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Observability for systems with more outputs than inputs and asymptotic observers

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1 Introduction, statement of the results

We deal with general controlled and observed smooth nonlinear systems:

$$(\Sigma)\quad \begin{cases} \dot{x} = f(x,u_0) \\ y = h(x,u_0)\,. \end{cases}$$

We also consider the case, more common in practice, where h is independent of u_0.

For the sake of clarity, the precise assumptions that we make on these systems are stated and discussed in the next Sect. 2.1. However, let us recall the main notion analyzed in this paper.

An input $u_0 : [0,\Theta_{u_0}] \to T$, ($T$, the control space), being given, we say that $\Sigma = (f,h)$ is *observable for* u_0 if, for each couple of initial conditions (x_1,x_2), $x_1 \neq x_2$, there exists a time subinterval $[0,\theta]$ on which the solutions $\gamma_i(\tau) = x(\tau,u(\tau),x_i)$, $i = 1,2$, are well defined, and the set of all $\tau \in [0,\theta]$ such that $y_1(\tau) \neq y_2(\tau)$ has not zero measure (denoting by $y_i(\tau)$ the corresponding outputs, $y_i(\tau) = h(\gamma_i(\tau),u_0(\tau))$.

Σ is said *observable* (or L^∞-observable) if it is observable for each input $u_0 : [0,\Theta_{u_0}] \to T$, measurable, bounded.

Σ is said *C^k-observable*, $(k = 1,\ldots,\infty,\omega)$, if it is observable for each C^k input u_0.

For these systems, our goal is twofold:

a) to analyze the genericity of the observability property,

b) to examine the possibility of the synthesis of nonlinear observers.

A nonlinear observer for (Σ) is another control system (O_Σ) controlled by both the input $u_0(\tau)$ and the output $y(\tau)$ of (Σ). This system (O_Σ) is intended to "estimate" asymptotically the state of (Σ). But in this paper, we

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will assume that (O_Σ) is controlled by the output $y(\tau)$ and the input $u_0(\tau)$ of (Σ) together with some of the derivatives $u_j(\tau) = \frac{d^j u_0(\tau)}{(d\tau)^j}$ of the input. (This last requirement seems unavoidable). Roughly speaking, we require that there is a smooth embedding e, (depending on the inputs), of the state space X of (Σ) into the state space $\hat{X}$ of (O_Σ) such that the distance $d(e(x(\tau)), \hat{x}(\tau))$ tends to zero when τ goes to infinity ($\hat{x}$ is the state of (O_Σ)). If $d(e(x(\tau)), \hat{x}(\tau))$ tends exponentially to zero, the observer (O_Σ) is said to be "exponential". Precise definitions and statements of results are given in Sect. 4.

There are several ways to deal with the synthesis of nonlinear observers. In the linear case, the problem is solved by the standard Luenberger observer [Lu]. A major difficulty, that does not appear in the linear case, is the existence of "bad inputs" that make the system (Σ) unobservable.

For general nonlinear systems, the problem is far from solved. In our opinion, this is due mainly to the presence of these "bad inputs". Our results in this paper also strengthen this opinion, as the reader will see.

The most general method that we know, to deal with the construction of nonlinear observers, is the "high-gain observer method". (It is much more general than the "output injection method" developed in [K-I], [K-R], [H-G1], [H-G2], which applies to a very special class of systems only). This "high-gain observer" method, has been initiated in [G-H-O], [G-H-K], [G-K] and we will use it in this paper. (Also, let us point out that other constructions of observers, derived from the high-gain construction, are possible. Let us just mention the "High-gain extended-Kalman-filter method" presented in [D-B-G1], [D-B-G2]. In [D-B-G1], a nice application is shown).

First let us comment briefly on these previous results of [G-H-O], [G-H-K], [G-K]: they deal with the single-output case. It could appear as unnatural that the single-output case plays a particular role in the theory. This is completely clarified in this paper. In [G-H-O], [G-K], we dealt with single-output controlled systems, and we showed that a certain "strong observability" property is nongeneric (another stronger but closely related observability property). When this property holds, it has been shown that systems can be put in a very special canonical form. In the case of systems having this property of strong observability, we were able to construct nonlinear *exponential observers with arbitrary exponential decay of the estimation error*. In our other paper [G-H-K], we have studied the *uncontrolled single output case*. We have shown that the set of systems such that the state can be reconstructed on the basis of the knowledge of the output plus its $2d$ first derivatives is open, dense (d = dimension of the state space). As a consequence, the high gain construction applies generically, and hence *exponential observers exist generically*.

The first purpose of this paper is to generalize the previous result to controlled systems for which the *number of outputs*, d_y, *is strictly larger than the number of inputs*, d_u. For these systems, we will show the following results:

1. *The set of systems such that the state can be reconstructed from the outputs and their k first derivatives* (k *large enough*), *for inputs that are also k-times differentiable, contains a residual set.*

2. If a bound for the inputs and their derivatives up to order k is given a-priori, this set is *open, dense*. (The state space is assumed to be compact).

3. A bound on the k first derivatives of the inputs being given, *the set of systems having an observer* that works for every such input *contains an open dense set.*

4. C^∞ systems can be C^∞-approximated by *analytic* C^k-observable ones. This does not follow from our preceeding results since we do not know that the C^k-observable systems contain an open-dense set. Hence, we prove this approximation property directly.

The second part of our paper is devoted to the approximation of C^r systems by L^∞-observable ones. Using arguments of subanalyticity, we prove that, for *analytic* systems, the general "*L^∞-observability*" *is equivalent* to the "*C^ω-observability*" *property*. Hence, by the previous result, *any system can be approximated by an analytic observable one* (again, as soon as $d_y > d_u$).

Our paper is organized as follows: in Sect. 2, we state precisely our observability results. In Sect. 3, we prove our genericity theorems. In Sect. 4, we construct observers for the generic class of systems defined in 3. In Sect. 5, we prove this property of equivalence between C^ω-observability and general observability. The short Sect. 6 is devoted to our final result: the possibility to approximate by analytic observable systems. In an appendix, we state and prove all of our technical lemmas.

2 Assumptions, notations, preliminaries

2.1 Notations, assumptions

a Systems and topologies under consideration

In the paper, we will deal with nonlinear systems of the form:

$$(\Sigma)\quad \frac{dx}{dt} = f(x,u_0); \qquad y = h(x,u_0)\,,$$

or of the form:

$$(\Sigma)\quad \frac{dx}{dt} = f(x,u_0); \qquad y = h(x)\,,$$

where $x \in X$, a *compact* C^∞ d_x-dimensional connected manifold, $h : X \times T \to R^{d_y}$, $u_0 \in T$, $T = I^{d_u}$, where I is some compact interval of R, or more generally, T is a compact, d_u-dimensional "corner manifold" in R^{d_u}. The compactness assumption for X and T will be relaxed only in Sect. 5, and when needed, the assumption of real analyticity will be made. f and h are assumed to be jointly C^r in x and u_0, for r fixed, sufficiently large.

The topology that we consider for our systems (Σ) is the topology of C^r uniform convergence on $X \times T$.

We shall denote the class of these systems endowed with this C^r topology, S^r. In the particular case where h depends only on x, the set of systems will be denoted by s^r. We denote by H^r the set of C^r maps $h(x,u_0) : X \times T \to R^{d_y}$,

by H_0^r the set of C^r maps $h(x) : X \to R^{d_y}$ and by F^r the set of C^r parametrized vector fields $f(x, u_0)$ on $X \times T$. $S^r = F^r \times H^r$, $s^r = F^r \times H_0^r$. H^r, H_0^r and F^r are the Banach spaces of C^r sections of the bundles over $X \times T$, X and $X \times T$ respectively: $B_H = X \times T \times R^{d_y}$, $B_{H_0} = X \times R^{d_y}$, $B_F = TX \times T$. s^r, H_0^r are also closed subspaces of the Banach spaces S^r, H^r respectively.

All along the paper, $T_X f(x) \in \mathrm{End}(T_x X)$ will denote the linearized vector field of a vector field f at a point x such that $f(x) = 0$.

$\times_X$ will mean the standard fiber product of bundles over X. For bundles $\Pi_E : E \to X \times T$ and $\Pi_F : F \to X$, we set $E \underline{\times_X} F = \{(e, f) \,|\, \mathrm{pr}_1 \circ \Pi_E(e) = \Pi_F(f)\}$, with $\mathrm{pr}_1 : X \times T \to X$.

The bundles of k-jets of C^r-sections of B_F, B_H, B_{H_0} are respectively denoted by $J^k F, J^k H, J^k H_0$. The sets of k-jets of systems form the fiber product $J^k S = J^k F \times_{X \times T} J^k H$ and $J^k s = J^k F \underline{\times_X} J^k H_0$. $J^k s$ and $J^k H_0$ are also subbundles of $J^k S$ and $J^k H$ respectively. The evaluation jet mapping is denoted by ev_k,

$$ev_k : S^r \times X \times T \to J^k S, \quad (\text{resp. } s^r \times X \times T \to J^k s)$$

$$ev_k(\Sigma, x, u_0) = j^k \Sigma(x, u_0) = (j^k f(x, u_0), j^k h(x, u_0)), j^k \Sigma(x, u_0)$$

denoting the k-jet of Σ at (x, u_0).

A few comments about our systems are in order:

i) The compactness assumption for X can certainly be avoided. We would certainly get similar results in the Whitney topology. However, for the purpose of construction of observers, compactness is required. Moreover, if X is not compact, the problem of construction of observers is not well-posed. For these reasons, we restrict to the compact case (except in Sect. 5, the result of which is independent).

ii) We have considered two kinds of systems: the usual ones, where the outputs are independent of the inputs and also the more general class of systems whose outputs depend explicitly on the inputs. The reason to consider these less common systems is that x and u play a more symmetric role. There is however another reason to consider this dependence: in a previous paper [G-K], we have dealt with analytic single output systems. In that case, of course, (as soon as there are nontrivial inputs), observability is not generic, but we were able to characterize the systems that are observable for all inputs. In this characterization, it is shown that it is totally unnatural to consider systems whose outputs do not depend on the inputs.

The statements of our results are the same for S^r and s^r. So are our proofs since the dependence of h in the control plays no role in them. But the results for S^r are not consequences of those for s^r or vice-versa.

For technical reasons, we will have to consider other spaces of systems, typically denoted by S, F, H, that will sometimes replace S^r, F^r, H^r or s^r, F^r, H_0^r. We require that S, F, H, are related to S^r, F^r, H^r or s^r, F^r, H_0^r, as follows:

A1. S, F, H, are subspaces of S^r, F^r, H^r, or s^r, F^r, H_0^r. They are Banach spaces for a *stronger* norm.

A2. For all $(x_1,u_1)\neq(x_2,u_2)$, $(x_i,u_i)\in X\times T$, $i=1,2$, there is a $\Sigma\in S$ (resp. s) such that the k-jets $j^k\Sigma(x_i,u_i)$ at the points (x_i,u_i) have arbitrary prescribed values.

For the proof of our observability Theorems 2.1, 2.2, 2.3 below, we will need only the obvious case where $S=S^r$ (resp. s^r). For the proof of our Theorem 2.4 (the real analytic, case), we will make the different choice $S=S_K$ or s_K, where S_K,s_K are described in the following (for the details, we refer to [G-M-T]).

We consider the situation where $X\times T,\Sigma$, are real analytic. $\tilde{X}$ denotes a complex manifold, O_X (resp. $O_{\tilde{X}}$) denotes the sheaf of germs of real analytic (resp. holomorphic) functions on X (resp. $\tilde{X}$). A complexification of (X,O_X) is a complex manifold $(\tilde{X},O_{\tilde{X}})$ together with an anti-involution (σ,σ^*) : $(\tilde{X},O_{\tilde{X}})\to(\tilde{X},O_{\tilde{X}})$ and a homomorphism $(\rho,\rho^*):(X,O_X\otimes_R C)\to(F,O_{\tilde{X}}\mid F)$ where F is the set of fixed points of σ.

Complexifications of real analytic, Hausdorff, paracompact manifolds exist, their germs along X are isomorphic. The complexification is a natural correspondence in the sense that $\tilde{X}\times\tilde{T}$ is a complexification of $X\times T$ in a natural way, and so is the tangent bundle $T\tilde{X}$ with respect to the tangent bundle TX of X.

We consider a complexification $\widetilde{X\times T}$ of $X\times T$. By Grauert's paper [G], (see also [GMT]), $\widetilde{X\times T}$ contains a neighbourhood of $X\times T$ which is a *Stein* manifold. Replacing $\widetilde{X\times T}$ by this neighbourhood, we can assume that $\widetilde{X\times T}$ is Stein.

Let $\mathbb{K}$ be the class of closures K of open connected, σ-invariant relatively compact neighbourhoods of $X\times T$ in $\widetilde{X\times T}$. Let us denote by $BF(K)$ (resp. $VF(K)$), the set of all functions $h(x,u_0)$ (resp. parametrized vector-fields $f(x,u_0)$) which are defined and continuous on K and holomorphic on $\mathrm{Int}(K)$. $BF(K)$ endowed with the sup-norm becomes a Banach space. $\widetilde{BF}(K)$ (resp. $\widetilde{VF}(K)$) is the real Banach subspace of conjugation-invariant elements of $BF(K)$ (resp. $VF(K)$). The restriction mapping $\mathrm{Res}_K: h\to h_{\mid X\times T}$ (resp. $f\to f_{\mid X\times T}$), maps $\widetilde{BF}(K)$ (resp. $\widetilde{VF}(K)$) continuously and injectively into $C^\infty(X\times T)$ (resp. $VF^\infty(X\times T)$). The subset $C^\omega(X\times T)$ (resp. $VF^\omega(X\times T)$) of all analytic functions (resp. parametrized vector-fields) on $X\times T$ is the inductive limit $\mathrm{Lim}_{\to\mathbb{K}}(\widetilde{BF}(K))$ (resp. $\mathrm{Lim}_{\to\mathbb{K}}(\widetilde{VF}(K))$).

This means that for any $h\in C^\omega(X\times T)$ (resp. $f\in VF^\omega(X\times T)$), $\exists K\in\mathbb{K}$, and a $h^\circ\in\widetilde{BF}(K)$ (resp. $f^\circ\in\widetilde{VF}(K)$) such that $h=\mathrm{Res}_K(h^\circ)$ (resp. $f=\mathrm{res}_K(f^\circ)$).

We denote by $S_K,F_K,H_K,S_K=F_K\times H_K$, $S_K\subset S_r$, the Banach spaces formed by the restrictions $\mathrm{Res}_K(\Sigma)=(\mathrm{Res}_K(f),\mathrm{Res}_K(h))$ of elements $\Sigma=(f,h)$ of $\widetilde{VF}(K)\times\widetilde{BF}(K)$, with the norm induced by $\widetilde{BF}(K)\times\widetilde{VF}(K)$.

Let us state some properties of S_K needed later on:

(i) the norm on S_K is *stronger* than the norm on S^r,

(ii) for any finite set of points $p_i=(x_i,u_i),1\leqq i\leqq N$ in $X\times T$, and elements $j_i\in J^kS(x_i,u_i)$, any $K\in\mathbb{K}$, there is a $\Sigma^\circ\in S_K$ such that $j^k\Sigma^\circ(x_i,u_i)=j_i,1\leqq i\leqq N$, (see [C1], [C2] or [G-M-T]),

(iii) the evaluation map $ev_k : S_K \times X \times T \to J^k S$,

$$(\Sigma, x, u_0) \to j^k \Sigma(x, u_0) \text{ is a } C^\omega \text{ mapping .}$$

Hence, in particular, the assumptions A1, A2 above are met. s_K, H_{0_K} are also defined in the obvious way and have the same properties.

b Notations

For a system Σ, we define the k^{th} dynamical extension (Σ^k) of (Σ) and the k^{th} dynamical extension f^k of f as follows. f^k is the vector field on $X \times T \times R^{(k-1)d_u} = X \times T \times (R^{d_u} \times \cdots \times R^{d_u})$ [$(k$-1 factors)]:

$$f^k(x, u_0, \ldots, u_{k-1}) = \sum_{i=1}^{d_x} f_i(x, u_0) \frac{\partial}{\partial x_i} + \sum_{i=0}^{k-2} \sum_{j=1}^{d_u} u_{i+1,j} \frac{\partial}{\partial u_{i,j}} ,$$

and $\Sigma^k = (f^k(x, u_0, \ldots, u_{k-1}) + b^k u_k, h(x, u_0))$, where $b^k u_k = \sum_{i=1}^{d_u} b_i^k u_{k,i}$, $b_i^k = \frac{\partial}{\partial u_{k-1,i}}$ and $u_k = (u_{k,i}) \in R^{d_u}$ is the control variable of (Σ^k).

We will have to consider several maps associated to f^k or (Σ^k). Denoting by L the Lie-derivative operator, we define:

$$R_k^* : X \times T \times R^{(k-1)d_u} \times S \to R^{(k\,d_y)} \times R^{(k\,d_u)}$$

$$(x, u, \Sigma) = (x, u_0, \ldots, u_{k-1}, (f, h)) \to (h(x, u_0), L_{f^k} h(x, u), \ldots,$$

$$(L_{f^k})^{k-1} h(x, u), u) .$$

(The successive derivatives of our outputs and inputs, $y_j(t)$ and $u_j(t)$, $j = 0, \ldots, k-1$ have their value in R^{d_y} and R^{d_u}).

$$R_k : X \times T \times R^{(k-1)d_u} \times S \to R^{(k\,d_y)}$$

$$(x, u, \Sigma) \to (h(x, u_0), L_{f^k} h(x, u), \ldots, (L_{f^k})^{k-1} h(x, u)) .$$

Notice that $R_k^*(x, u, \Sigma) = (R_k(x, u, \Sigma), u)$.

For Σ fixed in S, we define:

$$R_{k,\Sigma}^* : X \times T \times R^{(k-1)d_u} \to R^{(k\,d_y)} \times R^{(k\,d_u)} ,$$

$$R_{k,\Sigma}^*(x, u) = R_k^*(x, u, \Sigma)$$

and

$$R_{k,\Sigma} : X \times T \times R^{(k-1)d_u} \to R^{(k\,d_y)} ,$$

$$R_{k,\Sigma}(x, u) = R_k(x, u, \Sigma) .$$

ΔX denoting the diagonal in $X \times X$, we define also:

$$T_k^* : X \times X \backslash \Delta X \times T \times R^{(k-1)d_u} \times S \to R^{(k\,d_y)} \times R^{(k\,d_y)} \times R^{(k\,d_u)},$$

$$T_k^*(x_1, x_2, u, \Sigma) = (R_k(x_1, u, \Sigma), R_k(x_2, u, \Sigma), u),$$

$$T_k : X \times X \backslash \Delta X \times T \times R^{(k-1)d_u} \times S \to R^{(k\,d_y)} \times R^{(k\,d_y)},$$

$$T_k(x_1, x_2, u, \Sigma) = (R_k(x_1, u, \Sigma), R_k(x_2, u, \Sigma)),$$

$$T_{k,\Sigma}^*(x_1, x_2, u) = T_k^*(x_1, x_2, u, \Sigma),$$

$$T_{k,\Sigma}(x_1, x_2, u) = T_k(x_1, x_2, u, \Sigma).$$

In the remaining of the paper, we will be interested with the following facts

(F): $R_{k,\Sigma}^*$ is an embedding from $X \times T \times R^{(k-1)\,d_u}$ into $R^{(k\,d_y)} \times R^{(k\,d_u)}$.

Since X and I are compact, by the definition of $R_{k,\Sigma}^*$, this is equivalent to:

(F1): $R_{k,\Sigma}^*$ is one-to-one; and
(F2): $R_{k,\Sigma}^*$ is an immersion.

The fact (F1) is equivalent to the fact that the map $T_{k,\Sigma} : X \times X \backslash \Delta X \times T \times R^{(k-1)\,d_u} \to R^{(k\,d_y)} \times R^{(k\,d_y)}$ avoids the diagonal in $R^{(k\,d_y)} \times R^{(k\,d_y)}$.

2.2 Statement of our observability results

In the next section, we will prove the following results:

Theorem 2.1 *The set of systems such that* (F2) *is true, i.e., $R_{k,\Sigma}^*$ is an immersion,* **contains** *an open, dense subset of S^r (resp. s^r), for $k \geqq 2d_x$.* □

Theorem 2.2 *The set of systems such that* (F) *is true, i.e., $R_{k,\Sigma}^*$ is an embedding, contains a residual subset of S^r (resp. s^r), for $k \geqq 2d_x + 1$.* □

A bound $B > 0$ on the derivatives of the inputs being given, denote by I_B the interval $[-B, B]$.

Theorem 2.3 *The set of systems such that the restriction of $R_{k,\Sigma}^*$ to $X \times T \times I_B^{(k-1)\,d_u}$ is an embedding, is open, dense in S^r (resp. s^r) for $k \geqq 2d_x + 1$.* □

Theorem 2.4 *When $X \times T$ is analytic, the set of* **analytic** *systems such that $R_{k,\Sigma}^*$ is an embedding, is dense in S^r (resp. s^r) for $k \geqq 2d_x + 1$.* □

Again let us make a few comments on these results:

i) It can be seen very easily from our paper [G-K] that in the case where the number of inputs is equal to the number of outputs, these theorems are all false.

ii) The set of systems such that (F2) is true (Theorem 2.1) is *not open.* Again this follows from our paper [G-K]: take a 2 outputs, 1 input system, such that one of the two output functions is identically zero and the single output remaining system has the strong observability property of [G-K]. In any

neighbourhood of this system, there are systems that do not satisfy (F2). This is easily seen by keeping the zero output and making a generic perturbation of the remaining of the system. Another reason is that the map:

$$S^r \to C^{r-k+1}(X \times T \times R^{(k-1)d_u}, R^{(k\,d_y)} \times T \times R^{(k-1)d_u})$$

$$\Sigma \to R^*_{k,\Sigma}$$

is *not* continuous for the Whitney topology.

iii) Theorem 2.4 is not a consequence of Theorems 2.1, 2.2: our techniques do not prove that there exists an open dense subset of systems satisfying (F).

3 Proof of the observability theorems

The considerations in this Sect. 3 apply to both cases S^r and s^r. In order to avoid to clutter our text with "respectively..." and other alternatives, and for the sake of clarity, we will take S to mean either S or s in all proofs below. H_0 will be considered as the subspace of H of all functions in H independent of u_0. Everywhere below, when the difference between H and H_0 is not explicitly stated, for $h \in H_0$ and $p = (x_0, u_0)$, $h(p)$ or $d^n_X h(p; \ldots)$ will mean $h(x_0), d^n_X h(x_0; \ldots)$, and all the expressions $d^p_X d^q_U h$ with $q > 0$ are to be taken equal to 0. Note also that the bad sets for $J^k s$ (see below) are just the intersections of the bad sets for $J^k S$ with $J^k s$.

3.1 Openness and density of immersivity

3.1.1 The Bad sets

All our bad sets will be partially algebraic or semialgebraic subbundles of vector bundles:

Definition 3.1 *A subbundle B of a vector bundle E is called* **partially algebraic (semi algebraic)** *(PA and PSA respectively), if its typical fiber is an algebraic (resp. semi algebraic) subset of the vector space which is the typical fiber of E.* □

Definition 3.2 *$\hat{B}_1(k)$ (resp. $\hat{B}_2(k)$) is the subset of $J^k S \times R^{(k-1)d_u}$ of all $(j^k\Sigma(x,u_0), u')$ such that: 1) $f(x,u_0) \neq 0$ (resp. $f(x,u_0) = 0$), 2) for $\hat{B}_2(k)$ only: the couple $(d_x h(x,u_0), T_X f(x,u_0))$ is observable (see Appendix A.1 for the notion of observability for linear systems), 3) rank $T_x R_{k,\Sigma}(x,u_0,u') < d_x$, $(u' = u_1, \ldots, u_{k-1})$).* □

Definition 3.3 *$B_3(k)$ is the subset of $J^k S$ of all $j^k(x,u_0)$ such that $f(x,u_0) = 0$ and $(d_x h(x,u_0), T_X f(x,u_0))$ is unobservable.* □

Definition 3.4 *$B_{\text{im}}(k)$ is the union $B_1(k) \cup B_2(k) \cup B_3(k)$, $B_i(k)$, $i = 1,2$, being the canonical projection of $\hat{B}_i(k)$ into $J^k S$.* □

The following two lemmas are obvious:

Lemma 3.0 *$\hat{B}_1(k), \hat{B}_2(k), B_1(k), B_2(k), B_3(k), B_{\text{im}}(k)$ are PSA in their respective vector bundles.* □

Lemma 3.1 *$R^*_{k,\Sigma}$ is an immersion if and only if $j^k\Sigma$ avoids $B_{\text{im}}(k)$.* □

3.1.2 Estimates of the codimension of $B_{\text{im}}(k)$

Clearly, if B is a PSA subset of a vector bundle E, $\text{codim}(B,E)$, the codimension of B in E, is equal to the codimension of the typical fiber of B in the typical fiber of E.

a) **Codim**$(B_3(k), J^kS)$: let $J^kS(p)$, $p = (x, u_0)$, be a typical fiber of J^kS and let $V \subset J^kS(p)$ be the vector subspace $V = \{j^k\Sigma(p) \mid f(p) = 0\}$. V has codimension d_x in $J^kS(p)$.

Let $\lambda : V \to (T^*_xX)^{d_y} \times \text{End}(T_xX)$ be the mapping: $\lambda j^k\Sigma(p) = (d_xh(p), T_xf(p))$. If $\mathbb{U} \subset (T^*_xX)^{d_y} \times \text{End}(T_xX)$ denotes the set of unobservable couples (C,A), then, the typical fiber $B_3(k;p)$ of $B_3(k)$ is $\lambda^{-1}(\mathbb{U})$. But, by appendix A.1, $\text{codim}(\mathbb{U}, (T^*_xX)^{d_y} \times \text{End}(T_xX)) = d_y$. Hence, $\text{codim}(B_3(k;p), V) = d_y$ and $\text{codim}(B_3(k;p), J^kS(p)) = \text{codim}(B_3(k;p), V) + \text{codim}(V, J^kS(p)) = d_x + d_y$.

b) **Codim**$(B_1(k), J^kS)$: let G be the open subset of $J^kF \underset{X}{\times} TX \times R^{(k-1)d_u}$ of all $(j^kf(p), \Lambda, u')$ such that:

(i) $f(p) \neq 0$, (again, $p = (x, u_0)$),
(ii) $\Lambda \neq 0$.

Let $\mu : J^kH \times_{X\times T} G$ (resp. $J^kH_0 \times_{X\times T} G$) $\to R^{k\,d_y}$, $\mu(j^kh(p), j^kf(p), \Lambda, u') = T_xR_{k,\Sigma}(p,u')\Lambda$. $\Sigma = (f,h)$.

Clearly, $\hat{B}_1(k)$ is the canonical projection of $\mu^{-1}(0)$ into $J^kS \times R^{(k-1)d_u}$. If we show that μ is a submersion, it will follow that:

$$\text{codim}(\mu^{-1}(0), J^kS \underset{X}{\times} TX \times R^{(k-1)d_u}) = k\,d_y\,,$$

and since $\mu^{-1}(0)$ is conical in $J^kS \underset{X}{\times} TX \times R^{(k-1)d_u}$,

$$\text{codim}(\hat{B}_1(k), J^kS \times R^{(k-1)d_u}) = k\,d_y + 1 - d_x\,,$$

hence:

$$\text{codim}(B_1(k), J^kS) = k(d_y - d_u) + 1 + d_u - d_x\,.$$

To see that μ is a submersion: for any fixed $(j^kf(p), \Lambda, u') \in G$, the mapping: $j^kh(p)$ (resp. $j^kh(x_0)$) $\to \mu(j^kh(p), j^kf(p), \Lambda, u')$ (resp. $\mu(j^kh(x_0), j^kf(p), \Lambda, u')$) is the linear mapping Θ_Λ of Lemma 1 (iii), Appendix A2. This mapping is surjective, hence μ is a submersion.

c) **Codim**$(B_2(k), J^kS)$: (the most difficult case). $\hat{B}_2(k)$ is the subset of $J^kS \times R^{(k-1)d_u}$ of all $(j^k\Sigma(p), u')$ such that: 1) $f(p) = 0$, 2) the linear system $(d_Xh(p), T_xf(p))$ on $T_{x_0}X$ is observable, $p = (x_0, u_0)$, 3) the tangent mapping $T_xR_{k,\Sigma}(p,u') : T_{x_0}X \to R^{k\,d_y}$ has a nonzero kernel.

In the study of $\hat{B}_2(k)$, we will need the following:

Lemma 3.2 *Let W be a finite dimensional vector space, $A: W \to W$, $C: W \to R^1$ two linear mappings. If there exists a nonzero vector $\Lambda \in W$ such that*: (i) $CA^n\Lambda = 0$ *for* $0 \leqq n \leqq t$, t *some integer*, (ii) *the vectors* $\{A^n\Lambda \mid 0 \leqq n \leqq t+1\}$ *in W are linearly dependant, then the system (C,A) is not observable.* (*see Appendix* A1 *for observability of linear systems*). □

Proof. (ii) implies that there exists an integer m, $0 \leqq m \leqq t$, such that $A^{m+1}\Lambda$ belongs to the linear span, $\mathrm{Span}\{A^i\Lambda \mid 0 \leqq i \leqq m\}$. Then, $A^n\Lambda \in \mathrm{Span}\{A^i\Lambda \mid 0 \leqq i \leqq m\}$ for all integers n. Since $\mathrm{Span}\{A^i\Lambda \mid 0 \leqq i \leqq m\} \subset \mathrm{Ker}\, C$, $A^n\Lambda \in \mathrm{Ker}\, C$ for all n. Hence, (C,A) is not observable. □

To study $\hat{B}_2(k)$ more conveniently, we shall split it into subsets which are easier to handle:

Definition 3.5 (i) *For every integer ρ, $1 \leqq \rho \leqq k-1$, $\hat{B}_6(k,\rho)$ (resp. $\hat{B}_7(k,\rho)$) will be the subset of $\hat{B}_2(k)$ of all $(j^k\Sigma(p), u')$ such that*: (1) $u_1 = u_2 = \cdots = u_{\rho-1} = 0$, (2) $d_U f(p; u_\rho) \neq 0$ (*resp.* $d_U f(p; u_\rho) = 0$).

(ii) *Let $B_6(k,\rho)$, $B_7(k,\rho)$ denote the canonical projections of $\hat{B}_6(k,\rho)$, $\hat{B}_7(k,\rho)$ respectively, into J^kS.* □

Lemma 3.3 (i) $\hat{B}_2(k) = \bigcup_{\rho \geqq 1}(\hat{B}_6(k,\rho) \cup \hat{B}_7(k,\rho))$, $B_2(k) = \bigcup_{\rho \geqq 1}(B_6(k,\rho) \cup B_7(k,\rho))$, (ii) $\hat{B}_6(k,\rho), \hat{B}_7(k,\rho)$ (*resp.* $B_6(k,\rho), B_7(k,\rho)$) *are PSA subbundles of* $J^kS \times R^{(k-1)d_u}$ (*resp.* J^kS), (iii) $\hat{B}_6(k,\rho) = \hat{B}_7(k,\rho) = B_6(k,\rho) = B_7(k,\rho) = \varnothing$ *if* $\rho > d_x$. □

Proof. (i) and (ii) are trivial. (iii) By Lemma 2, Appendix A2, for any $(j^k\Sigma(p), u') \in \hat{B}_6(k,\rho) \cup \hat{B}_7(k,\rho)$ there exists a $\Lambda \in T_{x_0}X$, $\Lambda \neq 0$, such that: $d_X h(p; T_x f(p)^i \Lambda) = 0$ for $0 \leqq i \leqq \rho - 1$, $(p = (x_0, u_0))$. Applying Lemma 3.2 with $t = \rho - 1$, $W = T_{x_0}X$, $A = T_x f(p)$, $C = d_X h(p)$, we get that $(d_X h(p), T_x f(p))$ is not observable since the vectors $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho - 1$, are necessarily linearly dependent ($\dim T_{x_0}X = d_x < \rho$). This is a contradiction.

Now, we shall estimate the codimensions of $B_6(k,\rho)$ and $B_7(k,\rho)$.

d) **Codim**$(B_7(k,\rho), J^kS)$: Let $\tilde{B}_8(k,\rho)$ be the subset of $J^kS \underset{X}{\times} P(TX) \times R^{d_u}$ of all $(j^k\Sigma(p), 1, w)$ such that 1) $f(p) = 0$, 2) $d_U f(p,w) = 0$, 3) $d_X h(p; T_x f(p)^i \Lambda) = 0$, $0 \leqq i \leqq \rho - 1$ and $d_X h(p; T_x f(p)^\rho \Lambda) + d_X d_U h(p; \Lambda \otimes w) = 0$, for all $\Lambda \in 1$, 4) the vectors $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$ are linearly independent. ($P(T_xX)$ is the projective space associated to T_xX). We claim that the canonical projection $B_8(k,\rho)$ of $\tilde{B}_8(k,\rho)$ into J^kS contains $B_7(k,\rho)$. An element $e \in B_7(k,\rho)$ is the image of an element $(j^k\Sigma(p), u') \in \hat{B}_7(k,\rho)$. By condition 3) in the definition of $\hat{B}_2(k)$, there exists a $\Lambda \in T_{x_0}X$, $\Lambda \neq 0$, such that $T_x R_{k,\Sigma}(p,u')\Lambda = 0$. Let 1 be the class of Λ in $P(T_{x_0}X)$. If we show that the triple $(j^k\Sigma(p), 1, u_\rho)$ belongs to $\tilde{B}_8(k,\rho)$, it will follow that $B_8(k,\rho) \supset B_7(k,\rho)$. So, we have to check that $(j^k\Sigma(p), 1, u')$ satisfies the conditions 1–4 defining $\tilde{B}_8(k,\rho)$. But conditions 1 and 2 of the definition of $\tilde{B}_8(k,\rho)$ follow from condition 1) of the Definition 3.2 of $\hat{B}_2(k)$ and (ii) of the Definition 3.5 of $\hat{B}_7(k,\rho)$. Condition 3) follows from the fact that $T_x R_{k,\Sigma}(p,u')\Lambda = 0$ and Lemma 2-(2), Appendix A2. Finally condition 4) follows from the

condition 3) of the definition of $\tilde{B}_8(k,\rho)$ just proved and Lemma 3.2 applied with $t = \rho - 1$, $W = T_{x_0}X$, $A = T_x f(p)$, $C = d_X h(p)$ and Λ. We get a contradiction with condition 2) of Definition 3.2.

Clearly, $\tilde{B}_8(k,\rho), B_8(k,\rho)$ are PSA subbundles of the bundles $J^k S \underset{X}{\times} P(TX) \times R^{d_u}$ and $J^k S$ respectively. The inclusion $B_8(k,\rho) \supset B_7(k,\rho)$ implies that:

1) $\operatorname{codim}(B_7(k,\rho), J^k S) \geqq \operatorname{codim}(B_8(k,\rho), J^k S)$.

We have also:

2) $\operatorname{codim}(B_8(k,\rho), J^k S) \geqq \operatorname{codim}(\tilde{B}_8(k,\rho), J^k S \underset{X}{\times} P(TX) \times R^{d_u}) - d_u - d_x + 1$.

To estimate $\operatorname{codim}(\tilde{B}_8(k,\rho), J^k S \underset{X}{\times} P(TX) \times R^{d_u})$, let us note that $\tilde{B}_8(k,\rho)$ is the projection of the conical subset $\Phi^{-1}(0)$ of $J^k S \underset{X}{\times} TX \times R^{d_u}$ where $\Phi : G \to R^{d_x + (1+\rho)d_y}$ is defined as follows: G is the open PSA subbundle of $J^k S \underset{X}{\times} TX \times R^{d_u}$ of all $(j^k \Sigma(p), \Lambda, w)$ such that (i) $f(p) = 0$, (ii) the vectors $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$, are linearly independent. $\Phi(j^k \Sigma(p), \Lambda, w) = (d_U f(p; w), d_X h(p; T_x f(p)\Lambda, \ldots, d_X h(p; T_x f(p)^i \Lambda), \ldots, d_X h(p, T_x f(p)^{\rho-1}\Lambda), d_X h(p, T_x f(p)^\rho \Lambda) + d_X d_U h(p, \Lambda \otimes w))$.

Clearly, Φ is a submersion since the $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$, are linearly independent. Hence:

$$\operatorname{codim}(\Phi^{-1}(0), G) = d_x + (1+\rho) d_y \,.$$

Clearly, G is a submanifold of $J^k S \underset{X}{\times} TX \times R^{d_u}$ of codimension d_x. Hence:

$$\operatorname{codim}(\Phi^{-1}(0), J^k S \underset{X}{\times} TX \times R^{d_u}) = 2d_x + (1+\rho)\, d_y \,,$$

$$\operatorname{codim}(\tilde{B}_8(k,\rho), J^k S \underset{X}{\times} P(TX) \times R^{d_u}) = 2d_x + (1+\rho)\, d_y \,.$$

Using (1) and (2):

3) $\operatorname{codim}(B_7(k,\rho), J^k S) \geqq d_x + (1+\rho) d_y - d_u + 1$.

e) **Codim**$(B_6(k,\rho), J^k S)$: Let $G \subset J^k F \underset{X}{\times} TX \times R^{(k-1)d_u}$ be the subset of all $(j^k f(p), \Lambda, u')$ such that: (i) $f(p) = 0$, (ii) $u_1 = u_2 = \cdots = u_{\rho-1} = 0$, (iii) the vectors $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$, are linearly independent, (iv) $d_U f(p, u_\rho) \neq 0$.

G is clearly a PSA conical subbundle of $J^k F \underset{X}{\times} TX \times R^{(k-1)d_u}$ and a submanifold of the same set, of codimension:

4) $\operatorname{codim}(G, J^k F \underset{X}{\times} TX \times R^{d_u}) = d_x + (\rho - 1)\, d_u$,

Let $\mu : J^k H \times_{X \times T} G$ (resp. $J^k H_0 \times_{X \times T} G$) $\to R^{k\, d_y}$:

$$\mu(j^k h(p), j^k f(p), \Lambda, u') = T_x R_{k,\Sigma}(p, u'; \Lambda), \quad (\Sigma = (f, h)) \,.$$

$\hat{B}_6(k,\rho)$ is contained in the image of $\mu^{-1}(0)$ by the canonical projection: $J^k S \underset{X}{\times} TX \times R^{(k-1)d_u} \to J^k S \times R^{(k-1)d_u}$.

To see this, let $(j^k \Sigma(p), u') \in \hat{B}_6(k,\rho)$. Then by condition 3) of Definition 3.2, there exists a $\Lambda \in T_{x_0} X$, ($p = (x_0, u_0)$!), $\Lambda \neq 0$, such that $T_x R_{k,\Sigma}(p, u', \Lambda) = 0$.

Let us show that $(j^k\Sigma(p),\Lambda,u')$ belongs to $\mu^{-1}(0)$. All we have to do is to check that $(j^k f(p),\Lambda,u') \in G$, where $\Sigma = (f,h)$. Now, $f(p) = 0$ by condition 1 of the Definition 3.2 of $\hat{B}_2(k)$, $u_1 = u_2 = \cdots = u_{\rho-1} = 0$, and $d_U f(p,u_\rho) \neq 0$ by the definitions (i) and (ii) of Definition 3.5 for $\hat{B}_6(k,\rho)$. To check that the $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$ are linearly independent, note that by Lemma 2-(2) of the appendix A2, the fact that $T_x R_{k,\Sigma}(p,u',\Lambda) = 0$ implies that $d_X h(p, T_x f(p)^i \Lambda) = 0$ for $0 \leqq i \leqq \rho - 1$. Then Lemma 3.2 and Condition 2) of the Definition 3.2 for $\hat{B}_2(k)$ imply that the $T_x f(p)^i \Lambda$, $0 \leqq i \leqq \rho$, are linearly independent. Hence,

$$\begin{aligned}&\operatorname{codim}(\hat{B}_6(k,\rho), J^k S \times R^{(k-1)d_u}) \\ &\quad \geqq \operatorname{codim}(\mu^{-1}(0), J^k S \underline{\times_X} TX \times R^{(k-1)d_u}) + 1 - d_x \,.\end{aligned}$$

Lemma 2-(1), Appendix A2, implies immediately that μ is a submersion, hence:

$$\operatorname{codim}(\mu^{-1}(0), J^k H \times_{X\times T} G) = k\, d_y \,,$$

$$(\text{resp. } \operatorname{codim}(\mu^{-1}(0), J^k H_0 \times_{X\times T} G) = k\, d_y) \,.$$

Using (4):

$$\operatorname{codim}(\mu^{-1}(0), J^k S \underline{\times_X} TX \times R^{(k-1)d_u}) = k\, d_y + d_x + (\rho - 1)d_u \,,$$

$$\operatorname{codim}(\hat{B}_6(k,\rho), J^k S \times R^{(k-1)d_u}) \geqq k\, d_y + d_x + (\rho - 1)d_u - d_x + 1 \,,$$

$$(5)\quad \operatorname{codim}(B_6(k,\rho), J^k S) \geqq k\, d_y + 1 + (\rho - k)d_u.$$

(3), (5), Lemma 3.3 (i) imply that:

$$\operatorname{codim}(B_2(k), J^k S) \geqq \min\{d_x + (1+\rho)d_y + 1 - d_u, k\, d_y + 1 + (\rho - k)d_u \mid \rho \geqq 1\} \,.$$

Hence:

$$\operatorname{codim}(B_2(k), J^k S) \geqq \min\{d_x + 2d_y + 1 - d_u, k\,(d_y - d_u) + 1 + d_u\} \,.$$

Combining the results of 3.1.2.a-b-c-d-e:

$$\begin{aligned}\operatorname{Codim}(B_{\text{im}}(k), J^k S) \geqq \min\{&d_x + d_y, k\,(d_y - d_u) + 1 - d_x + d_u \,, \\ &d_x + 2d_y + 1 - d_u, k\,(d_y - d_u) + 1 + d_u\} \,.\end{aligned}$$

3.1.3 Proof of the openness and density of immersivity

Clearly, for $k \geqq 2d_x$, $\operatorname{codim} B_{\text{im}}(k) > d_x + d_u$, since $d_y > d_u$. The same applies to the closure $\operatorname{cl}(B_{\text{im}}(k))$, which is still PSA (see [L]).

By Abraham's theorems (see [A-R]) on transversal density and openness of nonintersection, the set of $\Sigma \in S$ such that $j^k\Sigma$ avoids $\operatorname{Cl}(B_{\text{im}}(k))$ is open dense.

Of course, we have used our assumptions A1, A2 of Sect. 2.1.a which imply that the map $j^k : S \to C^r(X \times T, J^k S)$ is a C^r representation and that $ev_k : S \times X \times T \to J^k S$ is a submersion.

Remark. In the case where $d_y = d_u$, $B_{\text{im}}(k)$ has codimension $d_u - d_x + 1$. This shows that the set on which $R^*_{k,\Sigma}$ is not immersive may have generically full codimension. This is in agreement with our paper [G-K]. (In that case, to get openness, we have to use an argument of openness of transversality to a closed stratified set, such as in [G-M]).

3.2 Density of injectivity

3.2.1 The bad sets

Definition 3.6 *Let $B_4(k)$ denote the subset of $J^kS \times J^kS$ of all couples $(j^k\Sigma(p), j^k\Sigma(q))$ such that*: 1) $p \neq q$, 2) $f(p) = f(q) = 0$, 3) $h(p) = h(q)$. □

Definition 3.7 (i) *Let $\hat{B}_5(k)$ be the subset of $(J^kS)^2 \times R^{(k-1)d_u}$ of all tuples $(j^k\Sigma(p), j^k\Sigma(q), u')$ such that*: 1) $p \neq q$, 2) $f(p) \neq 0$ *or* $f(q) \neq 0$, 3) $R_{k,\Sigma}(p, u') = R_{k,\Sigma}(q, u')$. □

(ii) *$B_5(k)$ will denote the canonical projection of $\hat{B}_5(k)$ into $(J^kS)^2$.*

We have again the obvious lemmas:

Lemma 3.4 *$B_4(k)$, $\hat{B}_5(k)$, $B_5(k)$ are respectively PSA subbundles of $(J^kS)^2_*$, $(J^kS)^2_* \times R^{(k-1)d_u}$, $(J^kS)^2_*$, where $(J^kS)^2_*$ (resp. $(J^kS)^2_* \times R^{(k-1)d_u}$) is the restriction of $(J^kS)^2$ (resp. $(J^kS)^2 \times R^{(k-1)d_u}$) to $(X \times X \backslash \Delta X) \times \Delta T$, where $\Delta X, \Delta T$ are the diagonals in $X \times X$ and $T \times T$.* □

Lemma 3.5 *Let $Z \subset (X \times X \backslash \Delta X)$ and let $\Sigma \in S$ be such that the mapping: $Z \times T \to (J^kS)^2_* : (x, y, u_0) \to (j^k\Sigma(x, u_0), j^k\Sigma(y, u_0))$ avoids $B_{\text{in}}(k) = B_4(k) \cup B_5(k)$. Then, $R_{k,\Sigma}(x, u_0, u') \neq R_{k,\Sigma}(y, u_0, u')$ for all $(x, y, u_0, u') \in Z \times T \times R^{(k-1)d_u}$.* □

3.2.2 Estimation of the codimension of $B_{\text{in}}(k)$

a) **Codimension of** $B_4(k)$ in $(J^kS)^2_*$ (or equivalently in $(J^kS)^2$):
It is obvious that this codimension is $2d_x + d_y$.

b) **Codimension of** $\hat{B}_5(k)$ in $(J^kS)^2_* \times R^{(k-1)d_u}$ and $B_5(k)$ in $(J^kS)^2_*$:
We will treat only the case $f(x, u_0) \neq 0$. The case $f(y, u_0) \neq 0$ is similar.

Let $(x, y, u_0, u_0) \in (X \times X \backslash \Delta X) \times \Delta T$. The typical fiber $\hat{B}_5(k; x, y, u_0)$ of $\hat{B}_5(k)$ in $J^kS(x, u_0) \times J^kS(y, u_0) \times R^{(k-1)d_u}$ is characterized by the properties: (i) $f(x, u_0) \neq 0$, (ii) $R_{k,\Sigma}(x, u_0, u') - R_{k,\Sigma}(y, u_0, u') = 0$.

Let G be the subset of $J^kS(x, u_0) \times J^kS(y, u_0) \times R^{(k-1)d_u}$ of all tuples $(j^k\Sigma(x, u_0), j^k\Sigma(y, u_0), u')$ such that $f(x, u_0) \neq 0$ and let $\chi : G \to R^{kd_y}$ be the mapping: $\chi(j^k\Sigma(x, u_0), j^k\Sigma(y, u_0), u') = R_{k,\Sigma}(x, u_0, u') - R_{k,\Sigma}(y, u_0, u')$. Then, $\hat{B}_5(k; x, y, u_0) = \chi^{-1}(0)$. χ is an algebraic mapping, affine in $j^kh(x, u_0)$. Now, by the Lemma 1, Appendix A2, for fixed $u' \in R^{(k-1)d_u}$ and $j^kf(x, u_0)$, the linear mapping $j^kh(x, u_0) \in J^kH(x, u_0) \to R_{j^kf(x,u_0), j^kh(x,u_0)}(x, u_0, u')$ is surjective. This shows that $\chi : G \to R^{kd_y}$ is a submersion. Since $\hat{B}_5(k; x, y, u_0) = \chi^{-1}(0)$, $\text{codim}(\hat{B}_5(k; x, y, u_0), J^kS(x, u_0) \times J^kS(y, u_0) \times R^{(k-1)d_u}) = kd_y$.

Hence, $\operatorname{codim}(\hat{B}_5(k),(J^kS)^2_* \times R^{(k-1)d_u}) = k\,d_y$. It follows that

$$\operatorname{codim}(B_5(k),(J^kS)^2_*) \geqq k(d_y - d_u) + d_u.$$

c) **codim**$(B_{\text{in}}(k),(J^kS)^2_*)$:

$$\operatorname{codim}(B_{\text{in}}(k),(J^kS)^2_*) \geqq \min(2d_x + d_y, k(d_y - d_u) + d_u)\,.$$

3.2.3 Proof of the density of injectivity

Let $k \geqq 2dx + 1$. Then, $\operatorname{codim}(B_{\text{in}}(k),(J^kS)^2_*) \geqq 2d_x + d_u + 1$. Then, another application of Abraham's transversal density theorem (with the same details (that we omit) as in 3.1.3) shows that the set of all $\Sigma \in S$ such that $(j^k\Sigma(x,u_0), j^k\Sigma(y,u_0))$ avoids $B_{\text{in}}(k)$ for all $(x,y) \in Z \subset (X \times X \backslash \Delta X)$ and all $u_0 \in T$ is residual. If Z is compact, it is open, by openness of the nonintersection property on compact sets.

3.3 Proof of observability theorems 2.1, 2, 3, 4

Just applying the results of the previous Sects. 3.1, 3.2, to the case where $S = S^r$ (resp. $S = s^r$), *proves the Theorems* 2.1, 2.2.

Proof of Theorem 2.3. The set of embeddings of $(X \times T \times I_B^{(k-1)d_u})$ into $R^{k\,d_y} \times R^{k\,d_u}$ is open. On the other hand, the map:

$$S^r \to C^{r-k+1}(X \times T \times I_B^{(k-1)d_u}, R^{k\,d_y} \times R^{k\,d_u})\,,$$

$$\Sigma \to \bar{R}^*_{k,\Sigma}\,,$$

where $\bar{R}^*_{k,\Sigma}$ is the restriction of $R^*_{k,\Sigma}$ to $X \times T \times I_B^{(k-1)d_u}$, is continuous (by compactness).

Hence, the set of (Σ) in S^r (resp. s^r) such that $\bar{R}^*_{k,\Sigma}$ is an embedding is open. It is dense by Theorem 2.2. □

Proof of Theorem 2.4. We consider a fixed C^r system Σ_0. We approximate it in the C^r topology by a real analytic one, Σ_1. By our Sect. 2.1, Σ_1 is in one of our Banach spaces S_K (resp. s_K). The Banach spaces S_K (resp. s_K) satisfy the assumptions A1, A2 of Sect. 2.1, by their properties i, ii, iii of the same section. We just apply the results of Sects. 3.1, 3.2 to the case where $S = S_K$ (resp. s_K), to get the result.

4 Construction of the observer

4.1 Statement of the results

X, I and $\Sigma = (f,h)$ being given on $X \times I^{d_u}$, a C^k observer of Σ relative to the bound B on the derivatives of the control, is a triple $O_\Sigma = (R^n, F_{y,u,v}, \tau)$ where

$F_{y,u,v}$ is a parametrized family of C^r vector fields on X, depending smoothly on y, u, v and $\tau : X \times I^{d_u} \times R^{(k-1)d_u} \to R^n \times R^{k d_u}$ is an embedding of the form (setting $u = (u_0, \ldots, u_{k-1})$):

$$\tau(x,u) = \tau(x,u_0,u_1,\ldots,u_{k-1}) = (\phi(x,u_0,u_1,\ldots,u_{k-1}),u_0,u_1,\ldots,u_{k-1})$$

and the differential equation:

$$\text{(4.0)} \qquad \begin{cases} (\dot{x},\dot{u}) = f^k(x,u) + b^k u_k \\ \dot{z} = F_{h(x,u_0),u,u_k}(z) \end{cases},$$

(where f^k and b^k have been defined in (2.1.b)), satisfies for any $(x_0,z_0) \in X \times R^n$, any $u_0(t)$, k times differentiable with all the norms of the first k derivatives $(u_1(t),\ldots,u_{k-1}(t),u_k(t))$ bounded by B:

$\mathrm{Lim}_{t\to+\infty} \| \phi(x(u,u_k,t,x_0),u(t)) - Z(t,y,u,u_k,z_0) \| = 0$, in which $x(u,u_k,t,x_0)$ and $Z(t,y,u,u_k,z_0)$ denote the solutions of (4.0) at time t, associated to the control (u,u_k) and the initial conditions (x_0,z_0).

Let E_t be the projection set of $Z_t = Z(t,y,u,u_k,z_0)$ on $\phi(X,u(t))$:

$$E_t = \{x^* \in \phi(X,u(t)) \mid \|Z_t - x^*\| = \inf_{x\in X} \|Z_t - \phi(x,u(t))\|\} .$$

$(\phi_{u(t)})^{-1}(E_t)$ estimates $x(t)$ assymptotically in the sense that, for any metric δ on X (compatible with the topology of X):

$\mathrm{Lim}_{t\to+\infty} \delta(x(t),(\phi_{u(t)})^{-1}(E_t)) = 0$. (By compactness of X, I and boundedness of $(u_1,\ldots,u_{k-1})$). Also, for t large enough, E_t contains only one point.

Remark. The practical interest of an observer is that it allows asymptotic estimation of the state $x(t)$ of the system Σ on the basis of the knowledge of the outputs and the inputs, without differentiating the outputs. Differentiating inputs has physical meaning since usually, the inputs (with their derivatives) are chosen by the user and therefore are perfectly known.

Our results are the following:

Theorem 4.1 *As soon as $R^*_{k,\Sigma} : X \times I^{d_u} \times I_B^{(k-1)d_u} \to R^{k d_y} \times R^{k d_u}$ is an embedding for some bound $B > 0$ on the derivatives of u up to order k, there does exist a C^k observer for Σ (relative to B), with $n = k d_y$.* □

I.e. since in Sect. 3 we have proven that, for any arbitrary bound B, the set of $\Sigma \in S^r$ (resp. s^r) such that $R^*_{k,\Sigma}$ is an embedding (for k large enough) is open, dense, we conclude:

Theorem 4.2 *The set of systems that have a C^k observer, relative to the bound $B > 0$ is open dense (as soon as the number of outputs is strictly larger than the number of inputs).* □

Also, we have, with Theorem 2.2:

Theorem 4.3 *The set of systems that have, relative to every bound $B > 0$, a C^k observer (depending on this bound) is residual (with again $d_y > d_u$).* □

Our proof, moreover, is *constructive*. It is a slight improvement of the "High-gain" technique developed in our papers [G-H-O], [G-H-K], [G-K]. Moreover, the estimation rate can be made exponential arbitrary, i.e:

For any θ real positive, one can construct an observer such that:

$$\|\phi(x(u,u_k,t,x_0),u(t)) - Z(t,y,u,u_k,z_0)\| \leqq k e^{-\theta t} . \tag{4.1}$$

Or, for any Riemann metric δ on X:

$$\delta(x(t),(\phi_{u(t)})^{-1}(E_t)) \leqq k_\delta e^{-\theta t} . \tag{4.2}$$

4.2 Proof of Theorem 4.1 and construction of the observer

We consider $R_{k,\Sigma} : X \times I^{d_u} \times R^{(k-1)d_u} \to R^{k d_y}$, which is assumed to be such that $R_{k,\Sigma,u} = R_{k,\Sigma}(x,u)$ is an embedding for each $u = (u_0,\ldots,u_{k-1}) \in I^{d_u} \times R^{(k-1)d_u}$ (eventually $I^{d_u} \times I_B^{(k-1)d_u}$ only).

We consider the maps $\phi^i = (L_{f^k})^k h^i$ and $\psi^{i,j} = (L_{b_j^k})(L_{f^k})^{k-1} h^i$, where h^i is any component of h, $i = 1,\ldots,d_y$.

$$\phi^i, \psi^{i,j} : X \times I^{d_u} \times R^{(k-1)d_u} \to R .$$

Θ being one of these maps, consider:

$$\overline{\Theta} : W = R^*_{k,\Sigma}(X \times I^{d_u} \times R^{(k-1)d_u}) \subset R^{k d_y} \times R^{k d_u} \to R ,$$

$\Theta(x,u) = \overline{\Theta} \circ (R^*_{k,\Sigma})(x,u)$, (recall that $R^*_{k,\Sigma}(x,u) = (R_{k,\Sigma}(x,u),u)$). $\overline{\Theta}$ is well defined since $R^*_{k,\Sigma}$ is an embedding. Since W is closed, by the Whitney extension theorem (see [A-R] p. 120), $\overline{\Theta}$ can be extended to a C^r map defined on all of $R^{k d_y} \times R^{k d_u}$.

Consider W_B, the closed subset of W obtained when the derivatives $u_1,\ldots,u_{k-1}$ of the control are restricted to be bounded by B. $\overline{\Theta}$ can be made globally Lipschitz on all of $R^{k d_y} \times R^{k d_u}$ just by multiplication by a C^∞ function which is equal to 1 on a neighbourhood of W_B and equal to zero at infinity, without modifying its values on W_B.

Consider the system on $R^{k d_y}$:

$$\overline{\Sigma_k} : \left\{ \begin{array}{l} \dot{z}_i = z_{i+1} (z_i = (z_i^1,\ldots,z_i^{d_y}),\ i \leqq k-1) \\ \dot{z}_k = \bar{\phi}(z,u) + \bar{\psi}(z,u) u_k \\ \qquad \text{or} \\ \dot{z} = H(z,u,u_k) \end{array} \right\}$$

Σ^k embeds into $\overline{\Sigma^k}$ in the following sense: for any input $u(t)$, k times differentiable, such that $\|u^{(i)}(t)\| \leqq B$, the trajectories of Σ^k and $\overline{\Sigma^k}$ are mapped one to each-other by the mapping: $(x(t),u(t)) \to R_{k,\Sigma}(x(t),u(t))$. (Of course, $(L_{b^k})(L_{f^k})^l h^i = 0$ when $l < k-1$).

We consider now the candidate "High-gain, Luenberger-type" observer: $(O_k)\dot{\mu} = H(\mu,u,u_k) - K_\lambda(C \cdot \mu - y)$, where $K_\lambda : R^{d_y} \to R^{k d_y}$, $K_\lambda = \Delta_\lambda K$,

$\Delta_\lambda = \text{block-diag}(\lambda \text{Id}^{d_y}, \ldots, \lambda^k \text{Id}^{d_y})$, Id^{d_y} the identity $d_y \times d_y$ matrix, λ to be chosen later, $C = (\text{Id}^{d_y}, 0, \ldots, 0)$, $C : R^{k\,d_y} \to R^{d_y}$.

One can prove that, provided that λ is chosen large enough and that K is chosen such that $(A - KC)$ is a stable matrix (where A is the d_y-antishift matrix:

$$A = \begin{bmatrix} 0 & \text{Id}^{d_y} & 0.. & ..0 \\ 0.. & ..0 & . & 0 \\ 0.. & . & ..0 & \text{Id}^{d_y} \\ 0 & ... & ... & 0 \end{bmatrix}$$

– such a K does exist by observability of the linear pair (C,A) and by the standard linear theory) the relation 4.1. above holds for an arbitrary real θ. This will end the proof.

The proof of this last fact is nothing but a restatement in this case of the basic proof of [G-H-O], [G-H-K], [G-K]:

Considering $\overline{\Sigma^k}$ and O_k:

$$\overline{\Sigma^k} : \dot{z} = H(z, u, u_k)\,,$$

$$O_k : \dot{\mu} = H(\mu, u, u_k) - K_\lambda(C \cdot \mu - y)\,,$$

with $z = (z_1', \ldots, z_k')'$, $z_i = (z_i^1, \ldots, z_i^{d_y})'$, $\mu = (\mu_1', \ldots, \mu_k')'$, $\mu_i = (\mu_i^1, \ldots, \mu_i^{d_y})'$. We set $\varepsilon = \mu - z$ and we make the changes of variables:

$$z_i^* = \frac{1}{\lambda^i} z_i, \qquad \mu_i^* = \frac{1}{\lambda^i}\mu_i, \qquad \varepsilon_i^* = \frac{1}{\lambda^i}\varepsilon_i\,.$$

The equation of ε^* writes:

$$\begin{aligned} \text{for } i < k\text{: } \frac{d}{dt}(\varepsilon_i^*) = \frac{1}{\lambda^i}\frac{d}{dt}(\varepsilon_i) &= \frac{1}{\lambda^i}\varepsilon_{i+1} - \frac{1}{\lambda^i}K_{\lambda,i}\varepsilon_1 \\ &= \frac{1}{\lambda^i}\varepsilon_{i+1} - K_i\varepsilon_1 = \lambda\varepsilon_{i+1}^* - \lambda K_i\varepsilon_1^* \\ &= \lambda(\varepsilon_{i+1}^* - K_i\varepsilon_1^*)\,. \end{aligned}$$

For $i = k$:

$$\frac{d}{dt}(\varepsilon_k^*) = \frac{1}{\lambda^k}\frac{d}{dt}(\varepsilon_k) = \frac{1}{\lambda^k}(\bar{\phi}(\mu, u) + \bar{\psi}(\mu, u)u_k - \bar{\phi}(z, u) - \bar{\psi}(z, u)u_k - \lambda^k K_k\varepsilon_1)$$

$$\begin{aligned} \frac{d}{dt}(\varepsilon_k^*) = -\lambda K_k\varepsilon_1^* + \frac{1}{\lambda^k}(&\bar{\phi}(\Delta_\lambda\mu^*, u) - \bar{\psi}(\Delta_\lambda z^*, u)u_k \\ &- (\bar{\phi}(\Delta_\lambda z^*, u) - \bar{\psi}(\Delta_\lambda\mu^*, u)u_k)) \end{aligned}$$

The functions $\bar{\phi}^* = \frac{1}{\lambda^k}\bar{\phi}(\Delta_\lambda z, u)$ and $\bar{\psi}^* = \frac{1}{\lambda^k}\bar{\psi}(\Delta_\lambda z, u)$ are globally Lipschitz with respect to z uniformly with respect to u and have the same

Lipschitz constant L as $\bar{\phi}$ and $\bar{\psi}$. Hence:

$$\frac{d}{dt}(\varepsilon^*) = \lambda(A - KC)\varepsilon^* + b^*((\bar{\phi}^* + \bar{\psi}^* u_k)(\mu^*, u) - (\bar{\phi}^* + \bar{\psi}^* u_k)(z^*, u)) ,$$

with $b^* = (0, \ldots, 0, \mathrm{Id}^{d_y})'$.

Choosing a positive definite solution S of the matrix equation:

$$(A - KC)'S + S(A - KC) = -\mathrm{Id} ,$$

which is possible by Lyapunov, we get:

$$\begin{aligned}\frac{d}{dt}(\varepsilon^{*\prime} S \varepsilon^*) &= -\lambda \varepsilon^{*\prime}\varepsilon^* + 2\varepsilon^{*\prime} S b^* \\ &\quad \times ((\bar{\phi}^* + \bar{\psi}^* u_k)(\mu^*, u) - (\bar{\phi}^* + \bar{\psi}^* u_k)(z^*, u)) \\ &\leqq -\lambda \varepsilon^{*\prime}\varepsilon^* + h\|\varepsilon^*\|^2 ,\end{aligned}$$

for a certain $h > 0$, independent of λ.

The result follows for λ large enough. □

Comment. As we mentioned in the introduction, other constructions are possible, that are all based upon the same principle. For instance, the "high-gain extended Kalman filter" construction is specially interesting, since it can have a stochastic interpretation, and it is also practically more tractable, less sensitive to uncertainty. See [D-B-G2] and, for a nice application, [D-B-G1].

5 Equivalence between differential observability and observability in the analytic case

In this section, we consider analytic systems and we prove that for these systems, C^ω-observability implies L^∞-observability. In fact, these notions are equivalent.

To prove this, we need first to introduce some technical tools, mainly the tangent cone to a subanalytic set.

Remark. The result of this section (Theorem 5.1): equivalence between differential observability and general observability, is false in the C^∞ case. Counterexamples are known.

5.1 Preliminaries

We start with X, a real analytic manifold, $Z \subset X$ a subanalytic subset, Z_{reg} the subset formed by regular points of Z, that is, the set of $x \in Z$ such that there exists an open neighbourhood O of x in X such that $O \cap Z$ is a real analytic connected manifold. Z_{reg} is a disjoint union of analytic manifolds and it is open and dense in Z. Z_{reg} is also subanalytic in X (see [B-M]). This is the key point for the proof of the facts p1 to p6 below.

Let us now define $TC(Z)$, the tangent cone to Z: TZ_{reg}, the tangent bundle to Z_{reg} is well defined since Z_{reg} is smooth. Denote by $\Pi_X : TX \to X$ the canonical projection, then we define:

$$TC(Z) = \Pi_X^{-1}(Z) \cap \overline{TZ}_{\text{reg}}$$

$\overline{TZ}_{\text{reg}}$ is the closure of TZ_{reg} in TX.

Now, we state the main properties of $TC(Z)$:

P1. $TC(Z)$ is subanalytic in TX. If Z is closed then $TC(Z) = \overline{TZ}_{\text{reg}}$.

P2. If Z is smooth, $TC(Z) = TZ$.

P3. If $Y \subset Z$ is subanalytic open in Z, then the restriction:

$$TC(Z)_{|Y} = TC(Y)\,.$$

$TC(Z)_{|Y} = \Pi_Z^{-1}(Y)$, where $\Pi_Z : TC(Z) \to Z$ denotes the canonical projection (restriction of $\Pi_X : TX \to X$ to $TC(Z)$).

In particular, since Z_{reg} is open in Z,

$$TC(Z)_{|Z_{\text{reg}}} = TC(Z_{\text{reg}}) = TZ_{\text{reg}}\,.$$

P4. If $Z = \{Z_\alpha \mid \alpha \in A\}$ is a stratification of Z meeting the condition (a) of Whitney, then, $TC(Z) \supset TZ_\alpha$ for all $\alpha \in A$.

P5. If $x : [\alpha, \beta] \to X$ is an absolutely continuous curve, the image of which is contained in Z, then the set of $t \in [\alpha, \beta]$ such that $\frac{dx(t)}{dt}$ exists and is contained in $TC(Z)$ has full measure.

P6. The dimension of $TC(Z)$ is $2 \dim Z$.

The proof of these facts is postponed to Sect. 5.4.

5.2 Observability

We consider an analytic system (Σ) on X as in the previous sections ($\dot{x} = f(x,u)$, $y = h(x,u)$). We just assume that u takes its values in a semi-analytic compact set U (which is the case for the systems considered previously: $T = I^{d_u}$). The output space Y is just assumed to be an analytic manifold, *X is not assumed to be compact.*

Let $X_*^{(2)} = X \times X \backslash \Delta X, \Delta X$ being the diagonal of $X \times X$, and let $h \times h : X \times X \times U \to Y \times Y$ be the analytic map $(x_1, x_2, u) \to (h(x_1,u), h(x_2,u))$.

We define now two decreasing sequences of subanalytic subsets of $X_*^{(2)} \times U$ and $X_*^{(2)}$ respectively, as follows:

$$\hat{\Gamma}_0 = X_*^{(2)} \cap (h \times h)^{-1}(\Delta Y) = \{(x_1, x_2, u) \mid h(x_1,u) = h(x_2,u), x_1 \neq x_2\}\,,$$

$\Gamma_0 = \text{Pr}_{1,2}\hat{\Gamma}_0, \text{Pr}_{1,2} : X \times X \times U \to X \times X$ is the projection $(x_1, x_2, u) \to (x_1, x_2)$. $\text{Pr}_{1,2}$ is a *proper* map. It is clear that $\hat{\Gamma}_0$ is semianalytic in $X \times X \times U$, closed in $X_*^{(2)} \times U$. Γ_0 is subanalytic in $X \times X$, closed in $X_*^{(2)}$, which is itself an analytic manifold.

Assume that $\hat{\Gamma}_n, \Gamma_n$ have been defined for $n \leqq m$, $\hat{\Gamma}_n$ subanalytic closed in $X_*^{(2)} \times U$, $\Gamma_n = \text{Pr}_{1,2}(\hat{\Gamma}_n)$, subanalytic closed in $X_*^{(2)}$.

Let us denote by $\hat{f} : X_*^{(2)} \times U \to TX_*^{(2)} \subset TX \times TX$, the analytic map $(x_1, x_2, u) \to (f(x_1, u), f(x_2, u))$. Then,

$$\hat{\Gamma}_{m+1} = \hat{f}^{-1}(TC(\Gamma_m)) \cap \hat{\Gamma}_m$$
$$\Gamma_{m+1} = \mathrm{Pr}_{1,2}(\hat{\Gamma}_{m+1}) .$$

It is clear that $\hat{\Gamma}_{m+1}$ is subanalytic closed in $X_*^{(2)} \times U$ and Γ_{m+1} is subanalytic closed in $X_*^{(2)}$. Also, $\hat{\Gamma}_0 \supset \hat{\Gamma}_1 \supset \cdots \supset \hat{\Gamma}_n \supset \cdots, \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots$.

We will show that as soon as $n \geqq 2 \dim X + 1 = N$, $\Gamma_n = \Gamma_N$ and $\hat{\Gamma}_n = \hat{\Gamma}_{N+1}$ for $n \geqq N + 1$.

First of all, let us recall the notion of local dimension ([Hi]): if Z is a subanalytic subset of a real analytic manifold X, and if $x \in X$, $\dim_x(Z) = \inf_V \{\dim(V \cap Z_{\mathrm{reg}}) \mid V$ open neighbourhood of $x\}$, with the convention that $\dim \varnothing = -1$. If $Z' \subset Z$ is a subanalytic subset of X, then for all $x \in X$:

$$\dim_x Z' \leqq \dim_x Z .$$

We need the following fundamental lemma:

Lemma 5.1 *If for some integer $m \in \{-1, 0, 1, 2, \ldots\}$ and for a $z \in X_*^{(2)}$, $\dim_z \Gamma_m = \dim_z \Gamma_{m+1}$ then for all $n \geqq m$, $\dim_z \Gamma_n = \dim_z \Gamma_m$.* □

The proof of this lemma is postponed to the next Sect. 5.3.

Corollary 5.1 *As soon as $n > \dim \Gamma_0$, $\Gamma_{n+1} = \Gamma_n$.* □

Proof. By the Lemma 5.1, for all $z \in X_*^{(2)}$ as soon as $n > \dim_z \Gamma_0$, $\dim_z \Gamma_{n+1} = \dim_z \Gamma_n$. Since $\dim \Gamma_0 = \sup_{z \in X} \dim_z \Gamma_0$, $\dim_z(\Gamma_{n+1}) = \dim_z(\Gamma_n)$ for all $z \in X_*^{(2)}$ and all $n > \dim \Gamma_0$. This implies that $\Gamma_{n+1} = \Gamma_n$ since the Γ_n are closed in $X_*^{(2)}$.

Corollary 5.2 *As soon as $n > 2 \dim X + 1$, $\hat{\Gamma}_{n+1} = \hat{\Gamma}_n$.* □

Proof. $\hat{\Gamma}_{n+1} = \hat{f}^{-1}(TC(\Gamma_n)) \cap \hat{\Gamma}_n$. Since $n > \dim \Gamma_0 + 1$, Corollary 6.1 shows that $\Gamma_n = \Gamma_{n-1}$. Therefore $\hat{\Gamma}_{n+1} = \hat{f}^{-1}(TC(\Gamma_{n-1})) \cap \hat{\Gamma}_n = \hat{f}^{-1}(TC(\Gamma_{n-1})) \cap \hat{\Gamma}_{n-1}$ because $\hat{\Gamma}_n = \hat{f}^{-1}(TC(\Gamma_{n-1})) \cap \hat{\Gamma}_{n-1}$. Hence $\hat{\Gamma}_{n+1} = \hat{\Gamma}_n$.

Setting $\Gamma = \bigcap_{n \geqq 0} \Gamma_n$ and $\hat{\Gamma} = \bigcap_{n \geqq 0} \hat{\Gamma}_n$, the Corollary 5.2 implies obviously the following proposition:

Proposition 5.1 (i) *$\Gamma = \Gamma_m$ for $m > 2 \dim X$;*
(ii) *$\hat{\Gamma} = \hat{\Gamma}_m$ for $m > 2 \dim X + 1$.* □

The basic fact about Γ is:

Proposition 5.2 *Assume that there exist two trajectories $(\bar{x}_i, \bar{u}) : [0, \bar{T}] \to X \times U$, $i = 1, 2$, of the system such that $\bar{x}_1(0) \neq \bar{x}_2(0)$ and $h(\bar{x}_1, \bar{u}) = h(\bar{x}_2, \bar{u})$ for almost all $t \in [0, \bar{T}]$. Then $\Gamma \neq \varnothing$ and $\hat{\Gamma} \neq \varnothing$.* □

The proof of Proposition 5.2 is also postponed to the next Sect. 5.3.

By definition and by Proposition 5.1, Γ and $\hat{\Gamma}$ satisfy:

(a) $\hat{f}(\hat{\Gamma}) \subset TC(\Gamma)$;

(b) $\Gamma = \mathrm{Pr}_{1,2}(\hat{\Gamma})$.

Pick any $(x_1,x_2) \in \Gamma_{\mathrm{reg}}$. There is a u such that $(x_1,x_2,u) \in \hat{\Gamma}$. (a), (b) above and the property P3 of the $TC(\Gamma)$ imply that, for any such u:

(c) $\hat{f}(x_1,x_2,u) \in T_{(x_1,x_2)}\Gamma_{\mathrm{reg}}$.

Also, since $\mathrm{Pr}_{1,2}(\hat{\Gamma}) = \Gamma$ and $\mathrm{Pr}_{1,2}$ is proper, it is known ([Hi1], [Ha1], [Ha], [G-P], for instance) that the restriction $\mathrm{Pr}_{1,2}|\hat{\Gamma}$ of Pr1,2 to $\hat{\Gamma}$ can be stratified, that is: for any stratum S of $\Gamma, \hat{\Gamma} \cap \mathrm{Pr}_{1,2}^{-1}(S)$ is a union of strata in $\hat{\Gamma}$ and the restriction of $\mathrm{Pr}_{1,2}$ to such a stratum $\hat{S}, \mathrm{Pr}_{1,2}|\hat{S}$, is an analytic submersion. We pick two such strata $S,\hat{S},S$ being of maximum dimension $\dim \Gamma$ (so that S is contained in Γ_{reg}). Considering the submersion $\mathrm{Pr}_{1,2}|\hat{S} \to S$, we see that we can chose an analytic section:

$\hat{u} : V \to \hat{S}$, $\hat{u}(x_1,x_2) = (x_1,x_2,u(x_1,x_2))$, where V is an open neighbourhood of (x_1°,x_2°) in S.

Also, by (c) above, we know that, for all $(x_1,x_2) \in V, H(x_1,x_2) = (f(x_1,u(x_1,x_2)), f(x_2,u(x_1,x_2))) \in T_{(x_1,x_2)}V$. Therefore, H is an analytic vector field on V. Let $\gamma = (x_1(t),x_2(t))$, $t \in [0,T]$, be an integral curve of H on V. Setting $v(t) = u(x_1(t),x_2(t))$ for $t \in [0,T]$, one has by construction: $h(x_1(t),v(t)) = h(x_2(t),v(t))$ since $V \subset S \subset \Gamma_0$. On the other hand, $x_1(t)$ and $x_2(t)$ are, by construction, distinct trajectories of our system, relative to the input $v(t)$. Hence, $v(t)$ is an analytic function defined on the time interval $[0,T]$, that "makes our system unobservable".

Finally we have shown that, if our system is "unobservable" for some input, it is "unobservable" for another C^ω input.

Our conclusions are summarized in the next theorem:

Theorem 5.1 *For the systems Σ under consideration (either $\Sigma \in S^\omega$ or $\Sigma \in s^\omega$), the following properties are equivalent:*

(i) *Σ is L^∞-observable,*

(ii) *Σ is C^ω-observable.* □

5.3 Proof of Lemma 5.1 and Proposition 5.2

1. Proof of Lemma 5.1. It is sufficient to show that $\dim_z \Gamma_{m+2} = \dim_z \Gamma_{m+1}$. There is an open neighbourhood V_0 of z in X such that for every neighbourhood V of z contained in $V_0, \dim(V \cap \Gamma_n^{\mathrm{reg}}) = \dim_z \Gamma_n$ for $n = m, m+1, m+2$. Since $\dim_z \Gamma_m = \dim_z \Gamma_{m+1}$, $V \cap \Gamma_{m+1}$ contains a submanifold ω_V of dimension $\dim_z \Gamma_m$ and open in Γ_m. If $y \in \omega_V$, since $y \in \Gamma_{m+1}$, there is a $u_y \in U$ such that $(y,u_y) \in \hat{\Gamma}_{m+1}$. Therefore, $\hat{f}(y,u_y) \in TC_y(\Gamma_m)$. By the property P2, Sect. 5.1, $TC_y(\Gamma_m) = TC_y(\omega_V) = T_y(\omega_V) = TC_y(\Gamma_{m+1})$. (Since $\omega_V \subset \Gamma_{m+1} \subset$

Γ_m, ω_V is also open in Γ_{m+1}). Finally:

$$\hat{f}(y,u_y) \in TC(\Gamma_{m+1}) ,$$

$$(y,u_y) \in \hat{\Gamma}_{m+1} ,$$

which shows that $(y,u_y) \in \hat{\Gamma}_{m+2}$ and hence $y \in \Gamma_{m+2}$.

Therefore, for each open neighbourhood V of z, contained in V_0, $\omega_V \subset \Gamma_{m+2}$. On the other hand, there exists an open neighbourhood W of z, $W \subset V_0$, such that $\dim_z \Gamma_{m+2} = \dim(W \cap \Gamma_{m+2})$. Since $\omega_V \subset \Gamma_{m+2}$ and $\dim(W \cap \Gamma_{m+1}) = \dim_z \Gamma_{m+1}$ (by the definition of V_0), one has:

$$\begin{aligned} \dim_z \Gamma_m = \dim \omega_V &\leqq \dim_z \Gamma_{m+2} = \dim(W \cap \Gamma_{m+2}) \\ &\leqq \dim(W \cap \Gamma_{m+1}) = \dim_z \Gamma_{m+1} . \end{aligned}$$

Finally, $\dim_z \Gamma_m = \dim_z \Gamma_{m+2} = \dim_z \Gamma_{m+1}$. □

2. Proof of Proposition 5.2. Since the $\bar{x}_i$, $i = 1, 2$ are continuous and $\bar{x}_1(0) \neq \bar{x}_2(0)$, there exists a T, $0 < T \leqq \bar{T}$ such that $\bar{x}_1(t) \neq \bar{x}_2(t)$ for all $t \in [0,T]$. Let us denote by $\bar{z} : [0,T] \to X_*^{(2)}$ the curve $\bar{z}(t) = (\bar{x}_1(t), \bar{x}_2(t))$.

We will show by induction on n that for almost all $t \in [0,T]$, $(\bar{z}(t), \bar{u}(t)) \in \hat{\Gamma}_n$. This will imply the proposition.

For $n = 0$, it results from the assumption of the proposition. Assume that it is proved for $n \leqq m$. Then $\bar{z}$ is an absolutely continuous curve, taking its values in Γ_m. By the property P5, Sect. 5.1, for almost all $t \in [0,T]$:

$$\frac{d\bar{z}}{dt} \in TC(\Gamma_m) ,$$

this shows that for almost all $t \in [0,T]$,

$$\hat{f}(\bar{z}(t), \bar{u}(t)) \in TC(\Gamma_m) ,$$

hence for almost all $t \in [0,T]$, $(\bar{z}(t), \bar{u}(t)) \in \hat{f}^{-1}(TC(\Gamma_m)) \cap \hat{\Gamma}_m = \hat{\Gamma}_{m+1}$. □

5.4 Proof of properties p1 to p6 of TC(Z)

1. Proof P1. (See also [D-W]). The question being local, we can assume that $X = R^n$, $TX = X \times X$, $Z \subset X$. $\Pi_X^{-1}(Z) = Z \times X$ is therefore subanalytic as a product of subanalytic sets. Hence it is sufficient to show that $\overline{TZ}_{\text{reg}}$ is subanalytic, which is implied by the fact that TZ_{reg} is subanalytic. Remember that Z_{reg} is subanalytic, and is an analytic submanifold of X, could be not of pure dimension.

So, let us consider now Z, like Z_{reg} above, that is: Z is subanalytic in X, and Z is smooth.

Let S be the unit sphere of $X = R^n$. Consider INC, the subset of $X \times X \times S$ defined by $\text{INC} = \{(x,y,v) \mid x, y \in X,\ v \in S,\ (y-x) \wedge v = 0,\ \langle (y-x), v \rangle \geqq 0\}$ ($\langle \cdot , \cdot \rangle$ denotes the Euclidian scalar product on R^n). INC is semianalytic, it is

also a smooth submanifold of $X \times X \times S$, with a boundary. Let $p : \mathrm{INC} \to X \times X$ be the restriction to INC of the canonical projection $\mathrm{Pr}_{1,2} : X \times X \times S \to X \times X$. $Z \times Z$ is subanalytic in $X \times X$. ΔX, the diagonal of $X \times X$ is analytic in $X \times X$. Therefore, $Z*^{(2)} = Z \times Z \setminus (\Delta X \cap Z \times Z) = Z \times Z \setminus \Delta Z$ is subanalytic. Hence $p^{-1}(Z*^{(2)}) = \mathrm{Pr}_{1,2}^{-1}(Z_*^{(2)}) \cap \mathrm{INC} = \mathrm{Sec}(Z)$ is still subanalytic. Set $\mathrm{Dir}(Z) = \overline{\mathrm{Sec}(Z)} \cap p^{-1}(\Delta Z)$. Both $\overline{\mathrm{Sec}(Z)}$ and $p^{-1}(\Delta Z) = \{(z,z,v) \mid z \in Z,\ v \in S\}$ are subanalytic hence so is $\mathrm{Dir}(Z)$ ($\mathrm{Dir}(Z)$ is nothing but the set of (z,z,v) such that v is a tangent direction to Z at z : (z,z,v) belongs to $\mathrm{Dir}(Z)$ iff there is a sequence $(x_n, y_n) \to (z,z)$, $x_n \neq y_n$, $x_n, y_n \in Z$, $\frac{y_n - x_n}{\|y_n - x_n\|} \to v$).

Consider now $\mu : X \times X \times S \times R \to X \times X = TX$,

$$\mu(x, y, v, t) = (x, v\,t)$$

Restricted to $\Delta X \times S \times R$, μ is proper. $R_+ \subset R$ is semianalytic, therefore, $\mu(\mathrm{Dir}(Z) \times R_+)$ is subanalytic. But, clearly, $\mu(\mathrm{Dir}(Z) \times R_+) = TZ$. TZ is subanalytic. □

2. *Proof P2.* If Z is smooth (that is, if all $z \in Z$ has a neighbourhood in X, $V(z)$ such that $Z \cap V(z)$ is a manifold), then, $TC(Z) = TZ$:

If Z is smooth, $Z_{\mathrm{reg}} = Z$, $TZ_{\mathrm{reg}} = TZ$.

$$TC(Z) = \overline{TZ}_{\mathrm{reg}} \cap \Pi_X^{-1}(Z) = \overline{TZ} \cap \Pi_X^{-1}(Z) = TZ\ .$$

3. *Proof P3.*

$$TC(Z)_{|Y} = \Pi_Z^{-1}(Y) = \Pi_X^{-1}(Y) \cap TC(Z)$$

$$= \Pi_X^{-1}(Y) \cap \Pi_X^{-1}(Z) \cap \overline{TZ}_{\mathrm{reg}} = \Pi_X^{-1}(Y) \cap \overline{TZ}_{\mathrm{reg}}$$

$$TC(Y) = \Pi_X^{-1}(Y) \cap \overline{TY}_{\mathrm{reg}}\ .$$

But $Y_{\mathrm{reg}} = Z_{\mathrm{reg}} \cap Y$, hence $TY_{\mathrm{reg}} \subset TZ_{\mathrm{reg}}$, $\overline{TY}_{\mathrm{reg}}, \subset \overline{TZ}_{\mathrm{reg}}$, $TC(Y) \subset TC(Z)_{|Y}$. Otherwise, if $\xi_y \in \Pi_X^{-1}(Y) \cap TC(Z)$, $y \in Y$ and there is a sequence $\xi y_n \to \xi_y$, $\xi y_n \in TZ_{\mathrm{reg}}$, $y_n \in Z_{\mathrm{reg}}$, $y_n \to y$. Therefore, for n sufficiently large, since Y is open in Z, $y_n \in Y_{\mathrm{reg}}$, $\xi y_n \in TY_{\mathrm{reg}}$, hence $\xi_y \in \overline{TY}_{\mathrm{reg}}$. At the end, $\xi_y \in \Pi_X^{-1}(Y) \cap \overline{TY}_{\mathrm{reg}} = TC(Y)$. □

4. *Proof P4.* Let Z_α be a maximal stratum of Z (i.e. there is no Z_β, $Z_\alpha \neq Z_\beta$, $Z_\alpha \subset \overline{Z_\beta}$). Then, $Z_\alpha \subset Z_{\mathrm{reg}}$ and by P3, $TZ_\alpha \subset TZ_{\mathrm{reg}} \subset TC(Z)$. If Z_α is not maximal, i.e. $Z_\alpha \subset \overline{Z_\beta}$, then by Whitney condition (a), $T(Z_\alpha) \subset \overline{T(Z_\beta)}$. Hence $T(Z_\alpha) \subset \overline{TZ}_{\mathrm{reg}}$. Also, since Z_α is smooth, $T(Z_\alpha) \subset \Pi_X^{-1}(Z_\alpha)$, therefore $T(Z_\alpha) \subset TC(Z)$.

5. *Proof P5.* Let δ be the considered time interval. Let $\{Z_\alpha \mid \alpha \in A\}$ be a Whitney stratification of Z by analytic manifolds that are also subanalytic in X. For each α in A, let δ_α be the set of $t \in \delta$ such that $x(t) \in Z_\alpha$. The set of points of δ_α that are not accumulation points of δ_α has measure zero. Let $\hat{\delta}_\alpha$ be the set of $t \in \delta_\alpha$ that are accumulation points of δ_α.

The set of t such that $\frac{dx(t)}{dt}$ exists has full measure in δ. Pick such a t. $t \in \delta_\alpha$ for some α. If $t \in \hat{\delta}_\alpha$, then $\frac{dx(t)}{dt} \in T_{x(t)}Z_\alpha$, since Z_α is smooth. But we know by P4 that $T(Z_\alpha) \subset TC(Z)$. $\frac{dx(t)}{dt} \in TC(Z)$. □

6. Proof P6. $\dim Z = \dim Z_{\text{reg}}$.

$$\dim TC(Z) = \dim TZ_{\text{reg}} = 2 \dim Z_{\text{reg}} .$$

6 The approximation theorem

Observe that, if Σ, analytic, is as in the previous Sects. 3, 4, such that $R^*_{k,\Sigma}$ is an embedding, then, Σ is observable for all C^ω inputs (Σ is observable for all C^k inputs, which is stronger):

A C^k input $u(t)$ being given on some interval $[0, \Theta]$; $x_1, x_2, x_1 \neq x_2$ being given initial conditions, assume that the corresponding outputs are equal on some time subinterval $[0, \tau]$, then, their $k-1$ first derivatives at time zero are also equal, and they, with the $u_j(0)$, are just the components of $R^*_{k,\Sigma}$ by definition. This is impossible since $R^*_{k,\Sigma}$ is injective. Σ is observable for u.

In fact, the fact that $R^*_{k,\Sigma}$ is an embedding expresses that $P_{\Sigma,u}$, the "*initial state to output map*" (the map which to the initial state associates the output trajectory) is *an embedding*, for the considered k-times differentiable input, although Σ observable means that $P_{\Sigma,u}$ is only injective, but for all L^∞ inputs u.

The results of Sects. 3, 5, show that, for $d_y > d_u$:

1. any C^r system Σ^0 can be C^r approximated by an (analytic) C^ω-observable one Σ^1 (Sect. 3),
2. Σ^1 is in fact observable (for all L^∞ inputs) (Sect. 5).

Consequently:

Theorem 6.1 (*approximation by observable analytic systems*). *Any system* $\Sigma^0 \in S^r$ (*resp.* s^r) *can be* C^r *approximated by an observable one* $\Sigma^1 \in S^r$ (*resp.* s^r) (*observable for all* L^∞ *inputs*), *that moreover can be chosen analytic and such that* $R^*_{k,\Sigma}1$ *is an embedding.* □

7 Appendix

A.1 Unobservable linear systems

$K = R$ or C. The set $\mathbb{L}_K$ of linear systems on K is the set of couples $L = (C, A)$ of linear maps $C \in \text{Lin}(K^{d_x}, K^{d_y})$, $A \in \text{End}(K^{d_x})$. $\mathbb{L}_K = \text{Lin}(K^{d_x}, K^{d_y}) \times \text{End}(K^{d_x})$.

$L = (C, A)$ is said observable if $V_L = \bigcap_{i \geqq 0} \text{Ker}\, CA^i = \{0\}$.

The set of unobservable systems is denoted by $\mathbb{U}_K$. $\mathbb{U}_K$ is algebraic.

If L is unobservable, V_L is nonzero and V_L is an invariant subspace contained in the kernel of C. Hence, V_L contains a complex eigenvector P of A,

such that $CP = 0$. Conversely, if $CP = 0$ for a complex eigenvector P of A, $CA^{ip} = 0$ and V_L is nonzero. Hence, $L \in \mathbb{U}_K$ if and only if $CP = 0$, P a complex eigenvector of A.

Let $\mathbb{V}_K \subset \mathbb{U}_K$, $\mathbb{V}_K = \{L \in \mathbb{L}_K \mid C \text{ vanishes on a } K\text{-eigenvector of } A\}$. $\mathbb{V}_C = \mathbb{U}_C$, $\mathbb{V}_R \subset \mathbb{U}_R$.

Let us prove that codim $\mathbb{V}_K = d_y$. This will imply that codim $\mathbb{U}_K = d_y$.

Let $\mathbb{V}_K^*$ be the subspace of $\mathbb{L}_K \times P(K^{d_x})$, $P(K^{d_x})$ the d_{x-1} projective space, of all couples (L, l), such that $Cl = 0$, $Al \subset l$. If $\Pi_2 : \mathbb{L}_K \times P(K^{d_x}) \to P(K^{d_x})$ is the projection, on the second factor, $\Pi_2^{-1}(l) \cap \mathbb{V}_K^* = \{A \in \mathrm{End}(K^{d_x}) \mid Al \subset l\} \times \{C \in \mathrm{Lin}(K^{d_x}, K^{d_y}) \mid Cl = 0\}$.

$\mathrm{Codim}\,(\Pi_2^{-1}(l) \cap \mathbb{V}_K^*, \mathbb{L}_K) = d_x + d_y - 1$. Hence, codim $\mathbb{V}_K^* = d_x + d_y - 1$. The projection $\Pi_1(\mathbb{V}_K^*) = \mathbb{V}_K$ has therefore codimension $dx + dy - 1 - (dx - 1) = dy$.

A.2 Lemmas

If V is a vector space, $\mathrm{Sym}^a(V)$ denotes the space of symmetric tensors of degree a on V, which can be canonically identified with the homogeneous polynomials of degree a on V^*, dual of V. The symbol $\odot$ means "symmetric tensor product".

The map $R_{k,\Sigma}$ of Sect. 2.1.b depends only on the k-jet of Σ, hence, it induces a map $R_j k_\Sigma$.

Let $(x, u_0, u') \in X \times T \times R^{(k-1)d_u}$ and set $p = (x, u_0)$. Let $j^k f(p) \in J^k F(p)$ be such that $f(p) \neq 0$.

Lemma 1 *Recall that $J^k H_0$ is the subspace of $J^k H$ formed by the $j^k h$ such that h depends on x only.*

(i) *The mapping $\Theta_0 : J^k H_0(x) \to R^{k\,d_y}$, $j^k h(x) \to R_j k_\Sigma(x, u_0, u')$ is linear and surjective.*

(ii) *The mapping $\Theta_1 : J^k H_0(x) \to (R^{k\,d_y} \otimes T_x^* X)$, $j^k h(x) \to T_x R_{k,\Sigma}(x, u_0, u')$ is linear and surjective.*

(iii) *For any $\Lambda \in T_x X, \Lambda \neq 0$, the mapping,*

$$\Theta_\Lambda : J^k H(p)(\textit{resp. } J^k H_0(x)) \to R^{k\,d_y},\ (p = (x, u_0)),$$

$j^k h(p)$ (resp. $j^k h(x)$) $\to T_x R_{k,\Sigma}(p, u'; \Lambda)$ is linear and surjective.

Note. It is not true that the mapping: $J^k H_0(x) \to J^1(X \times T, R^{k\,d_y})$, $j^k h(x) \to j^1 R_j k_\Sigma(p, u')$ is surjective.

Proof. Let f be a representant of $J^k f(p)$ ($p = (x, u_0)$). Take a coordinate system $(O, x_1, \ldots, x d_x)$ for X at x, then:

$$(L_{fk})^n h(p, u') = d_x^n h(x; f(p)^{\odot n}) + \sum_{a=0}^{n-1} d_x^a h(x; R_{a,n}(j^k f(p), u')),$$

$R_{a,n}$ is a universal polynomial mapping: $J^k f(p) \times R^{(k-1)d_u} \to \mathrm{Sym}^a(T_x X)$. This gives immediately (i), since $\Theta_0(h) = (L_{fk})^n h(p, u')$.

Let $e_1,\dots,e_{d_x}$, be a basis of T_xX such that $f(p)=e_1$. Then,

$$d_x(L_{f^k})^n h(p,u')(e_i) = d_x^{n+1}h(x;e_i\odot e_1^{\odot n}) + \sum_{a=1}^{n} d_x^a h(x;Q_{a,n}(j^{n-a}f(p),u',e_i))\,,$$

where $Q_{a,n}: J^{n-a}F(p)\times R^{(k-1)d_u}\to \mathrm{Sym}^a(T_xX)$ is a universal mapping.

Since the elements $e_i\odot e_1^{\odot n}$, $i=1,\dots,d_x$ are linearly independent in $\mathrm{Sym}^a(T_xX)$, the surjectivity of Θ_1 follows. (iii) is an obvious consequence of (ii). □

Lemma 2 1) *Let $(j^kf(y),\Lambda,u')$ be an element of $J^kF\underset{X}{\times}TX\times R^{(k-1)d_u}$ such that* (i) $f(y)=0$, (ii) $u_1=\cdots=u_{\rho-1}=0$ ($\rho\geqq 1!$), (iii) *the vectors $T_xf(y)^i\Lambda$, $0\leqq i\leqq\rho$ are linearly independent*, (iv) $d_Uf(y,u_\rho)\neq 0$. *Then, the linear mapping $\Theta_\Lambda: J^kH(y)$ (resp. $J^kH_0(x_0)$, $y=(x_0,u_0)$) $\to R^{k\,d\,y}, j^kh(y)$ (resp. $j^kh(x_0)$) $\to T_xR_{k,\Sigma}(y,u';\Lambda)$, $\Sigma=(f,h)$, is surjective.*

2) *for $0\leqq N<\rho$:*

$$d_X(L_{f^k})^N h(y,u';\Lambda)=\frac{1}{N!}d_Xh(y;T_xf(y)^N\Lambda),\quad (resp.\ \frac{1}{N!}d_Xh(x_0;T_xf(y)^N\Lambda))\,,$$

for $N=\rho$:

$$d_X(L_{f^k})^\rho h(y,u';\Lambda)=\frac{1}{\rho!}[d_Xh(y;T_xf(y)^\rho\Lambda)+d_Xd_Uh(y;\Lambda\otimes u_\rho)]\,,$$

$$(resp.\ \frac{1}{\rho!}d_Xh(x_0;T_xf(y)^\rho\Lambda)).$$

Proof. To prove 1) we have to compute Θ_Λ, at least partially. In the following considerations, h can be taken in H or in H_0 as a subspace of H by considering a function $h\in H_0$ as a function of (x,u_0) independent of u_0. This has no effect on our formulas other than simplifying them without making the crucial terms disappear: all terms containing $d_X^pd_U^qh$, $q>0$, are to be taken zero. If $h\in H_0$, when we write $h(y)$ or $d_X^nh(y;\dots)$ for. $y=(x_0,u_0)$, we mean $h(x_0)$ or $d_X^nh(x_0;\dots)$.

To do this, let $(O,x^1,\dots,x^{d_x})$ be a coordinate system of X at x_0 ($y=(x_0,u_0)$!). It is not easy to compute the Lie derivatives $d_X(L_{f^k})^Nh$, so we proceed differently and use the "flow" of f^k and computations with formal power series.

There is an open set O_1, a number $\varepsilon>0$ and a smooth mapping $\varphi:]-\varepsilon,\varepsilon[\times O_1\to O$, $(t,x)\to\varphi(t,x)$, such that: 1) $x_0\in O_1\subset O$, 2) $\varphi(0,x)=x$ for $x\in O_1$, 3) $\frac{d}{dt}\varphi(t,x)=f(\varphi(t,x),V(t))$ for all $(t,x)\in]-\varepsilon,\varepsilon[\times O_1$, where $V(t)$ is the polynomial mapping $V(t)=\frac{1}{\rho!}u_\rho t^\rho+\cdots+\frac{1}{(k-1)!}u_{k-1}t^{k-1}$. For later purposes, we set $v_i=\frac{1}{i!}u_i$. Hence $V(t)=v_\rho t^\rho+\cdots+v_{k-1}t^{k-1}$. To compute $d_X(L_{f^k})^Nh$, we use the following simple observations: let $u=(u_0,u')$. If $h\in H$

(resp. H_0):

(1) $(L_{f^k})^N h(x,u) = \frac{\partial^N}{(\partial t)^N} h(\varphi(t,x), V(t))_{|t=0} = h_N(x,u)$ say,

(2) $d_X((L_{f^k})^N h)(x,u;\Lambda) = d_X h_N(x,u;\Lambda)$ for $\Lambda \in T_x X \approx R^{d_x}$ using the coordinate system. (Note that $h_0 = h$!).

Let $\varphi_N(x,u) = \frac{1}{N!}\frac{\partial^N}{(\partial t)^N}\varphi(t,x)_{|t=0}$ for $x \in O_1$. Condition 2) on φ implies that $\varphi_0(x,u) = x$, for $x \in O_1$.

The Taylor series in t of φ is:

$$j_t^\infty(\varphi) = \sum_{n=0}^{\infty} t^n \varphi_n ,$$

that of $h(\varphi(t,x), V(t))$:

$$j_t^\infty(h(\varphi, V)) = \sum_{n=0}^{\infty} t^n h_n .$$

Let us compute h_N and $d_X h_N$. Let $\hat{\varphi}$ denote the formal power series in t, $\sum_{N=1}^{\infty} t^N \varphi_N$. Then as a formal power series in t:

$$j_t^\infty(h(\varphi, V)) = \sum_{p,q \geqq 0} \frac{1}{p!}\frac{1}{q!} d_X^p d_U^q h(\hat{\varphi}, \ldots, \hat{\varphi}, V, \ldots, V) . \tag{3}$$

($p\hat{\varphi}'$s and qV's) .

For a multi-index $\alpha \in N^1$, $|\alpha| = \alpha_1 + \cdots + \alpha_1$. Hence, for $x \in O_1$:

$$h_N(x,u) = \sum \left\{ \frac{1}{p!}\frac{1}{q!} d_X^p d_U^q h(x, u_0; \varphi_\alpha(x,u) \otimes v_\beta) \mid p \geqq 0, q \geqq 0, \right.$$

$$\left. \alpha \in N^p,\ \beta \in N^q, |\alpha| + |\beta| = N \right\},$$

where $\varphi_\alpha(x,u)$ (resp. v_β) will denote the symmetric tensor product $\varphi_{\alpha_1}(x,u) \odot \cdots \odot \varphi_{\alpha_p}(x,u)$ (resp. $v_{\beta_1} \odot \cdots \odot v_{\beta_q}$) of the vectors $\varphi_{\alpha_1}(x,u), \ldots, \varphi_{\alpha_p}(x,u) \in R^{d_x}$ (resp. $v_{\beta_1}, \ldots, v_{\beta_q} \in R^{d_u}$).

Then, for any $x \in O_1$, any $\Lambda \in T_x X \approx R^{d_x}$:

$$d_X h_N(x,u;\Lambda) = \mathrm{I}_N + \mathrm{II}_N , \tag{4}$$

$$\mathrm{I}_N = \sum \left\{ \frac{1}{p!}\frac{1}{q!} d_X^{p+1} d_U^q h(x, u_0; \Lambda \otimes \varphi_\alpha(x,u) \otimes v_\beta) \mid p \geqq 0, q \geqq 0 , \right. \tag{5}$$

$$\left. \alpha \in N^p, \beta \in N^q, |\alpha| + |\beta| = N \right\},$$

$$\mathrm{II}_N = \sum \left\{ \frac{1}{p!}\frac{1}{q!} d_X^p d_U^q h(x, u_0; \varphi_{\alpha_1} \odot \cdots \odot d_X \varphi_{\alpha_i}(x,u;\Lambda) \odot \cdots \odot \varphi_{\alpha_p}(x,u) \right.$$

$$\left. \otimes v_\beta) \mid p \geqq 1, q \geqq 0, 1 \leqq i \leqq p, \alpha \in N^p, \beta \in N^q, |\alpha| + |\beta| = N \right\}.$$

It is easy to see that:

$$(6)\quad \mathrm{II}_N = \sum\left\{ \frac{1}{(p-1)!}\frac{1}{q!} d_X^p d_U^q h(x,u_0; d_X\varphi_r(x,u;\Lambda)\otimes\varphi_\gamma(x,u)\otimes v_\beta) \right.$$
$$\left. | p \geqq 1, q \geqq 0, 1 \leqq r,\gamma \in N^{p-1}, \beta \in N^q, r+|\gamma|+|\beta| = N \right\}.$$

Before proceeding further, we have to determine the φ_N and $d_X\varphi_N$ for small N. The claim is that:

$$(7)\quad \varphi_N(x_0,u) = 0,\ 0 \leqq N \leqq \rho,\ \varphi_{\rho+1}(x_0,u) = \frac{1}{\rho+1} d_U f\left(x_0,u;\frac{1}{\rho!}u_\rho\right).$$

$$(8)\quad d_X\varphi_N(x_0,u;\Lambda) = \frac{1}{N!} T_x f(x_0,u)^N \Lambda,\quad 0 \leqq N \leqq \rho.$$

To see this, write the equation $\frac{d}{dt}\varphi(t,x) = f(\varphi(t,x),V(t))$ as a formal power series in t: recall that $\hat{\varphi} = \sum_{N=1}^{\infty} t^N \varphi_N$.

$$\sum_{N=1}^{\infty} N\varphi_N(x,u)t^{N-1} = \sum_{p,q\geqq 0} d_X^p d_U^q f(x,u;\hat{\varphi}(t,x)^{\odot p}\otimes V(t)^{\odot q}).$$

Hence:

$$(9)\quad N\varphi_N(x,u) = \sum\left\{ \frac{1}{p!}\frac{1}{q!} d_X^p d_U^q f(x,u_0;\varphi_\alpha(x,u)\otimes v_\beta) \mid p \geqq 0, \right.$$
$$\left. q \geqq 0, \alpha \in N^p, \beta \in N^q, |\alpha|+|\beta| = N-1 \right\}.$$

Since $v_i = 0$ if $i < \rho$, $|\beta| \geqq q\rho$. This shows that if $N \leqq \rho$:

$$(10)\quad \varphi_1(x,u) = f(x,u_0),$$

$$N\varphi_N(x,u_0) = \sum\left\{ \frac{1}{p!} d_X^p f(x,u_0;\varphi_\alpha(x,u)) \mid \alpha \in N^p, |\alpha| = N-1 \right\}.$$

Because $f(x_0,u_0) = 0$, an easy induction on N shows that $\varphi_N(x_0,u) = 0$ if $1 \leqq N \leqq \rho$. In case $N = \rho+1$, no terms with $p \geqq 1$ can appear since for any $\alpha \in N^p$, $|\alpha| \leqq \rho$ and hence, $\alpha_i \leqq \rho$ for all i, $1 \leqq i \leqq p$. Hence:

$$(\rho+1)\varphi_{\rho+1}(x_0,u) = d_U f\left(x_0,u;\frac{1}{\rho!}u_\rho\right),\quad \text{(note that } v_\rho = \frac{1}{\rho!}u_\rho).$$

Deriving (10), we get for $\Lambda \in T_xX \approx R^{d_x}$:

$$(11)\quad N d_X\varphi_N(x,u_0;\Lambda) = \mathrm{III}_N + \mathrm{IV}_N,$$

$$\mathrm{III}_N = \sum\left\{ \frac{1}{p!} d_X^{p+1} f(x,u_0;\Lambda\otimes\varphi_\alpha(x,u)) \mid \alpha \in N^p, |\alpha| = N-1 \right\},$$

$$\mathrm{IV}_N = \sum \left\{ \frac{1}{(p-1)!} d_X^p f(x, u_0; d_X \varphi_r(x,u;\Lambda) \otimes \varphi_\gamma(x,u)) \right.$$

$$\left. | \gamma \in N^{p-1}, r \geqq 1, |\gamma| + r = N - 1 \right\}.$$

Since $\varphi_N(x_0,u) = 0$ if $N \leqq \rho$, for $x = x_0$, this last formula reduces to: $N d_X \varphi_N(x_0,u;\Lambda) = d_X f(x_0,u_0; d_X \varphi_{N-1}(x_0,u;\Lambda))$.

An easy induction gives: $d_X \varphi_N(x_0,u;\Lambda) = \frac{1}{N!} d_X f(x_0,u_0)^N \Lambda$, for $0 \leqq N \leqq \rho$.

Let us go back to $d_X h_N$. In the light of formulas (7) and (8), the maximum value of p appearing in a nonzero term in $I_{N_{|x=x_0}}$ is $n = [\frac{N}{\rho+1}]$, integer part of $\frac{N}{\rho+1}$ since $p(\rho+1) \leqq |\alpha| \leqq N$. In a nonzero term in $I_{N_{|x=x_0}}$ with $p = n$, $q\rho \leqq |\beta| \leqq N - n(\rho+1) \leqq \rho$. Hence $q = 0$ or $q = 1$. Finally, we get:

If $N - n(\rho+1) < \rho$,

$$\mathrm{I}_{N_{|x=x_0}} = \frac{1}{n!} \sum \{ d_X^{n+1} h(x_0,u_0; \Lambda \otimes \varphi_\alpha(x_0,u_0)) \,|\, \alpha \in N^n, |\alpha| = N \}$$

$$+ \sum \left\{ \frac{1}{p!} \frac{1}{q!} d_X^{p+1} d_U^q h(x_0,u_0; \Lambda \otimes \varphi_\alpha(x_0,u_0) \otimes v_\beta) \right.$$

$$\left. | \; p \leqq n-1, \; \alpha \in N^p, \beta \in N^q, |\alpha| + |\beta| = N \right\},$$

if $N - n(\rho+1) = \rho$,

$$\mathrm{I}_{N_{|x=x_0}} = \frac{1}{n!} \sum \{ d_X^{n+1} h(x_0,u_0; \Lambda \otimes \varphi_\alpha(x_0,u_0)) \,|\, \alpha \in N^n, |\alpha| = N \}$$

$$+ \frac{1}{n!} d_X^{n+1} d_U h(x_0,u_0; \Lambda \otimes \varphi_{\rho+1}(x_0,u_0)^{\odot n} \otimes v_\rho)$$

$$+ \sum \left\{ \frac{1}{p!} \frac{1}{q!} d_X^{p+1} d_U^q h(x_0,u_0; \Lambda \otimes \varphi_\alpha(x_0,u_0) \otimes v_\beta) \right.$$

$$\left. | \; p \leqq n-1, \; \alpha \in N^p, \beta \in N^q, |\alpha| + |\beta| = N \right\}.$$

The maximum value of p in a nonzero term in $\mathrm{II}_{N_{|x=x_0}}$ is $n+1$, since $(p-1)(\rho+1) \leqq |\gamma| \leqq N$. In a nonzero term in $\mathrm{II}_{N_{|x=x_0}}$ with $p = n+1$, $q\rho \leqq |\beta| \leqq N - |\gamma| \leqq N - n(\rho+1) \leqq \rho$. If $N - n(\rho+1) < \rho$ then $q = 0$. If $N - n(\rho+1) = \rho, q\rho \leqq |\beta| \leqq N - |\gamma| - r \leqq \rho - 1$ since $r \geqq 1$. We get, setting $N - n(\rho+1) = \varepsilon$:

$$\mathrm{II}_{N_{|x=x_0}} = A_N + B_N \,,$$

$$A_N = \frac{1}{n!} \sum \{ d_X^{n+1} h(x_0,u_0; d_X \varphi_r(x_0,u;\Lambda) \otimes \varphi_\gamma(x_0,u))$$

$$| \; 1 \leqq r \leqq \varepsilon, \; \gamma \in N^n, |\gamma| + r = N \}$$

$$B_N = \sum \left\{ \frac{1}{(p-1)!}\frac{1}{q!} d_X^p d_U^q h(x_0,u_0; d_X\varphi_r(x_0,u;\Lambda)\otimes\varphi_\gamma(x_0,u)\otimes v_\delta) \right.$$
$$\left. | \, 1 \leqq r \leqq \varepsilon,\ p \leqq n, \gamma \in N^{p-1}, \delta \in N^q, r+|\gamma|+|\delta| = N \right\}.$$

Putting $\mathrm{I}_{N_{|x=x_0}}$, $\mathrm{II}_{N_{|x=x_0}}$ together and using the formulas (7), (8):

let N be an integer between 0 and $k-1$, let $n = [\frac{N}{\rho+1}]$ and $\varepsilon = N - n(\rho+1)$, $0 \leqq \varepsilon \leqq \rho$.

If $\varepsilon < \rho$ and $\Lambda \in T_{x_0}X \approx R^{d_x}$,

(12) $$d_X h_N(x_0,u_0;\Lambda) = d_X^{n+1} h(x_0,u_0;\psi_N) + \sum \{ d_X^p d_U^q h(x_0,u_0;\Psi_{N,p,q}) | \, 1 \leqq p \leqq n,\ q \leqq N \},$$

if $\varepsilon = \rho$ and $\Lambda \in T_{x_0}X \approx R^{d_x}$,

(13) $$d_X h_N(x_0,u_0;\Lambda) = d_X^{n+1} h(x_0,u_0;\psi_N) + d_X^{n+1} d_U h \left(x_0,u_0;\Lambda\otimes\frac{1}{n!}\varphi_{\rho+1}(x_0,u)^{\odot n}\otimes v_\rho \right) + \sum \{ d_X^p d_U^q h(x_0,u_0;\Psi_{N,p,q}) \, | \, 1 \leqq p \leqq n,\ q \leqq N \},$$

where:

(14) $$\psi_N = \psi_N(j^1 f(y),\Lambda,u') = \sum_{r=0}^{\varepsilon} \frac{1}{r!} T_X f(y)^r \Lambda \otimes \frac{1}{n!} U_n(\varepsilon - r),$$

(15) $$U_n(i) = \sum \{ \varphi_\gamma(x_0,u) \, | \, \gamma \in N^n, |\gamma| = n(\rho+1)+i \}.$$

In particular:

(16) $$U_n(0) = \varphi_{\rho+1}(x_0,u)^{\odot n} = \frac{1}{(\rho+1)^n} \left[d_U f \left(x_0,u_0;\frac{1}{\rho!}u_\rho \right) \right]^{\odot n},$$

(17) $$\Psi_{N,p,q} = \Psi_{N,p,q}(j^N f(y),\Lambda,u') = \frac{1}{(p-1)!}\frac{1}{q!} \sum \{ d_X\varphi_r(x_0,u;\Lambda)\otimes\varphi_\gamma(x_0,u)\otimes v_\delta \, | \, r \geqq 0, \gamma \in N^{p-1}, \delta \in N^q,\ r+|\gamma|+|\delta| = N \}.$$

Clearly, ψ_N is a polynomial mapping:

$$J^k S(y) \times T_{x_0}X \times R^{(k-1)d_u} \to \mathrm{Sym}^{n+1}(T_{x_0}X), \text{linear in } \Lambda,$$

$\Psi_{N,p,q}$ is a polynomial mapping:

$$J^k S(y) \times T_{x_0}X \times R^{(k-1)d_u} \to \mathrm{Sym}^p(T_{x_0}X) \otimes \mathrm{Sym}^q(R^{d_u}).$$

In view of the expressions (14), (15), (16) of the ψ_N and the assumptions (iii) and (iv) of part 1 of the lemma, the elements $\{\psi_N(j^1 f(y), \Lambda, u') \mid n(\rho+1) \leqq N \leqq n(\rho+1)+\rho\}$ in $\mathrm{Sym}^{n+1}(T_{x_0}X)$ are linearly independent. Since $\Theta_\Lambda(j^k h(y)) = (d_X h_0(x_0, u; \Lambda), d_X h_1(x_0, u; \Lambda), \ldots, d_X h_{k-1}(x_0, u; \Lambda))$, the formulas (12), (13) show that Θ_Λ is surjective. The part 2 of the lemma follows from formulas (12), (13): if $n=0$, $U_0(i)=0$ for $i>1$, $U_0(0)=1$ and $\Psi_{N,p,q}=0$ since $1 \leqq p \leqq n=0$, which is impossible. □

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