Math. Z. 223, 387 – 395 (1996)

Mathematische Zeitschrift

Least area Tori and 3-manifolds of nonnegative scalar curvature

Mingliang Cai, Gregory J. Galloway
Department of Mathematics and Computer Science, University of Miami, USA

Received 14 April 1994; in final form 20 March 1995

1. Introduction

Gibbons [Gi] and Schoen and Yau [SY] showed that if Σ is a compact orientable stable minimal surface in an orientable 3-manifold M of positive scalar curvature $S > 0$ then Σ must be a sphere. This result played a fundamental role in the seminal study of Schoen and Yau of the topology of 3-manifolds of nonnegative scalar curvature. If M is merely assumed to have nonnegative scalar curvature $S \geqq 0$ then Σ can also be a torus, but only under special circumstances. As was observed in Fisher–Colbie–Schoen [FCS], if Σ is a stable minimal torus in an orientable 3-manifold M of nonnegative scalar curvature then Σ must be flat and totally geodesic, and the scalar curvature and normal Ricci curvature of M along Σ must vanish. To loosely paraphrase: if a torus $\Sigma \subset M$ (where M has $S \geqq 0$) is infinitesimally of least area then M splits infinitesimally along Σ. The problem of establishing a noninfinitesimal version of this result, which we address in this paper, has remained unresolved. The aim is to show that if Σ is locally of least area then a neighborhood of Σ splits along Σ. We are lead to formulate the following conjecture.

Conjecture. *Let M^3 be an orientable 3-manifold of nonnegative scalar curvature. Suppose Σ is a torus in M which is locally of least area, i.e., which is of least area among all nearby surfaces isotopic to it. Then M splits in a neighborhood of Σ, i.e., there is a neighborhood U of Σ which is isometric to $(-\varepsilon, \varepsilon) \times \Sigma$ and, hence, which is flat.*

Our main theorem settles this conjecture in the analytic case.

Theorem A. *If M^3 is analytic, the conjecture is true.*

Although our proof requires analyticity, it does provide information in the C^∞ case. Some of this information is summarized in Theorem B below.

This work has been partially supported by NSF grant DMS-9204372.

Let $\{\Sigma_t\}$ be the normal unit speed geodesic variation of Σ, i.e., for small t, Σ_t is the surface obtained by pushing Σ a signed distance t along its normal geodesics in the direction of a given unit normal N. Extend N to be the unit normal field to all of the Σ_t's. Let $B = B_t$, $H = H_t$, and $K = K_t$ denote, respectively, the second fundamental form, mean curvature, and Gaussian curvature of Σ_t. Let ∇, S and Ric denote, respectively, the Levi–Civita connection, scalar curvature and Ricci curvature of M.

As follows easily from Eqs. (13) and (14) in Sect. 2, to prove the conjecture it suffices to show that $H_t = 0$ for all t sufficiently small. Hence, in view of the analyticity assumption, to prove Theorem A it suffices to show that all t-derivatives of $H = H_t$ vanish along Σ, $\frac{\partial^n H}{\partial t^n}|_{t=0} = 0$. (In fact this is the only way in which analyticity is used.) In the course of showing this we will actually prove the following.

Theorem B. *Let M^3 be an orientable C^∞ 3-manifold of nonnegative scalar curvature. With respect to the variation $\{\Sigma_t\}$ of Σ described above, we have at $t = 0$ (i.e., along Σ)*

$$\nabla_N^{(n)} B = \frac{\partial^n H}{\partial t^n} = \frac{\partial^n K}{\partial t^n} = \frac{\partial^n S}{\partial t^n} = \frac{\partial^n \mathrm{Ric}(N)}{\partial t^n} = 0\,,$$

for all nonnegative integers n.

In Sect. 2 we present the proofs of Theorems A and B, and consider some global consequences. We also briefly discuss how the conjecture relates to certain aspects of the theory of black holes (see [G2] for a more detailed discussion).

Recall, as noted above, the work of Schoen and Yau [SY] applies to 3-manifolds of nonnegative scalar curvature, $S \geqq 0$. As such, Schoen and Yau must themselves contend with the torus case. In their work the torus case is handled by certain global topological assumptions: in their applications, M^3 is compact (without boundary) and the torus is incompressible. Thus, they can make use of the fact that if M^3 has nonnegative scalar curvature then either it is flat or it can be conformally deformed to a manifold of strictly positive scalar curvature. This technique is not directly applicable to the situation considered here.

2. Proofs and consequences

We begin by establishing a basic identity which is essential to our proof of Theorem B. Let N be a smooth vector field on a smooth Riemannian 3-manifold M^3. Let $R(N)$ be the smooth vector field defined by,

$$\mathrm{Ric}(N, Y) = \langle R(N), Y\rangle \quad \text{for all } Y\,,$$

where $\langle\,,\rangle$ is the metric and Ric is the Ricci tensor. Now suppose N is *hypersurface orthogonal*, i.e., suppose the normal plane field to N is integrable. Then N determines a foliation of M by smooth surfaces. Let $K : M \to \mathbb{R}$ denote the Gaussian curvature function of this foliation, i.e., for each $p \in M$, $K(p)$ is the Gaussian curvature at p of the leaf passing through p.

Lemma. *Let N be a hypersurface orthogonal unit geodesic vector field on a smooth Riemannian 3-manifold M^3. Then N obeys the following identity,*

$$\operatorname{div}[R(N) - (K + \operatorname{Ric}(N))N] = 0\,, \tag{1}$$

where $\operatorname{Ric}(N) = \operatorname{Ric}(N,N)$.

The lemma is used below to obtain an evolution equation for $K = K(x,t)$ (see Eq. (16)), which plays an essential role in our proof of Theorem A.

Proof of the Lemma. The lemma is a consequence of the contracted differential Bianchi identity,

$$R^j_{k;j} = \frac{1}{2}\frac{\partial S}{\partial x^k}\,, \tag{2}$$

where R_{ij} are the components of the Ricci tensor and S is the scalar curvature. From (2) we obtain,

$$\begin{aligned} \operatorname{div}(R(N)) &= (R^j_k N^k)_{;j} = R^j_{k;j} N^k + R^j_k N^k_{;j} \\ &= \frac{1}{2} N(S) + \sum_{i=1}^{2} \operatorname{Ric}(e_i, \nabla_{e_i} N)\,, \end{aligned} \tag{3}$$

where, at a given point p, $\{e_1, e_2\}$ is an orthonormal basis of the tangent space of the leaf through p. To interpret the second term on the right hand side of (3) we choose e_1 and e_2 to be principal directions for the leaf through p. Without loss of generality, we may assume that e_1 and e_2 can be smoothly extended to a neighborhood U of p in M such that at each point $q \in U$, e_1 and e_2 are orthonormal principal directions for the leaf through q. To see this, first note that, by continuity, it is sufficient to establish (1) on a dense subset of M. The set of nonumbilic points of all the leaves union the interior of the set of umbilic points is a dense subset of M at each point of which the desired extension holds. Thus on the neighborhood U of p we have,

$$\nabla_{e_i} N = \kappa_i e_i, \quad i = 1,2\,, \tag{4}$$

where for each $q \in U$, $\kappa_1(q)$ and $\kappa_2(q)$ are the principal curvatures of the leaf passing through q. For $1 \leqq i,\ j \leqq 3$, set $R_{ij} = \operatorname{Ric}(e_i, e_j)$ and let K_{ij} denote the sectional curvature of the plane spanned by e_i and e_j (where $e_3 = N$). Then by using (4) and, where appropriate, expressing Ricci curvatures in terms of sectional curvatures we obtain,

$$\begin{aligned} \sum_{i=1}^{2} \operatorname{Ric}(e_i, \nabla_{e_i} N) &= \kappa_1 R_{11} + \kappa_2 R_{22} \\ &= (\kappa_1 + \kappa_2)(K_{12} + R_{33}) - \kappa_1 K_{23} - \kappa_2 K_{13}\,. \end{aligned} \tag{5}$$

By a straight forward computation, which we carry out below, one has,

$$K_{i3} = -N(\kappa_i) - \kappa_i^2, \quad i = 1,2\,. \tag{6}$$

Substitution of these equations into (5) gives,

$$\begin{aligned}\sum_{i=1}^{2} \mathrm{Ric}(e_i, \nabla_{e_i} N) &= (\kappa_1 + \kappa_2)(K_{12} + \kappa_1\kappa_2 + R_{33}) + N(\kappa_1\kappa_2) \\ &= (K + \mathrm{Ric}(N))\mathrm{div}(N) + N(\kappa_1\kappa_2)\,,\end{aligned} \tag{7}$$

where to obtain the second equation we have used the Gauss equation. Substitution of (7) into (3) then gives,

$$\begin{aligned}\mathrm{div}(R(N)) &= (K + \mathrm{Ric}(N))\mathrm{div}(N) + N\left(\tfrac{1}{2}S + \kappa_1\kappa_2\right) \\ &= (K + \mathrm{Ric}(N))\mathrm{div}(N) + N(K + \mathrm{Ric}(N)) \\ &= \mathrm{div}[(K + \mathrm{Ric}(N))N]\,,\end{aligned}$$

which establishes (1) given (6).

To complete the proof of the lemma we compute $K_{13} = \langle R(e_1, N)N, e_1\rangle$. We have,

$$\begin{aligned}R(e_1, N)N &= \nabla_{e_1}\nabla_N N - \nabla_N \nabla_{e_1} N - \nabla_{[e_1, N]} N \\ &= -\nabla_N(\kappa_1 e_1) - \nabla_{\kappa_1 e_1 - \nabla_N e_1} N \\ &= (-N(\kappa_1) - \kappa_1^2)e_1 - k_1 \nabla_N e_1 + \nabla_{\nabla_N e_1} N\,.\end{aligned}$$

By taking the scalar product of the left and right hand sides of the above equation with e_1, and observing that $\nabla_N e_1$ is perpendicular to both e_1 and N (and hence is a multiple of e_2), we obtain the desired expression for K_{13}.

We proceed to the proof of Theorem B.

Proof of Theorem B. Let $\Phi : (-\varepsilon, \varepsilon) \times \Sigma \to M$ be the normal exponential map of Σ, $\Phi(t,x) = exp_x tN$, where N is a smooth unit normal field along Σ. Choose ε sufficiently small so that Φ is a diffeomorphism onto $U = \Phi((-\varepsilon,\varepsilon) \times \Sigma)$. For each $t \in (-\varepsilon, \varepsilon)$, let $\Sigma_t = \Phi(\{t\} \times \Sigma)$, i.e., Σ_t is the surface obtained by pushing Σ out along its normal geodesics a signed distance t in the direction of N. Extend N to be the unit normal field to the Σ_t's, $N = \Phi_*\left(\frac{\partial}{\partial t}\right)$. Below we will frequently identify, via Φ, points in U with points in $(-\varepsilon,\varepsilon) \times \Sigma$, so that, for instance, scalar fields $f : U \to \mathbb{R}$ may be viewed as functions of $t \in (-\varepsilon,\varepsilon)$ and $x \in \Sigma$, $f = f(t,x)$.

Let $B = B_t$ be the second fundamental form of Σ_t; thus for vectors $X, Y \in T_p\Sigma_t$, $B(X,Y) = \langle \nabla_X N, Y\rangle$. Let $K = K_t$ and $H = H_t = \mathrm{tr}\, B_t$ denote the Gaussian curvature and mean curvature, respectively, of Σ_t.

Let $\{e_1, e_2\}$ be a smooth orthonormal frame of Σ. Extend e_1 and e_2 to U by parallelly translating them along the normal geodesics to Σ. For $1 \leqq i,\ j \leqq 2$, set $\lambda_{ij} = B(e_i, e_j)$. One has,

$$\nabla_N^{(n)} B(e_i, e_j) = \frac{\partial^n \lambda_{ij}}{\partial t^n}, \quad 1 \leqq i,\ j \leqq 2\,. \tag{8}$$

A straight forward computation using the Gauss equation shows that the normal Ricci curvature $\mathrm{Ric}(N)$, scalar curvature S, B and K are related by,

$$\mathrm{Ric}(N) = \frac{1}{2}S - \frac{1}{2}|B|^2 + \frac{1}{2}H^2 - K\,, \tag{9}$$

where in terms of the components λ_{ij},

$$|B|^2 = \sum_{i,j=1}^{2} \lambda_{ij}^2 \quad \text{and} \quad H = \sum_{i=1}^{2} \lambda_{ii}\,. \tag{10}$$

Thus, in view of (8), (9) and (10), to prove Theorem B it is sufficient to establish the following claim.

Claim. *At* $t = 0$ (*i.e., along* Σ) *we have,*

$$\nabla_N^{(n)} B = 0 \ (\textit{i.e.}, \lambda_{ij}^{(n)} = 0, 1 \leqq i,j \leqq 2) \ \textit{and} \ S^{(n)} = K^{(n)} = 0\,, \tag{11}$$

for all nonnegative integers n (*where* $S^{(n)} = \frac{\partial^n S}{\partial t^n}$, *etc.*).

The proof of the claim requires several formulas. The mean curvature $H = H(t,x)$ of the foliation $\{\Sigma_t\}$ obeys the Ricatti equation,

$$\frac{\partial H}{\partial t} = -\mathrm{Ric}(N) - |B|^2\,,$$

which, taken together with (9), gives,

$$\frac{\partial H}{\partial t} = -\frac{1}{2}S - \frac{1}{2}|B|^2 - \frac{1}{2}H^2 + K\,. \tag{12}$$

For $t \in (-\varepsilon, \varepsilon)$, let $A(t) =$ area of Σ_t. The first and second variation formulas imply,

$$A'(t) = \int_{\Sigma_t} H\, dx_t\,, \tag{13}$$

and,

$$A''(t) = -\int_{\Sigma_t} (\mathrm{Ric}(N) + |B|^2 - H^2)\, dx_t\,.$$

Since, by the Gauss–Bonnet theorem, $\int_{\Sigma_t} K\, dx_t = 0$, substitution of (9) into the above equation gives,

$$A''(t) = -\frac{1}{2}\int_{\Sigma_t} (S + |B|^2 - H^2)\, dx_t\,. \tag{14}$$

The area form dx_t on Σ_t is related to the area form dx on Σ by, $dx_t = \mu(t,x)dx$, where μ satisfies, $\mu(0,x) = 1$. By choosing ε small enough, we may assume

$$\frac{1}{2} \leqq \mu(t,x) \leqq 2, \quad \text{for all } t \in (-\varepsilon,\varepsilon) \text{ and } x \in \Sigma\,.$$

Equation (14) can now be written as,

$$A''(t) = -\frac{1}{2}\int_\Sigma (S + |B|^2 - H^2)\mu\, dx \,. \tag{15}$$

If X is a vector field on U, let X^T denote the orthogonal projection of X onto each leaf Σ_t; $\operatorname{div} X$ and $\operatorname{div}_{\Sigma_t} X^T$ are related by,

$$\operatorname{div} X = \operatorname{div}_{\Sigma_t} X^T + \langle X, N\rangle H + \langle \nabla_N X, N\rangle \,.$$

Setting $X = R(N)$ in the above equation and simplifying yields,

$$\operatorname{div}_{\Sigma_t} R(N)^T = \operatorname{div}(R(N) - \operatorname{Ric}(N)N) \,.$$

By combining this equation with (1) we obtain,

$$\frac{\partial K}{\partial t} = -KH + \operatorname{div}_{\Sigma_t} R(N)^T \,. \tag{16}$$

Equations (12), (15) and (16) enable us to control the t-derivatives of B, S and K along Σ. During the course of the proof of the claim we make frequent use of the product rule,

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \,.$$

The proof of the claim is by induction on n. The case $n = 0$ is just the result of Fisher–Colbie–Schoen [FCS] referred to in the introduction. Now assume the claim is true for $0 \leqq n \leqq m - 1$,

$$\lambda_{ij}^{(n)}(0,x) = S^{(n)}(0,x) = K^{(n)}(0,x) = 0, \quad 0 \leqq n \leqq m-1 \,, \tag{17}$$

for all $x \in \Sigma$. We first show that $S^{(m)}(0,x) = 0$ for all x. Note that since the scalar curvature is nonnegative we must have $S^{(m)}(0,x) \geqq 0$ for all x.

By differentiating both sides of (12) with respect to t $(m-1)$-times and using (17) it follows that,

$$H^{(n)}(0,x) = 0, \quad 0 \leqq n \leqq m \,, \tag{18}$$

and hence by the product rule,

$$(H^2)^{(n)}(0,x) = 0, \quad 0 \leqq n \leqq 2m+1 \,. \tag{19}$$

Thus, by Taylor's theorem, there exists a bounded function $f = f(t,x)$, $|f(t,x)| \leqq C$, say, such that,

$$S - H^2 = \frac{S^{(m)}(0,x)}{m!} t^m + f(t,x) t^{m+1} \,.$$

Hence, from (15),

$$\begin{aligned}A''(t) &\leqq -\frac{1}{2}\int_\Sigma (S-H^2)\mu\,dx\\ &\leqq -\frac{t^m}{2m!}\int_\Sigma S^{(m)}(0,x)\mu(t,x)dx + \frac{t^{m+1}}{2}\int_\Sigma |f(t,x)|\mu(t,x)dx\\ &\leqq -\frac{t^m}{4m!}\int_\Sigma S^{(m)}(0,x)dx + C\cdot A(0)t^{m+1}\,.\end{aligned}$$

If $\int_\Sigma S^{(m)}(0,x)dx > 0$ then there exists $\delta > 0$ such that $A''(t) < 0$ for all $t \in (0,\delta)$. But since $A'(0) = 0$ (Σ is minimal), this contradicts the assumption that Σ is locally of least area. Hence, $\int_\Sigma S^{(m)}(0,x)dx = 0$ and, thus, $S^{(m)}(0,x) = 0$ for all x.

From (17) and the product rule we have $(|B|^2)^{(n)}(0,x) = 0$, $0 \leqq n \leqq 2m-1$. Applying a similar argument to that given above, one obtains $(|B|^2)^{(2m)}(0,x) = 0$, where one now uses the full strength of (19). In terms of components, $|B|^2 - H^2 = 2(\lambda_{12}^2 - \lambda_{11}\lambda_{22})$. Thus, along Σ we have,

$$(\lambda_{12}^2 - \lambda_{11}\lambda_{22})^{(2m)} = 0\,,$$

which, by the product rule and (17), implies,

$$(\lambda_{12}^{(m)})^2 - \lambda_{11}^{(m)}\lambda_{22}^{(m)} = 0\,.$$

Using $H^{(m)} = \lambda_{11}^{(m)} + \lambda_{22}^{(m)} = 0$ along Σ in the above equation we conclude $\lambda_{11}^{(m)}(0,x) = \lambda_{22}^{(m)}(0,x) = \lambda_{12}^{(m)}(0,x) = 0$ for all x, i.e., $\nabla_N^{(m)}B = 0$ along Σ.

It remains to show that $K^{(m)}(0,x) = 0$. In view of (16) and (18) it is sufficient to show that

$$(\operatorname{div}_{\Sigma_t} R(N)^T)^{(m-1)} = 0 \tag{20}$$

along Σ. To this end, we introduce Gaussian normal coordinates $x^0 = t, x^1, x^2$ on $U \approx (-\varepsilon,\varepsilon)\times\Sigma$. In these coordinates, $g_{00} = \langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\rangle = 1$ and $g_{0j} = \langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^j}\rangle = 0, 1 \leqq j \leqq 2$. Set $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$ and $b_{ij} = B(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), 1 \leqq i,j \leqq 2$; observe that $b_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial t}$. The transformation law for tensors and the product rule imply that $\lambda_{ij}^{(n)} = 0$, $0 \leqq n \leqq m-1$ if and only if $b_{ij}^{(n)} = 0$, $0 \leqq n \leqq m-1$. Hence, from (17),

$$b_{ij}^{(n)}(0,x) = 0, \quad 0 \leqq n \leqq m-1\,. \tag{21}$$

Using a standard coordinate expression for the divergence we have,

$$\operatorname{div}_{\Sigma_t} R(N)^T = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}R_{0j})\,, \tag{22}$$

where $g = \det(g_{ij})$, (g^{ij}) is the inverse of (g_{ij}), $R_{0j} = \mathrm{Ric}(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^j})$, and repeated indices run from 1 to 2 only. A straight forward computation yields,

$$R_{0j} = \frac{\partial b^k_j}{\partial x^k} - \frac{\partial b^k_k}{\partial x^j} + \Gamma^i_{ik} b^k_j - \Gamma^k_{ij} b^i_k \,, \tag{23}$$

where $b^k_j = b_{ji} g^{ik}$ and the Γ^k_{ij}'s are the Christoffel symbols. Equation (20) now follows from (21), (22), (23) and the product rule. This completes the proof of the claim and Theorem B.

Proof of Theorem A. Let the notation be as in the proof of Theorem B. The analyticity assumption and Theorem B imply that the mean curvature function vanishes identically on U, $H \equiv 0$. Hence, by (13), we have $A''(t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Equation (14) then implies that $B \equiv 0$ on U. Thus, N is parallel and Φ is an isometry.

By fairly routine continuation arguments one obtains the following global version of Theorem A.

Theorem C. *Let M be a connected orientable analytic* 3*-manifold of nonnegative scalar curvature, and suppose M contains a torus Σ which is locally of least area. Then M is flat. If M is geodesically complete then M is isometric to, or double covered by, either a flat* 3*-torus or $\mathbb{R} \times \Sigma$. If M is compact with mean convex boundary then M is isometric to, or double covered by,* $[0, \ell] \times \Sigma$.

Naturally, one would like to resolve whether or not the analyticity assumption in Theorem A is needed. Theorem A bears some relationship to a splitting result of Anderson and Rodriguez ([AR], Theorem 1) for 3-manifolds which contain complete noncompact least area surfaces. Their result does not require analyticity, but assumes nonnegative Ricci curvature. Unfortunately, their geometric method does not appear to carry over to the situation considered here. Still, we feel rather strongly that the analyticity assumption in Theorem A is unnecessary. Moreover, we feel it may be possible to resolve the C^∞ case within the framework of this paper. Another possible approach to the conjecture is to consider more general variations. For example, one can consider variations $\{\Sigma_t\}$ obtained by pushing Σ along its normal geodesics at unit speed in some conformally related metric. The conjecture would follow easily if one could establish the existence in a neighborhood of Σ of a conformal change of metric such that, in this new metric, the tori at constant distance from Σ are flat. What makes this problem difficult is that the conformal deformation affects not just the curvature of the tori, but also their position; deforming to zero curvature a *fixed* family of tori is simple.

A fundamental result in the theory of black holes due to Hawking (cf. [HE]) asserts that, for stationary black hole spacetimes which obey the dominant energy condition, black hole boundaries are topologically spherical. The proof of Hawking's theorem and the theorem of Gibbons [Gi] and Schoen and Yau [SY] discussed in the introduction are similar in several respects. A black hole boundary need not minimize area, but satisfies some other related criterion. In Hawking's proof the torus also arises as a borderline case. The argument given in [HE] to eliminate the torus case is not so clear. (For instance, one encounters a certain rescaling problem, related to the conformal deformation problem mentioned in the previous paragraph, which is not addressed.) It would be of interest to establish a rigidity result for toroidal black holes akin to Theorem A. These matters are discussed in greater detail in [G2].

Theorem A can be used to improve certain results concerning the topology of "bodies", i.e., compact 3-manifolds with boundary. In [G1] the second author obtained a result concerning the notion of topological censorship. Roughly speaking it was shown that, in the steady state limit, the topology of space outside the event horizon of a black hole must be trivial, i.e., eventually, nontrivial topology becomes hidden behind the event horizon. In this work it is assumed that the black hole is surrounded by an external mean convex sphere, or, more generally, an external mean convex boundary having an S^2 component. Theorem A can be used in the analytic case to remove the assumption in the main theorem of [G1] that the external boundary has an S^2 component.

References

[AR] Anderson, M.T., Rodriguez, L.: Minimal surfaces and 3-manifolds of nonnegative Ricci curvature. Math. Ann. **284**, 461–475 (1989)

[FCS] Fischer-Colbie, D., Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. Commun. Pure Applied Math. **33**, 199–211 (1980)

[G1] Galloway, G.J.: On the topology of black holes. Commun. Math. Phys. **151**, 53–66 (1993)

[G2] Galloway, G.J.: Least area tori, black holes and topological censorship. Contemp. Math. Proceedings of the Special Session on Mathematical Physics and Differential Geometry. (J.K. Beem and K.L. Duggal, eds.) **170**, 113–123 (1994)

[Gi] Gibbons, G.W.: The time symmetric initial value problem for black holes. Commun. Math. Phys. **27**, 87–102 (1972)

[HE] Hawking, S.W., Ellis, G.F.R.: The large scale structure of space-time. Cambridge University Press, Cambridge, 1973

[SY] Schoen, R., Yau, S.-T.: Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature. Ann. Math. **110**, 127–142 (1979)