_R| \leqq C_\alpha R^{-|\alpha|/2}, \quad \forall a, \tag{3.14}$$

and notice that

$$\begin{aligned} &\mathcal{N}(0, 1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}), L_2(Q_R)) \\ &\quad = \mathcal{N}(0, 1-\lambda\chi_R((H-E)^{-1}+\omega(H-E)^{-2})\chi_R, L_2(Q_R)) \\ &\quad \leqq \mathcal{N}(0, 1-\lambda\chi_R((H-E)^{-1}+\omega(H-E)^{-2})\chi_R, L_2(Q_{R+R^{1/2}})). \end{aligned}$$

Further, recall that $H-E : H_A^2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is invertible therefore for any $u \in L_2(Q_{R+R^{1/2}})$ there exists $v \in H_A^2(\mathbb{R}^n)$ such that $u = (H-E)v|_{Q_{R+R^{1/2}}}$. Hence,

$$\begin{aligned} &\mathcal{N}(0, 1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}); L_2(Q_R)) \\ &\quad \leqq \mathcal{N}(0, \mathfrak{A}^{(\omega)}(H-E,\lambda,R), H_A^2(Q_{R+R^{1/2}})), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{A}^{(\omega)}(H-E,\lambda,R)[\,\cdot\,] = \|(H-E)\cdot\|^2_{L_2(Q_{R+R^{1/2}})} - \lambda\langle((H-E)^{-1} \\ +\omega(H-E)^{-2})\chi_R(H-E)\cdot, \chi_R(H-E)\cdot\rangle_{L_2(\mathbb{R}^n)}, \end{aligned}$$

and $H_A^2(Q)$ is defined similarly to $H_A^2(\mathbb{R}^n)$.

It follows from (3.14) that

$$\|[\chi_R, H-E]H^{-1/2}\| \leqq CR^{-1/2},$$

and since

$$\begin{aligned}&(H-E)\chi_R(H-E)^{-1}\chi_R(H-E)-\chi_R(H-E)\chi_R\\&\quad=[H-E,\chi_R](H-E)^{-1}\chi_R(H-E)+\chi_R[\chi_R,H-E]\\&\quad=[H-E,\chi_R](H-E)^{-1}[\chi_R,H-E]+[H-E,\chi_R]\chi_R+\chi_R[\chi_R,H-E]\,,\end{aligned}$$

we obtain

$$\begin{aligned}&|\lambda\langle((H-E)^{-1}+\omega(H-E)^{-2})\chi_R(H-E)u,\chi_R(H-E)u\rangle_{L_2}\\&\quad-\lambda\|(-i\nabla+A(x))\chi_Ru\|^2-\lambda\langle(V-E)\chi_Ru,\chi_Ru\rangle_{L_2}|\\&\quad\leqq\nu(\|(H-E)u\|^2_{L_2}+\lambda^2\|u\|^2_{L_2})\,.\end{aligned}$$

where $\nu\to 0$ as $R\to+\infty$. Hence,

$$\mathfrak{A}^{\omega}(H-E,\lambda,R)\geqq\mathfrak{A}_\nu(H-E,\lambda,R)\,,$$

where

$$\begin{aligned}\mathfrak{A}_\nu(H-E,\lambda,R)[\,\cdot\,]&=(1-\nu)\|(H-E)\cdot\|^2_{L_2}-\lambda\|(i\nabla-A(x))\chi_R\cdot\|^2_{L_2}\\&\quad-\lambda\langle(V-E)\chi_R\cdot,\chi_R\cdot\rangle_{L_2}-\nu\lambda^2\|\cdot\|^2_{L_2}\,,\end{aligned}$$

and Lemma 2.3 gives

$$\begin{aligned}&\mathcal{N}(0,1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}),L_2(Q_R))\\&\quad\leqq\mathcal{N}(0,\mathfrak{A}_\nu(H-E,\lambda,R),H^2_A(Q_{R+R^{1/2}}))\,.\end{aligned}\tag{3.15}$$

Since $\mathbb{C}_0^\infty(\bar{Q}_{R+R^{1/2}})$ is dense in $H^2_A(Q_{R+R^{1/2}})$, we may substitute the former for the latter in (3.15). Next, applying the localization procedure which was used to derive (3.5), we obtain

$$\begin{aligned}&\mathcal{N}(0,\mathfrak{A}_\nu(H-E,\lambda,R),H^2_A(Q_{R+R^{1/2}}))\\&\quad\leqq\mathcal{N}(0,\mathfrak{A}_{\nu_1}(H-E,\lambda,R),\mathbb{C}_0^\infty(Q_R))\\&\qquad+\mathcal{N}(0,\|(H-E)\cdot\|^2-C\lambda^2\|\cdot\|^2,\mathbb{C}_0^\infty(\bar{Q}_{R+R^{1/2}}\setminus\bar{Q}_{R-R^{1/2}}))\,,\end{aligned}\tag{3.16}$$

where $\nu_1\to 0$ as $R\to+\infty$ and $C>0$ is large enough (but fixed).

Due to Lemma 2.9

$$\begin{aligned}&\mathcal{N}(0,\|(H-E)\cdot\|^2-C\lambda^2\|\cdot\|^2,\mathbb{C}_0^\infty(\bar{Q}_{R+R^{1/2}}\setminus\bar{Q}_{R-R^{1/2}}))\\&\quad\leqq C_1(1+\lambda^{n/2})R^{n-1/2}\,,\end{aligned}\tag{3.17}$$

and we deduce from (3.15)–(3.17)

$$\begin{aligned}&\mathcal{N}(0,1-\lambda((H-E)^{-1}+\omega(H-E)^{-2}),L_2(Q_R))\\&\quad\leqq\mathcal{N}(0,(1-\nu_1)(H-E)^2-\lambda(H-E)-\nu_1\lambda^2,\mathbb{C}_0^\infty(Q_R))\\&\qquad+C_1(1+\lambda^{n/2})R^{n-1/2}\,.\end{aligned}\tag{3.18}$$

Similarly to (3.16) and (3.17) we can find $C_2 > 0$ and $\nu_1 \to 0$ as $R \to +\infty$ such that

$$\mathcal{N}(0, \mathfrak{A}_{-\nu_1}, H^2(Q_R)) \leqq \mathcal{N}(0, (H-E)^2 - \lambda(H - E + \omega), \mathbb{C}_0^\infty(Q_R)) + C_2(1+\lambda^{n/2})R^{n-1/2} .$$

Next, by applying (3.13), we obtain

$$\begin{aligned} &\mathcal{N}(0, 1 - \lambda((H-E)^{-1} + \omega(H-E)^{-2}), L_2(Q_R)) \\ &\geqq \mathcal{N}(0, \mathfrak{A}_{-\nu_1}, H^2(Q_R)) - C_2(1+\lambda^{n/2})R^{n-1/2} . \end{aligned} \tag{3.19}$$

Let $\mathcal{D}_R$ be the domain of $H(Q_R)$. Since $\mathbb{C}_0^\infty(Q_R) \subset \mathcal{D}_R \subset H_A^2(Q_R)$, Lemma 2.4, (3.18) and (3.19) yield an estimate

$$\begin{aligned} &-C_1(1+\lambda^{n/2})R^{n-1/2} + \mathcal{N}(0, \mathfrak{A}_{-\nu_1}(H-E, \lambda, R), \mathcal{D}_R) \\ &\leqq \mathcal{N}(0, 1 - \lambda((H-E)^{-1} + \omega(H-E)^{-2}), L_2(Q_R)) \\ &\leqq C_1(1+\lambda^{n/2})R^{n-1/2} + \mathcal{N}(0, \mathfrak{A}_{\nu_1}(H-E, \lambda, R), \mathcal{D}_R) . \end{aligned} \tag{3.20}$$

If $\lambda_1 \leqq \lambda_2 \leqq \cdots$ are the eigenvalues of $H(Q_R)$ (counted with their multiplicities) then (3.20) can be rewritten as

$$\begin{aligned} &-C_1(1+\lambda^{n/2})R^{n-1/2} + \operatorname{card}\{k \mid (1+\nu_1)(\lambda_k - E)^2 - (\lambda_k - E)\cdot\lambda + \nu_1\lambda^2 < 0\} \\ &\leqq \mathcal{N}(0, 1 - \lambda((H-E)^{-1} + \omega(H-E)^{-2}), L_2(Q_R)) \\ &\leqq C_1(1+\lambda^{n/2})R^{n-1/2} + \operatorname{card}\{k \mid (1-\nu_1)(\lambda_k - E)^2 - (\lambda_k - E)\cdot\lambda - \nu_1\lambda^2 < 0\} , \end{aligned}$$

and (3.11) follows. □

3.2. Applying (3.11) to (3.10) and (3.9), we obtain for $k \in I_{++}$:

$$\begin{aligned} &\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1} - \varepsilon H^{-1})W^{1/2}, \mathbb{C}_0^\infty(U_{R,k})) \\ &\geqq N(E + \lambda w_{R,k}^-(1-\varepsilon), H(U_{R,k})) - N(E + \varepsilon\lambda w_{R,k}^-, H(U_{R,k})) \\ &\quad - C(1+\lambda w_{R,k}^-)^{n/2}R^{n-1/2} , \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\mathcal{N}(0, \lambda^{-1} - W^{1/2}((H-E)^{-1}(1+\varepsilon_1) + \varepsilon_2(H-E)^{-2})W^{1/2}, \mathbb{C}_0^\infty(U_{R,k})) \\ &\leqq N(E + \lambda w_{R,k}^+(1+\varepsilon_3), H(U_{R,k})) - N(E - \varepsilon_3\lambda w_{R,k}^+, H(U_{R,k})) \\ &\quad + C(1+\lambda w_{R,k}^+)^{n/2}R^{n-1/2} . \end{aligned} \tag{3.22}$$

Now, E is in a gap of H and there exists $C_\delta > 0$ such that $|\lambda w_{r,k}^+| \leqq C_\delta$ for $|k| > 1$. Therefore, for these k,

$$\varepsilon\lambda w_{r,k}^+ \to 0 \quad \text{as } \varepsilon \to 0 .$$

It is easy to show that the density of states is continuous at E in a gap, therefore

$$\sum_{k\in I_{++}} |N(E\pm\varepsilon\lambda w^{+}_{r,k};H(U_{r,k}))-N(E;H(U_{r,k}))| = o\left(\sum_{k\in I_{++}} N(E;H(U_{r,k}))\right),$$

as $\lambda\to\infty$ (and hence, $\varepsilon\to 0$). Using Lemma 2.9, we see that the RHS is $o(\lambda^{n/m})$, with the constant depending on δ, of course.

Thus,

$$\sum_{k\in I_{++}} N(E\pm\varepsilon\lambda w^{+}_{r,k};H(U_{r,k})) = \sum_{k\in I_{++}} N(E;H(U_{r,k})) + o(\lambda^{n/m}). \tag{3.23}$$

Applying (3.21) and (3.22) with $(\partial U)^{+}_{r,k}$ instead of $U_{r,k}$ and using Lemma 2.7, we obtain

$$\sum_{k\in I_{++}} \mathcal{N}(0;\lambda^{-1}-W^{1/2}((H-E)^{-1}+\varepsilon_1 H^{-1})W^{1/2};\mathbb{C}_0^{\infty}((\partial U)^{+}_{r,k})) = o(\lambda^{n/m}). \tag{3.24}$$

Note that the constants in o-terms depend on δ.

Now we have necessary bounds for all terms in (3.7) and (3.8), except for those with $k\in I_{00}\cup I_{--}$. To treat them we need the following lemma.

3.3.

Lemma 3.2. *Let $\omega\to+0$ as $R\to+\infty$, let $w>0$ and $0\leqq W(x)\leqq w$ on Q_R. Then*

$$\mathcal{N}(0,\lambda^{-1}-W^{1/2}((H-E)^{-1}+\omega(H-E)^{-2})W^{1/2},\mathbb{C}_0^{\infty}(Q_R)) \leqq C(\lambda w)^{n/2}R^n \tag{3.25}$$

with C independent of $R\geqq 1$, $\lambda>0$, w.

Proof. There exists $C_1>0$ such that

$$-(H-E)^{-1}-\omega(H-E)^{-2}\geqq -C_1(H(A,0)+1)^{-1},$$

therefore

$$\begin{aligned}&\mathcal{N}(0,\lambda^{-1}-W^{1/2}((H-E)^{-1}+\omega(H-E)^{-2})W^{1/2},\mathbb{C}_0^{\infty}(Q_R))\\ &\quad\leqq \mathcal{N}(0,\lambda^{-1}-C_1W^{1/2}(H(A,0)+1)^{-1}W^{1/2},\mathbb{C}_0^{\infty}(Q_R)).\end{aligned} \tag{3.26}$$

Let $e_+ : L_2(Q_R)\to L_2(\mathbb{R}^n)$ be the extension by zero operator, and define $\mathcal{A}_R : L_2(Q_R)\to L_2(Q_R)$ by $\mathcal{A}_R u = (H(A,0)+1)^{-1}e_+u|_{Q_R}$. This is a positive compact operator, therefore the Glazman lemma, the Birman–Schwinger principle and Lemma 2.3 give

$$\begin{aligned}&\mathcal{N}(0,\lambda^{-1}-W^{1/2}((H(A,0)+1)^{-1}W^{1/2},\mathbb{C}_0^{\infty}(Q_R)) = N_+(\lambda,\mathcal{A}_R^{-1},W)\\ &\quad\leqq N(\lambda w,\mathcal{A}_R^{-1}) = \mathcal{N}(0,\lambda^{-1}w^{-1}-(H(A,0)+1)^{-1}),\mathbb{C}_0^{\infty}(Q_R)).\end{aligned} \tag{3.27}$$

If $\lambda w < 1/2$ then the RHS of (3.27) is zero, and if $|\lambda w| > 1/2$ then applying Lemma 3.1 and Lemma 2.9 we obtain

$$\text{the RHS of (3.27)} \leqq C(\lambda w)^{n/2} R^n .$$

Now (3.26) gives (3.25). $\square$

3.4. Recall that $r = \delta\lambda^{1/m}$. By using an appropriate partition of unity on $U_{r,k}, |k| \leqq 1$, and (3.25), we can obtain

$$\sum_{|k|\leqq 1} \mathcal{N}(0; \lambda^{-1} - W^{1/2}((H-E)^{-1}(1+\varepsilon) + \varepsilon_1 H^{-1})W^{1/2}; \mathbb{C}_0^\infty(U_{r,k}))$$
$$\leqq C_2 \int_{|x|\leqq 2\delta\lambda^{1/m}} (\lambda(1+|x|)^{-m})^{n/2}\, dx \leqq C_3 \lambda^{n/m} \delta^{n(1-m/2)} . \tag{3.28}$$

Due to (1.2) $\lambda w_{r,k}^+ \to 0$ as $\delta \to 0$, uniformly in $k \in I_{--}$, hence Lemma 3.2 gives

$$\sum_{k\in I_{--}} \mathcal{N}(0; \lambda^{-1} - W^{1/2}((H-E)^{-1}(1+\varepsilon) + \varepsilon_1 H^{-1})W^{1/2}; \mathbb{C}_0^\infty(U_{r,k}^+)) \leqq c(\delta)\lambda^{n/m} \tag{3.29}$$

where $c(\delta) \to 0$ as $\delta \to 0$.

3.5. By gathering (3.7), (3.8), (3.21)–(3.25), (3.28) and (3.29), we obtain

$$\sum_{k\in I_{++}} (N(E + \lambda w_{r,k}^-(1-\varepsilon), H(U_{r,k})) - N(E, H(U_{r,k}))) - f_1(\delta)\lambda^{n/m} - f_2(\delta,\lambda)$$
$$\leqq N_+(\lambda; H-E; W) \leqq \sum_{k\in I_{++}} (N(E + \lambda w_{r,k}^+(1+\varepsilon), H(U_{r,k})) - N(E, H(U_{r,k})))$$
$$+ f_1(\delta)\lambda^{n/m} + f_2(\delta,\lambda) , \tag{3.30}$$

where $f_1(\delta) \to 0$ as $\delta \to 0$, and $f_2(\delta,\lambda) = o(\lambda^{n/m})$ for $\delta > 0$ fixed and $\lambda \to +\infty$.

For $\delta > 0$ fixed, there exists $C_\delta > 0$ such that $\operatorname{card} I_{++} \leqq C_\delta\, \forall\lambda$, and since due to (1.2)

$$|w_{r,k}^+ - \tilde{W}(x)| + |w_{r,k}^- - \tilde{W}(x)| \to 0 \quad \text{as } \lambda \to \infty ,$$

uniformly in $x \in U_{r,k}$ and $|k| > 1$, we deduce from (3.30)

$$\int_{U(C_0\lambda^{1/m}, \delta\lambda^{1/m})} dx \int_{E<t<E+\lambda(1-\varepsilon)\tilde{W}(x)} d\rho(t,H) - f_1(\delta)\lambda^{n/m} - f_2(\delta,\lambda)$$
$$\leqq N_+(\lambda; H-E; W)$$
$$\leqq \int_{U(C_0\lambda^{1/m}, \delta\lambda^{1/m})} dx \int_{E<t<E+\lambda(1+\varepsilon)\tilde{W}(x)} d\rho(t,H) + f_1(\delta)\lambda^{n/m} + f_2(\delta,\lambda) , \tag{3.31}$$

where $\varepsilon \to 0$ as $\lambda \to \infty$, and $U(C_0\lambda^{1/m}, \delta\lambda^{1/m})$ is a union of $U_{r,k}$ such that $|k| > 1$ and $U_{r,k} \subset B_{C_0\lambda^{1/m}}$.

In the LHS and RHS of (3.31), we make changes of variables $x \mapsto (\lambda(1-\varepsilon))^{1/m}x$ and $x \mapsto (\lambda(1+\varepsilon))^{1/m}x$ respectively, then divide by $\lambda^{n/m}$ and calculate the limit $\lim_{\delta\to 0}\lim_{\lambda\to+\infty}$; the result is

$$c^+(\tilde{W}) \leqq \lim_{\lambda\to+\infty} N_+(\lambda; H-E; W)\lambda^{-n/m} \leqq c^+(\tilde{W}),$$

and Theorem 1.1 has been proved.

4. Proof of Theorem 1.2

4.1. We need the following lemma.

Lemma 4.1. *Let $\omega \to +0$ as $R \to +\infty$, let $w > 0$ and $0 \leqq W(x) \leqq w$ on Q_R. Then for each N there exists C such that*

$$\mathcal{N}(0; \lambda^{-1} + W^{1/2}((H-E)^{-1} - \omega(H-E)^{-2})W^{1/2}; C_0^\infty(Q_R)) \leqq C(\lambda w)^{n/2N}R^n. \tag{4.1}$$

Proof. Let P_1 be a spectral projection $P_{(-\infty,E)}(H)$, and set $P_2 = I - P_1$. If $\omega > 0$ is small enough, we have

$$P_2((H-E)^{-1} - \omega(H-E)^{-2}) \geqq 0,$$

and therefore, for each N there exists $C_N > 0$ such that

$$\begin{aligned}(H-E)^{-1} - \omega(H-E)^{-2} &\geqq P_1((H-E)^{-1} - \omega(H-E)^{-2}) \\ &\geqq -C_N(H(A,0)+1)^{2N}.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathcal{N}(0; \lambda^{-1} + W^{1/2}((H-E)^{-1} - \omega(H-E)^{-2})W^{1/2}; \mathbb{C}_0^\infty(Q_R)) \\ &\leqq \mathcal{N}(0; \lambda^{-1} - C_N W^{1/2}(H(A,0)+1)^{-N}W^{1/2}; \mathbb{C}_0^\infty(Q_R)),\end{aligned}$$

and we finish the proof just as the proof of Lemma 3.2. □

4.2. The proof of Theorem 1.2 differs from the one of Theorem 1.1 in the following respects:

1) functions of the form $\mathcal{N}(0; 1-\lambda(\,\cdot\,); \cdot\,)$ and $\mathcal{N}(0; \lambda^{-1}-(\,\cdot\,); \cdot\,)$ should be replaced by functions of the form $\mathcal{N}(0; 1+\lambda(\,\cdot\,); \cdot\,)$ and $\mathcal{N}(0; \lambda^{-1}+(\,\cdot\,); \cdot\,)$, respectively;

2) instead of (3.11), we use (3.12);

3) instead of (3.25), we use (4.1).

The necessity of using more involved estimate (4.1) is as follows. If $m \in (0,2)$, then we can use (4.1) with $N = 1$, but for $m \geqq 2$, (4.1) with $N = 1$

will not give an analogue of (3.28) while (4.1) with $N > m/2$ gives

$$\sum_{|k|\leqq 1} \mathcal{N}(0; \lambda^{-1} + W^{1/2}((H-E)^{-1}(1-\varepsilon) - \varepsilon_1 H^{-1})W^{1/2}; \mathbb{C}_0^{\infty}(U_{r,k}))$$

$$\leqq C_2 \int_{|x|<2\delta\lambda^{1/m}} (\lambda(1+|x|)^{-m}))^{n/2N}\,dx \leqq C_3(\delta)\lambda^{n/m},$$

where $C_3(\delta) \to 0$ as $\delta \to 0$.

Similarly we transform the proof of an analogue of (3.29), and Theorem 1.2 has been proved.

5. Proof of Theorem 1.3

Now we use constructions in the proof of Theorem 1.1 with $r = \ln\ln\lambda$ and $R = C_0\lambda^{1/2}$. Noticing that $R^n(\ln\ln\lambda)^N = o(\lambda^{n/2}\ln\lambda)$ and

$$\int_{|x|\leqq 2C_0\lambda^{1/m}} (\lambda(1+|x|)^{-2})^{n/2} \sim c_1\lambda^{n/2}\ln\lambda$$

with some $c_1 \in \mathbb{R}$, we see that in all the estimates in the Proof of Theorem 1.1 $o(\lambda^{n/m})'s$ should (and could) be replaced with $o(\lambda^{n/2}\ln\lambda)'s$. Further, in (3.27) we now have

$$\leqq C_2 \int_{|x|\leqq 2\ln\ln\lambda} (\lambda(1+|x|)^{-2})^{n/2}\,dx = O(\lambda^{n/2}\ln\ln\lambda) = o(\lambda^{n/2}\ln\lambda),$$

and as a result we obtain instead of (3.30)

$$\sum_{k\in I_{++}} (N(E+\lambda w_{r,k}^{-}(1-\varepsilon), H(U_{r,k})) - N(E, H(U_{r,k}))) + o(\lambda^{n/2}\ln\lambda)$$

$$\leqq N_+(\lambda, H-E, W)$$

$$\leqq \sum_{k\in I_{++}} (N(E+\lambda w_{r,k}^{+}(1+\varepsilon), H(U_{r,k})) - N(E, H(U_{r,k}))) + o(\lambda^{n/2}\ln\lambda)\,. \tag{5.1}$$

In (5.1), terms with

$$E \pm w_{r,k}^{\pm}(1\pm\varepsilon) \leqq (\ln\ln\lambda)^4$$

give a contribution

$$O((\ln\ln\lambda)^{2n}R^n) = o(\lambda^{n/2}\ln\lambda)\,; \tag{5.2}$$

since $\operatorname{diam} U_{r,k} \leqq 4r = 4\ln\ln\lambda$ and the magnetic potential is uniform, the others admit an estimate

$$\mathcal{N}((E+\lambda w_{r,k}^{-})(1\pm\varepsilon_1); H(U_{r,k}))$$

$$\leqq \mathcal{N}((E+\lambda w_{r,k}^{\pm})(1\pm\varepsilon); H(U_{r,k}))$$

$$\leqq \mathcal{N}((E+\lambda w_{r,k}^{+})(1\pm\varepsilon_1); H(U_{r,k}))\,, \tag{5.3}$$

where $\varepsilon_1 \to 0$ as $\lambda \to +\infty$.

The asymptotics of the LHS and RHS of (5.3) can be computed by means of the well-known formula

$$\mathcal{N}(\mu, -\Delta; \mathbb{C}_0^\infty(Q_R)) \to (2\pi)^{-n}|v_n| \operatorname{meas} Q_R \cdot \mu^{n/2}, \tag{5.4}$$

as $\mu + R \to +\infty, \mu \geqq 1, R \geqq 1$ (see e.g. review [RSS]); here we also need a uniform estimate for the remainder in (5.4) and such estimate is provided by a general theorem of the approximate spectral projection method in [L]- see e.g. Theorem 7.1 there.

In the definition of I_{++} (after (3.7)), we take $\varepsilon_0 \to 0$ as $R \to +\infty$ sufficiently slowly so that uniformly in $k \in I_{++}$ and $x \in U_{r,k}$

$$|w_{r,k}^+ - \tilde{W}(x)| + |w_{r,k}^- - \tilde{W}(x)| \leqq \varepsilon_1 \tilde{W}(x), \tag{5.5}$$

and one easily deduces from (5.1)–(5.5)

$$N_+(\lambda, H - E, W) = (2\pi)^{-n}|v_n|\lambda^{n/2} \int\limits_{\ln\ln\lambda < |x| < C_0\lambda^{1/2}} \tilde{W}(x)^{n/2}\, dx + o(\lambda^{n/2}\ln\lambda). \tag{5.6}$$

Since $\tilde{W}(x)$ is positively homogeneous of degree -2, we deduce from (5.6)

$$\begin{aligned} &N_+(\lambda, H - E, W) \\ &\quad = (2\pi)^{-n}|v_n|\lambda^{n/2} \int\limits_{S_{n-1}} \tilde{W}(x)^{n/2}\, dS(x) \int\limits_{\ln\ln\lambda < r < C_0\lambda^{1/2}} r^{-1}\, dr + o(\lambda^{n/2}\ln\lambda) \\ &\quad = (2\pi)^{-n}|v_n|\frac{1}{2} \int\limits_{S_{n-1}} \tilde{W}(x)^{n/2}\, dS(x)\lambda^{n/2}\ln\lambda + o(\lambda^{n/2}\ln\lambda), \end{aligned}$$

and (1.8) has been proved.

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