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Elementary modules

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Introduction

Let A be a finite dimensional, connected wild hereditary k-algebra, k an algebraically closed field. We denote by A-reg the full subcategory of regular A-modules in A-mod. This category is closed under images and extensions, but, contrary to the tame situation, not closed under kernels and cokernels. A nonzero regular module E is called *elementary*, if there is no nontrivial regular submodule X, such that E/X is regular, too. Since the Auslander–Reiten translations τ and τ^- define an equivalence on A-reg, a module E is elementary, if and only if $\tau^i E$ is elementary, for all integers i.

It follows from the definition that each nonzero regular module X has a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$ with elementary composition factors X_i/X_{i-1}, hence the class $\mathscr{E}$ of elementary modules is the smallest class of regular modules, whose extension-closure is A-reg. By definition the elementary modules are exactly the quasi-simple regular modules, if the algebra A is tame.

We will show in part 2 that – parallel to the tame situation – there exist only finitely many Coxeter-orbits of dimension-vectors of elementary modules. Totally different to the tame case is, that a τ-sincere module E is elementary only if $\dim_k \operatorname{Ext}(E,E) \geqq 2$ holds (Theorem 3.4) and that there exist infinitely many algebras whose elementary modules all are stones, that is indecomposable modules without self-extensions, see part 4. In this case Theorem 2.1 then says that there are only finitely many τ-orbits of elementary modules.

Finally we show the occurrence of elementary modules in natural constructions: If B is (wild) concealed, if M is a regular B-modules then the one-point extension $B[M]$ is a tilted algebra only if M is elementary. Similarly, if the quiver $\mathscr{Q}$ is a wild star with vertices $\{0,\dots,n\}$ where 0 denotes the center of the star, if M is an indecomposable $k\mathscr{Q}$-module with $\underline{\dim}\, M =$

$(2, 1, \ldots, 1)$, then $k\mathcal{Q}[M]$ is a wild canonical algebra if and only if M is elementary.

Parts of this paper have their origin in the second author's thesis [11]. Especially Theorem 2.1 and Theorem 3.4 are central results of [11].

Notations

The word algebra always denotes a finite-dimensional, unitary, basic algebra over some algebraically closed field k. The letter A normally is reserved for wild hereditary, connected algebras. We call a module X a *brick* if $\mathrm{End}(X)$ is isomorphic to k. A brick without self-extensions is called a *stone*. The number of pairwise non-isomorphic simple A-modules will be denoted by $n(A)$.

By $\Omega(A)$ we denote the set of regular components of the Auslander–Reiten quiver $\Gamma(A)$ of A. An indecomposable regular module X is called τ-sincere, if $\tau^i X$ is sincere for all integers i.

If U is a quasi-simple module and r a natural number, we denote by $U(r)$ ($[r]U$) the indecomposable regular module with quasi-length r and quasi-socle (quasi-top) U.

If X is an indecomposable regular module, say $X = [m]U$ for U quasi-simple, we denote the wing of X (of length m) by $\mathcal{W}(X)$.

From the concept of perpendicular categories we use the following result (see [3], [15] or [14]): If A is a hereditary algebra and X a quasi-simple regular stone, then the right perpendicular category $X^{\perp}$, defined by the objects $\{Y \mid \mathrm{Hom}(X, Y) = \mathrm{Ext}^1(X, Y) = 0\}$ is an abelian subcategory of A-mod which is equivalent to a module category B-mod, and B is wild, connected and hereditary. Sometimes we write $(X, Y)({}^1(X, Y)$, respectively) instead of $\mathrm{Hom}_A(X, Y)(\mathrm{Ext}^1_A(X, Y)$, respectively).

In general we follow the notations used in [12].

1 Basic properties

Definition *Let A be a hereditary algebra. A regular A-module $E \neq 0$ is called elementary if there is no short exact sequence $0 \to U \to E \to V \to 0$ with U and V regular and nonzero.*

The proof of the following lemma is straightforward:

Lemma 1.1 *Let A be hereditary.*

(a) *If E is elementary, then so is $\tau^n E$ for all $n \in \mathbb{Z}$.*
(b) *Elementary modules are quasi-simple.*
(c) *If A is tame and E is quasi-simple regular, then E is elementary.*

Remarks. 1) If T is a preprojective tilting module with $\mathrm{End}(T) = B$, then the functor $\mathrm{Hom}(T, -)$ induces an equivalence between A-reg and B-reg. Hence the results on elementary modules hold for concealed algebras, too.

2) We will show below that all elementary modules are bricks. It is well known (see [9]) that for a wild hereditary algebra there always exist quasi-simple modules which are not bricks. So the converse of 1.1(b) is not true if A is wild.

3) One can construct examples of regular modules M having filtrations with totally different elementary composition factors.

Before summarising the basic properties of elementary modules, we need the following.

Lemma 1.2 *Let A be wild hereditary*

(a) *Let $X \neq 0$ be a regular module. Then there exists a positive integer N such that for all regular modules R and for all $f \in \mathrm{Hom}(\tau^l X, R)$ with $l \geqq N$ the kernel* $\ker f$ *is regular.*

(b) *Let Y be regular. If Y has no nontrivial regular factor-modules then so has $\tau^l Y$ for all $l \geqq 0$.*

Proof. a) As the dimensions $\dim_k \tau^{-l} P$ grow exponentially with l for P projective (see [2]) there exists $N \in \mathbb{N}$ such that $\dim_k \tau^{-l} P > \dim_k X$ for all $l \geqq N$ and for all projective modules $P \neq 0$. For $l \geqq N$ and R regular consider $0 \neq f \in \mathrm{Hom}(\tau^l X, R)$. We have the short exact sequence $0 \to \ker f \to \tau^l X \to \mathrm{Im} f \to 0$ with $\mathrm{Im} f$ regular and $\ker f$ without nonzero preinjective direct summand. Applying τ^{-l} we get $0 \to \tau^{-l} \ker f \to X \to \tau^{-l} \mathrm{Im} f \to 0$, which shows that $\ker f$ is regular.

b) If $\tau^l Y$ has a nontrivial regular factor-module Z then we get a short exact sequence $0 \to U \to \tau^l Y \to Z \to 0$. Application of τ^{-l} gives a contradiction.

Proposition 1.3 *Let A be representation-infinite and hereditary, let E be an indecomposable regular module. There are equivalent*

(1) *E is elementary.*

(2) *There exists an integer N such that $\tau^l E$ has no nontrivial regular factor-modules for all $l \geqq N$.*

(3) *There exists an integer M such that $\tau^{-l} E$ has no nontrivial regular submodules for all $l \geqq M$.*

(4) *If $Y \neq 0$ is a regular submodule of E then E/Y is preinjective.*

(5) *If $X \neq E$ is a submodule of E with E/X regular, then X is preprojective.*

Proof. (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are obvious. (1) $\Rightarrow$ (2) and dually (1) $\Rightarrow$ (3) follow from 1.2(b).

(1) $\Rightarrow$ (4): Suppose $E/Y = Z_1 \oplus Z_2$ with $Z_1 \neq 0$ regular and Z_2 preinjective. We get the following diagram

$$\begin{tikzcd}
 & & & Z_2 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & E/Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r] & Z_1 \arrow[r] \arrow[d] & 0 \\
 & Z_2 \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

hence K is regular, a contradiction, (1) $\Rightarrow$ (5) is dual.

Corollary 1.4 (a) *If E is elementary then E is a brick.*

(b) *If E is elementary and Y is regular with $\underline{\dim}\, Y = \underline{\dim}\, E$ then either Y is isomorphic to E or Y and E are orthogonal, that is* $\mathrm{Hom}(E, Y) = 0 = \mathrm{Hom}(Y, E)$.

Proof. (a) Follows immediately from 1.3(2).

(b) Suppose E and Y are not orthogonal, say $\mathrm{Hom}(Y, E) \neq 0$. As Y is regular, we have $\underline{\dim}\, \tau^l Y = \underline{\dim}\, \tau^l E$ for all $l \in \mathbb{Z}$. 1.3(3) then says that there is a surjective map $g : \tau^{-l} Y \to \tau^{-l} E$, thus g is an isomorphism.

Examples. (1) Let S be indecomposable regular such that $\underline{\dim}\, S$ or $\dim_k S$ is minimal among all non-zero regular modules. Then S is elementary.

(2) Let A be the path-algebra of a star $\mathcal{Q}$, not of Dynkin-type, where

$$\begin{tikzcd}[row sep=small]
 & \circ \arrow[dl] & \cdots \arrow[l] & \circ \arrow[l] \\
 & \circ \arrow[l] & \cdots \arrow[l] & \circ \arrow[l] \\
\mathcal{Q} = \circ & \vdots & & \vdots \\
 & \circ \arrow[ul] & \cdots \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

Let M be the following indecomposable module:

$$M = \begin{array}{ccccc} & & k & \longleftarrow \cdots \longleftarrow & k \\ & {}^{\alpha_1}\swarrow & & & \\ & {}^{\alpha_2}\swarrow & k & \longleftarrow \cdots \longleftarrow & k \\ k^2 & & & & \\ & & \vdots & & \vdots \\ & \nwarrow^{\alpha_r} & & & \\ & & k & \longleftarrow \cdots \longleftarrow & k \end{array}$$

It is easy to check that the following are equivalent:

(i) The one-point extension $A[M]$ is a canonical algebra in the sense of Ringel [12, 3.7].

(ii) $\alpha_i(k) + \alpha_j(k) = k^2$ for all $1 \leqq i \neq j \leqq r$.

(iii) M is elementary.

(3) If A is connected, wild and hereditary, say $A = k\mathcal{Q}$ with $n \geqq 3$ simples, then $\mathcal{Q}$ always has a full connected subquiver $\mathcal{Q}'$ with $|\mathcal{Q}'_0| = n - 1$ such that $B = k\mathcal{Q}'$ is representation-infinite (this follows e.g. from [5]). As almost all indecomposable preprojective A-modules are sincere, see [13, 1.2] almost all preprojective B-modules are regular in A-mod. Let E be an indecomposable preprojective B-module such that E is regular in A-mod but all proper predecessors of E in B-mod are preprojective in A-mod. Then E is elementary in A-mod. Especially any representation-infinite hereditary algebra with more than two simple modules has elementary stones.

2 The finiteness condition

For a tame hereditary algebra A the set of dimension vectors of elementary that is quasi-simple modules is finite. Of course this is no longer true, if A is wild, as $\underline{\dim}\,\tau^i E \neq \underline{\dim}\,\tau^j E$ for $i \neq j$. If E is elementary, if Φ is the Coxeter-transformation (corresponding to τ) then we get $\Phi^j(\underline{\dim}\,E) = \underline{\dim}\,\tau^j E$ for all $j \in \mathbb{Z}$. For $x \in \mathbb{Z}^n$ we call $(\Phi^j(x))_{j \in \mathbb{Z}}$ the Coxeter-orbit of x. The main result of this part is

Theorem 2.1 *If A is hereditary then there exist only finitely many Coxeter-orbits of dimension-vectors of elementary modules.*

Proof. The assertion is trivial, if A is not wild, so assume A is wild. As each regular component contains only finitely many non-sincere modules we can choose an (indecomposable) regular module R such that $\tau^{-n}R$ is sincere for all $n \geqq 0$. If X is elementary by [1, 3.1] and [6, 1.1] there exists $E = \tau^j X$ such that $\mathrm{Hom}(R, E) = 0$ but $\mathrm{Hom}(\tau^- R, E) \neq 0$. Take $0 \neq f \in \mathrm{Hom}(\tau^- R, E)$, let be

U (K, C, respectively) the image (the kernel, the cokernel) of f. Then we get the two short exact sequences

$$\begin{array}{ll} (1) & 0 \to K \to \tau^{-}R \to U \to 0 \\ (2) & 0 \to U \to E \to C \to 0. \end{array}$$

Applying the functor $\mathrm{Hom}(R,-)$ to (1) and (2), we get $\cdots \to \mathrm{Ext}^1(R,\tau^{-}R) \to \mathrm{Ext}(R,U) \to 0$ and $\cdots \to \mathrm{Hom}(R,E) \to \mathrm{Hom}(R,C) \to \mathrm{Ext}(R,U) \to \cdots$

From $\mathrm{Hom}(R,E) = 0$ we deduce

$$\dim_k \mathrm{Hom}(R,C) \leqq \dim_k \mathrm{Ext}(R,U) \leqq \dim_k \mathrm{Ext}(R,\tau^{-}R) = s\,.$$

As E is elementary, C is preinjective by 1.3(4), that is

$$C = \bigoplus_{i\in\mathbb{N}_0} \bigoplus_{j=1}^{n} \tau^i I(j)^{l_{i,j}}$$

where $I(1),\ldots,I(n)$ are the indecomposable injective modules and almost all $l_{i,j}$ are zero. The inequality $\dim_k \mathrm{Hom}(R,C) \leqq \dim_k \mathrm{Ext}(R,\tau^{-}R) = s$ hence implies

$$\sum_{i\in\mathbb{N}_0} \sum_{j=1}^{n} l_{i,j} \cdot \dim_k \mathrm{Hom}(\tau^{-i}R, I(j)) \leqq s\,.$$

As the components of the dimension vectors grow exponentially, there exists i_0 with $\dim \mathrm{Hom}(\tau^{-i}R, I(j)) \geqq s$ for all $i \geqq i_0$ and for all $j = 1,\ldots,n$, that is $l_{i,j} = 0$ for all $i \geqq i_0$ and for all j. As $\mathrm{Hom}(\tau^{-i}R, I(j)) \neq 0$ for all $i \geqq 0$ and for all j only finitely many $l_{i,j}$ satisfy this condition. Thus we get an upper bound $\underline{c}$ for $\underline{\dim}\, C$, only depending on R. Especially we have $\underline{\dim}\, E \leqq \underline{\dim}\, R + \underline{c}$. As there are only finitely many roots smaller or equal to $\underline{\dim}\, R + \underline{c}$ the assertion follows.

Example. If A is the path-algebra of the quiver $1 \leftrightarrows 2 \leftarrow 3$ all regular modules E with $\underline{\dim}\, E = (1,1,0)$ are elementary, as their dimension is minimal. The stone E' with $\underline{\dim}\, E' = (1,2,0)$ is elementary by an argument dual to example 3 in part 1. One can show that A has exactly two Coxeter-orbits of elementary modules, namely $(\Phi^i(1,1,0))_{i\in\mathbb{Z}}$ and $(\Phi^i(1,2,0))_{i\in\mathbb{Z}}$, for details see [11].

3 Perpendicular categories and τ-sincere elementary modules

If A is connected, wild hereditary and X is a quasi-simple regular stone, then the right perpendicular category $X^{\perp}$ is equivalent to C-mod, where C is connected, wild hereditary. If A has n simple modules, C has $n-1$ simples. If M is the minimal Ext-projective generator in $X^{\perp}$ then $X \oplus M$ is a tilting module and $H = \mathrm{Hom}_A(M,-) : X^{\perp} \to C$-mod, where $C = \mathrm{End}_A(M)$, is an equivalence. We use this notation for the rest of the paper. If X is a quasi-simple stone, then $[2]X$ is a brick, see [7, 1.6] and is contained in $X^{\perp}$.

Proposition 3.1 *Let A be wild hereditary, let X be a quasi-simple regular stone. Then we have*

(a) $Z = H([2]X)$ *is elementary in* C-mod.
(b) $\tau_C Z$ *has no nontrivial regular factor-modules.*
(c) $\tau_C^- Z$ *has no nontrivial regular submodules.*

For the proof we need some lemmas:

Lemma 3.2 *Let X be a regular stone and $U \in X^\perp$.*

(a) *If U is cogenerated by X then U is projective in $X^\perp$.*
(b) *If U is generated by τX, then U is injective in $X^\perp$.*

Proof. (a) As U is finitely generated, there exists a monomorphism $0 \to U \to X^r$.

For $Z \in X^\perp$ we apply the functor $\mathrm{Ext}(-, Z)$ and get $0 = \mathrm{Ext}(X^r, Z) \to \mathrm{Ext}(U, Z) \to 0$, thus U is projective in $X^\perp$. Part (b) of 3.2 is dual.

Lemma 3.3 *Let X be a quasi-simple regular stone.*

(a) *If $U \in X^\perp$ is an indecomposable submodule of $[2]X$, not projective in $X^\perp$, then $[2]X/U$ is injective in $X^\perp$.*

(b) *If $V \in X^\perp$ is an indecomposable factor-module of $[2]X$ say $V = [2]X/U$, not injective in $X^\perp$, then U is projective in $X^\perp$.*

Proof. (a) If $0 \to \tau X \xrightarrow{f} [2]X \xrightarrow{g} X \to 0$ denotes the Auslander–Reiten sequence, if $\varepsilon : U \to [2]X$ denotes the inclusion with cokernel $\pi : [2]X \to V$ we get the following diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & U \arrow[d, "\varepsilon"] & & \\ 0 \arrow[r] & \tau X \arrow[r, "f"] & {[2]X} \arrow[r, "g"] \arrow[d, "\pi"] & X \arrow[r] & 0 \\ & & V \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

Note that $V \in X^\perp$, as $X^\perp$ is an abelian subcategory. As $\mathrm{Hom}(U, \tau X) \cong D\,\mathrm{Ext}(X, U) = 0$ we have $0 \neq \varepsilon g \in \mathrm{Hom}(U, X)$. By [4, 4.1] the map εg then is injective or surjective, therefore surjective by 3.2 since U is not projective. Thus we have $\dim_k U > \dim_k X$. If V_0 is an indecomposable direct summand of V and $\pi_0 : [2]X \to V_0$ is the induced map, then $f\pi_0 : \tau X \to V_0$ is nonzero, since $\mathrm{Hom}(X, V_0)$ is zero. Again by [4, 4.1] $f\pi_0$ is injective or surjective. The dimension-argument shows that $f\pi_0$ is surjective and 3.2(b) then says V_0 is injective in $X^{\perp}$.

The proof of (b) is dual to (a).

Proof of 3.1. (a) Follows from (b), (c) is dual to (b), so it suffices to show (b): Suppose $\tau_C Z$ has an indecomposable regular factor-module R in $X^{\perp}$. Then we have the short exact sequence $0 \to K \to \tau_C Z \to R \to 0$; applying τ_C^- gives $0 \to \tau_C^- K \to Z \to \tau_C^- R \to 0$. By 3.3(b) $\tau_C^- K$ then is projective in C-mod. So $\tau_C^- K = 0$ that is $R \cong \tau_C Z$.

If A is a tame hereditary algebra then a quasi-simple regular A-module E is τ-sincere if and only if it is homogenous that is $\dim \mathrm{Ext}^1(E,E) = 1$. In contrast we get in the wild situation:

Theorem 3.4 *Let A be wild hereditary and E a τ-sincere elementary module. Then* $\dim_k \mathrm{Ext}(E,E) \geqq 2$ *holds.*

Proof. Let us first show that E has self-extensions. If E is a τ-sincere elementary stone, the right perpendicular category $E^{\perp}$ is contained in A-reg. Considering the universal short exact sequence

$$0 \to A \to X \to E^l \to 0$$

with $l = \dim_k \mathrm{Ext}^1(E,A)$ we see that $\mathrm{Hom}(U,E) \neq 0$ for each indecomposable projective module $U \in E^{\perp}$, that is $[2]E$ is sincere in $E^{\perp}$. If U is simple Ext-projective in $E^{\perp}$ like in the proof of 3.3 we can consider the diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & U \arrow[d, "\varepsilon"] & & \\
0 \arrow[r] & \tau E \arrow[r, "f"] & {[2]E} \arrow[r, "g"] \arrow[d, "\pi"] & E \arrow[r] & 0 . \\
& & V \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

If $h = \varepsilon g : U \to E$ is injective, by [10, 2.2] the cokernel $\operatorname{coker} h$ is regular, too, a contradiction to E being elementary. If h is surjective and V_0 is a factor-module of V, simple in $E^{\perp}$, as in 3.3 we get a short exact sequence

$$0 \to K \to \tau E \to V_0 \to 0$$

and again by [10, 2.2] K is regular in contrast to τE being elementary.

Hence E is a brick with self-extensions, that is $\operatorname{Hom}(E,\tau E)$ is non-zero. Moreover by 1.3(2) we may assume that each $f\in\operatorname{Hom}(E,\tau E)\setminus\{0\}$ is injective with cokernel C. Applying the functor $\operatorname{Hom}(E,-)$ to the short exact sequence $0\to E\to\tau E\to C\to 0$ we get

$$0\to(E,E)\to(E,\tau E)\to(E,C)\to{}^1(E,E)\to{}^1(E,\tau E)\to{}^1(E,C)\to 0\,.$$

Since E is elementary, C is preinjective, thus $\operatorname{Ext}^1(E,C)=0$ holds.

From $\dim_k\operatorname{Ext}^1(E,E)=1$ we would deduce $\operatorname{Hom}(E,E)\cong\operatorname{Hom}(E,\tau E)\cong k$ and $\operatorname{Ext}^1(E,E)\cong\operatorname{Ext}^1(E,\tau E)\cong k$ that is $\operatorname{Hom}(E,C)=0$, contrary to the assumption E being τ-sincere.

4 Elementary modules and exceptional components

If A is wild hereditary, following [8] we call a regular component $\mathscr{C}\in\Omega(A)$ exceptional, if for a quasi-simple module X in $\mathscr{C}$ there exists an integer $s\geqq 2$ with $\operatorname{Hom}(X,\tau^sX)\neq 0$ but $\operatorname{Hom}(X,\tau^{s+1}X)=0$. It was shown in [8] that in this case X is not τ-sincere and a stone. More precise, by [8] we may assume that $A=A'[M]$ is a one-point extension of some algebra A' by a projective module $M=\operatorname{rad}P(\omega)$ (ω the extension vertex). Moreover we may assume that $X\in A'$-mod and there is a short exact sequence

$$0\to X\to\tau^sX\to I(\omega)\to 0\,.$$

Proposition 4.1 *Let A be wild hereditary with an exceptional component $\mathscr{C}$ such that for X quasi-simple in $\mathscr{C}$ there is $s\geqq 2$ with* $\operatorname{Hom}(X,\tau^sX)\neq 0$, $\operatorname{Hom}(X,\tau^{s+1}X)=0=\operatorname{Hom}(X,\tau^{s+2}X)$. *If E is an elementary module in A-mod then* $\operatorname{Hom}(\tau^jE,I(\omega))=0$ *for some $j\in\mathbb{Z}$.*

Proof. As mentioned above, we may assume the existence of a short exact sequence

$$0\to X\to\tau^sX\to I(\omega)\to 0\,.$$

If E' is elementary there exists $i\in\mathbb{Z}$ such that for $E=\tau^iE'$ we have $\operatorname{Hom}(X,\tau E)=0$, but $\operatorname{Hom}(X,\tau^2E)\neq 0$. Applying $\operatorname{Hom}(E,-)$, we get

$$0\to\operatorname{Hom}(E,X)\to\operatorname{Hom}(E,\tau^sX)\to\operatorname{Hom}(E,I(\omega))\to\operatorname{Ext}^1(E,X)=0\,.$$

Let us show that $\operatorname{Hom}(E,\tau^sX)=0$. Assume there is $g\in\operatorname{Hom}(E,\tau^sX)\setminus\{0\}$ and take any $f\in\operatorname{Hom}(X,\tau^zE)\setminus\{0\}$. Then by 1.3 the composition $f\circ\tau^2g$ is a nonzero map from X to $\tau^{s+2}X$, a contradiction. Thus $\operatorname{Hom}(E,I(\omega))=0$ holds.

Exceptional components can be constructed in the following way: Let $\tilde{\mathscr{Q}}$ be an Euclidean quiver with more than two vertices and path-algebra B, let x be an extending vertex of $\tilde{\mathscr{Q}}$, that is $\mathsf{n}(x)=1$, when n is the dimension vector of the quasi-simple homogenous modules. Then each of the inhomogenous tubes of B gives rise to an exceptional component of the algebra $A = B[P(x)]$ in the following way:

If $\mathscr{T} \subset \Gamma(B)$ is a tube of rank s, if $X \in \mathscr{T}$ is the quasi-simple module with $\underline{\dim} X(x) = 1$, then we have $X, \tau_A X, \ldots, \tau_A^{s-1} X \in B$-mod, $\mathrm{Hom}(X, \tau^s X) \cong k$ with short exact sequence $0 \to X \to \tau^s X \to I(\omega) \to 0$ and $\mathrm{Hom}(X, \tau^{s+1} X) = 0$. We say that these exceptional components, corresponding to the inhomogenous tubes of B, are defined by B, see [8, Sect. 4].

If A has an exceptional component defined by a tube of rank s with $\mathrm{Hom}(X, \tau^s X) \neq 0$ but $\mathrm{Hom}(X, \tau^{s+1} X) = 0 = \mathrm{Hom}(X, \tau^{s+2} X)$ then elementary modules with self-extensions are modulo Auslander–Reiten shift quasi-simple homogenous in B-mod. With these notations we get

Lemma 4.2 *Let B be a tame hereditary algebra and $A = B[P(x)]$ a wild hereditary algebra with exceptional components defined by B. Suppose there is a quasi-simple homogenous B-module E which is elementary in A-mod. If $\mathscr{C}$ is an exceptional component in $\Gamma(A)$ defined by a tube $\mathscr{T} \subset \Gamma(B)$ of rank s, if X is quasi-simple in $\mathscr{C}$, then $\mathrm{Hom}(X, \tau^{s+3+j} X) \neq 0$ for all $j \geqq 0$ holds.*

Proof. By [7, 4.3] we may assume that the modules $X, \tau_A X = \tau_B X, \ldots, \tau_A^{s-1} X = \tau_B^{s-1} X$ form the mouth of the tube $\mathscr{T}$. Especially we get $\mathrm{Hom}(\tau E, \tau^s X) = 0 = \mathrm{Ext}(\tau E, \tau^s X)$.

First we will show that $\mathrm{Hom}(X, \tau^{2+j} E) \neq 0$ for all $j \geqq 0$. Applying the functor $\mathrm{Hom}(\tau E, -)$ to the short exact sequence

$$0 \to X \to \tau^s X \to I(\omega) \to 0$$

we get the long exact sequence

$$\cdots \to (\tau E, \tau^s X) \to (\tau E, I(\omega)) \to {}^1(\tau E, X) \to {}^1(\tau E, \tau^s X) \to \cdots .$$

From $E \cong \tau_B E$ and $(\underline{\dim} E)(x) = 1$ we get $\mathrm{Hom}(\tau E, I(\omega)) \cong k$ and therefore $k \cong \mathrm{Ext}(\tau E, X) \cong D\,\mathrm{Hom}(X, \tau^2 E)$.

As E has self-extensions $\mathrm{Hom}(\tau^2 E, \tau^{2+j} E) \neq 0$ for all $j \geqq 0$ holds. If $f: X \to \tau^2 E$ is nonzero and $g: \tau^2 E \to \tau^{2+j} E$ is nonzero, then the composition $fg: X \to \tau^{2+j} E$ is nonzero since E is elementary, see 1.3.

Next we prove that $\mathrm{Hom}(\tau^- E, \tau^s X)$ is nonzero. For this we apply $\mathrm{Hom}(\tau^- E, -)$ to the above short exact sequence and get

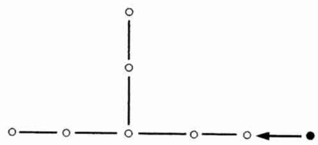
$$0 \to (\tau^- E, X) \to (\tau^- E, \tau^s X) \to (\tau^- E, I(\omega)) \to {}^1(\tau^- E, X) \to \cdots .$$

Now $\mathrm{Hom}(\tau^- E, I(\omega)) \neq 0$ holds: Otherwise we would have $\tau_A^- E = \tau_B^- E = E$, a contradiction. Moreover we have $\mathrm{Ext}^1(\tau^- E, X) \cong D\,\mathrm{Hom}(X, E) = 0$ hence $\mathrm{Hom}(\tau^- E, \tau^s X) \neq 0$ holds.

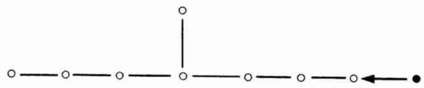
Finally, if $g \in \mathrm{Hom}(\tau^- E, \tau^s X)$ is nonzero and $f \in \mathrm{Hom}(X, \tau^{2+j} E)$ is nonzero, then the composition $f \circ \tau^{3+j} g$ is nonzero in $\mathrm{Hom}(X, \tau^{s+3+j} X)$ for all $j \geqq 0$.

Let denote by $\tilde{\tilde{E}}_n$ $(n = 6, 7, 8)$ and $\tilde{\tilde{D}}_n$ $(n \geqq 4)$ the following graphs with $n+2$ vertices

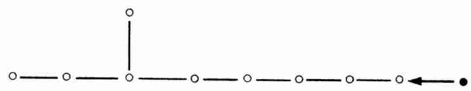
$\tilde{\tilde{E}}_6$



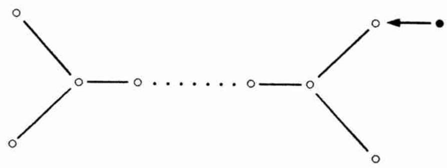
$\tilde{\tilde{E}}_7$



$\tilde{\tilde{E}}_8$



$\tilde{\tilde{D}}_n$



Theorem 4.3 *Let A be the path-algebra of a quiver of type $\tilde{\tilde{E}}_n$ $(n = 6, 7, 8)$ or $\tilde{D}_n$ $(n \geqq 6)$. Then all elementary modules are stones. Especially there exist only finitely many τ-orbits of elementary modules.*

Proof. We may suppose that A is a one-point extension of a tame algebra B with underlying graph $\tilde{E}_n$ $(n = 6, 7, 8)$ or $\tilde{D}_n$ $(n \geqq 6)$ where the black vertex in the above picture is the extension vertex. By [8] table, therefore the algebra A has an exceptional component $\mathscr{C}$ defined by a tube $\mathscr{T}$ of period s such that $\operatorname{Hom}(X, \tau^s X) \neq 0$, $\operatorname{Hom}(X, \tau^{s+1} X) = 0 = \operatorname{Hom}(X, \tau^{s+2} X)$. Therefore each τ-orbit of an elementary module contains a module $E \in B$-mod. Consulting again [8] table, Lemma 4.2 tells for the cases $\tilde{\tilde{E}}_7$, $\tilde{\tilde{E}}_8$ and $\tilde{\tilde{D}}_n$ $(n \geqq 6)$, that E has to be a stone.

For the case $\tilde{\tilde{E}}_6$ we need an argument from L. Unger: If E is a quasi-simple homogenous B-module, then we have in B-mod the short exact sequence $0 \to X \to E \to Y \to 0$ where X is indecomposable preprojective with

$$\underline{\dim} X = \begin{array}{ccccc} & & 0 & & \\ & & 1 & & \\ 1 & 2 & 2 & 1 & 0 \end{array}$$

and Y is indecomposable with

$$\underline{\dim}\, Y = \begin{matrix} & & 1 & & \\ & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{matrix} .$$

As A-modules the modules X and Y become regular and thus E is not elementary.

5 Tilted algebras

In the simplest case a wild tilted algebra A which is not concealed is of the form $A = A_0[M]$ with A_0 wild concealed and connected and ${}_{A_0}M$ regular (or dually the one-point coextension $[N]A_0$ with N regular). Our aim is to find properties of M in this case.

If M is a B-module for some algebra B and $\alpha: B \to B$ is a k-linear automorphism of B, the map $s: B \otimes_k M \to M$, given by $s(b \otimes m) = \alpha(b)m$ defines another B-module structure on M, which we denote by ${}_{\alpha}M$. The proof of the following lemma is straightforward.

Lemma 5.1 *Let B be a finite dimensional algebra.*

(a) *For a B-module M and $\alpha \in \mathrm{Aut}(B)$ the algebra $B[M]$ and $B[{}_{\alpha}M]$ are isomorphic.*

(b) *Let M and M' be B-modules and $\beta: B[M] \to B[M']$ an isomorphism with $\beta(B) = B$. If $\alpha \in \mathrm{Aut}(B)$ is the restriction of β to B, then $M' \cong_{\alpha} M$ holds.*

For an algebra C and two sets $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ of pairwise orthogonal primitive idempotents of C, there exists an inner automorphism $\alpha \in \mathrm{Aut}(C)$ and some permutation $\pi \in S(n)$ with $\alpha(f_i) = e_{\pi(i)}$. Using this fact, we deduce from (5.1)

Lemma 5.2 *Let A_0 be a concealed algebra, let M and M' be regular A_0-modules and let $\beta: A_0[M'] \to A_0[M]$ be an isomorphism. Then there exists an inner automorphism $\alpha \in \mathrm{Aut}(A_0[M])$ such that $\alpha\beta(A_0) = A_0$. Consequently we have $M \cong_{\gamma} M'$ for some $\gamma \in \mathrm{Aut}(A_0)$.*

Theorem 5.3 *Let A_0 be a concealed algebra, let E be a regular A_0-module such that $A_0[E]$ is a tilted algebra. Then we have*

(a) *E is elementary*

(b) *$\tau_{A_0}E$ has no trivial regular factor modules.*

Proof. Let $A_0[E]$ be tilted, say of type A. Then there exists a tilting module T in A-mod such that $\mathrm{End}(T) \cong A_0[E]$. As $A_0[E]$ has a preprojective component containing $n-1$ projective modules, where n denotes the number of simple A-modules, the tilting module T has a decomposition $T = X \oplus T_0$ such that $F(T_0)$ is preprojective in $\mathrm{End}(T)$-mod with $\mathrm{End}(T_0) = A_0$ and $F(X)$ is not preprojective, with $F = \mathrm{Hom}(T, -)$. By definition we have $T_0 \in X^\perp$. Clearly X is regular, even quasi-simple regular in A-mod, see [7, Sect. 2]. Following the notation of Sect. 3 we denote by M the minimal projective generator of $X^\perp$, by H the functor $H = \mathrm{Hom}(M, -): X^\perp \to C$-mod, where $C = \mathrm{End}(M)$.

As $T_0 \in X^\perp$, the module $H(T_0)$ is a tilting module in C-mod; moreover $H(T_0)$ is preprojective and A_0 is concealed of type C. From 3.1 we know that $Z = H([2]X)$ is elementary in C-mod and moreover $\tau_C Z$ is without nontrivial regular factor-modules. Hence $\tilde{Z} := \mathrm{Hom}(H(T_0), Z)$ is elementary in A_0-mod, too. As $\mathrm{Hom}(H(T_0), \tau_C Z) \cong \tau_{A_0}\tilde{Z}$ the module $\tau_{A_0}\tilde{Z}$ has only trivial regular factor-modules.

Applying the functor $\mathrm{Hom}(T_0, -)$ to the Auslander–Reiten sequence

$$0 \to \tau X \to [2]X \to X \to 0$$

we see $\mathrm{Hom}(T_0, X) \cong \mathrm{Hom}(T_0, [2]X)$ as A_0-module. Thus we have

$$A_0[E] \cong \begin{bmatrix} \mathrm{End}(T_0) & \mathrm{Hom}(T_0, X) \\ 0 & \mathrm{End}(X) \end{bmatrix} \cong \begin{bmatrix} A_0 & \mathrm{Hom}(T_0, [2]X) \\ 0 & k \end{bmatrix}$$

$$\cong \begin{bmatrix} \mathrm{End}(H(T_0)) & \mathrm{Hom}(H(T_0), Z) \\ 0 & k \end{bmatrix} \cong \begin{bmatrix} A_0 & \tilde{Z} \\ 0 & k \end{bmatrix}.$$

From 5.1 we deduce that $E \cong \gamma\tilde{Z}$ for some automorphism $\gamma \in \mathrm{Aut}(A_0)$. Hence with $\tilde{Z}$ also E has the desired properties.

Corollary 5.4 *Let A_0 be wild concealed and E elementary such that $A_0[E]$ is tilted. Then there exists a positive integer N with $A_0[\tau^{-i}E]$ is not tilted for all $i \geqq N$.*

Proof. Choose N such that $\tau^{-i+1}E$ has nontrivial regular factor-modules for all $i \geqq N$.

In contrast to 5.4 we have

Proposition 5.5 *Let A_0 be wild concealed and E elementary in A_0-mod such that $A_0[E]$ is tilted of type A. Then $A_0[\tau^i E]$ is tilted of type A for all $i \geqq 0$.*

Proof. It is enough to show $A_0[\tau E]$ is tilted provided $A_0[E]$ is. We use the same notation as in 5.3: We have an A-tilting module $T = T_0 \oplus X$ with $\mathrm{End}(T) = A_0[E]$, $\mathrm{End}(T_0) = A_0$, the minimal Ext-projective generator in $X^\perp$ is called M, its ring of endomorphisms is denoted by C and H denotes the equivalence

$$H = \mathrm{Hom}(M, -): X^\perp \to C\text{-mod}.$$

Moreover we know that $H(T_0)$ is a preprojective tilting module in C-mod. If P_ω denotes the projective $C' = \mathrm{End}(X \oplus M)$-module $\mathrm{Hom}(X \oplus M, P_\omega)$ one easily checks that $P_\omega \oplus \tau_C^- H(T_0)$ is tilting module in C'-mod. Note that $Z := \mathrm{rad}\, P_\omega = H([2]X)$. As in 5.3 we get

$$\mathrm{Hom}(\tau_C^- H(T_0) P_\omega) = \mathrm{Hom}(\tau_C^- H(T_0), Z) \cong \mathrm{Hom}(H(T_0), \tau_C Z).$$

Thus, as in [12, 4.7(4)] we get

$$\begin{aligned}\operatorname{End}(\tau_C^- H(T_0) \oplus P_\omega) &= \begin{bmatrix} \operatorname{End}(\tau_C^- H(T_0)) & \operatorname{Hom}(\tau_C^- H(T_0), P_\omega) \\ 0 & \operatorname{End}(P_\omega) \end{bmatrix} \\ &\cong \begin{bmatrix} A_0 & \operatorname{Hom}(H(T_0), \tau_C Z) \\ 0 & k \end{bmatrix} \cong A_0[\tau E] ,\end{aligned}$$

where the last isomorphism again uses 5.3. If we apply now the functor

$$G = (M \oplus P_\omega) \otimes - : C[Z]\text{-mod} \to A\text{-mod}$$

we see that $G(\tau_C^- H(T_0) \oplus P_\omega)$ is a tilting module in A-mod with endomorphism ring isomorphic to $A_0[\tau E]$.

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