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Perron–Frobenius theorem, large deviations, and random perturbations in random environments

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1 Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space with an invertible P-preserving transformation $\theta : \Omega \to \Omega$ and $X^\omega, \omega \in \Omega$ a family of compact subsets of a compact metric space X. For each $\omega \in \Omega$ and $x \in X^\omega$, let $P^\omega(x, \cdot)$ be a probability measure on $X^{\theta\omega}$ Borel measurably depending on x (but not necessarily on ω). By the Kolmogorov extension theorem, for each $x \in X^\omega$, there is a unique probability measure $\mathbf{P}_x^\omega$ on the product space $\mathbf{X}^\omega = X^\omega \times X^{\theta\omega} \times X^{\theta^2\omega} \times \cdots$ such that under $\mathbf{P}_x^\omega$ the coordinate maps $\{Z_n^\omega, n \geqq 0\}$ become a time inhomogeneous Markov chain starting at x and evolving according to $\{P^{\theta^n\omega}\}$, i.e., $Z_n^\omega \in \Gamma \subset X^{\theta^n\omega}$ with probability $P^{\theta^n\omega}(y, \Gamma)$ provided $Z_{n-1}^\omega = y \in X^{\theta^{n-1}\omega}$. In this way, one can think of $(\Omega, \mathscr{F}, P, \theta)$ as a random stationary environment for the Markov chains Z_n^ω. In the case when $X^\omega \equiv X$ is a countable set the ergodic theory for such Markov chains was studied in [Co1–2] and [Or], and the central limit theorem was derived in [Co3]. Under certain additional conditions, large deviation estimates were obtained in [Se] when $X^\omega \equiv X$ but X is only locally compact.

For $x \in X^\omega$ and $\Gamma \subset X^{\theta^n\omega}$, set

$$P^\omega(n, x, \Gamma) = \int \cdots \int P^\omega(x, dy_1) P^{\theta\omega}(y_1, dy_2) \cdots P^{\theta^{n-1}\omega}(y_{n-1}, \Gamma) .$$

Thus, $P^\omega(n, x, \cdot)$ is the n-step transition probability of the Markov chain Z_n^ω. The main assumption of this paper is that there exist random variables $N = N_\omega \in \mathbb{Z}_+ = \{1, 2, \ldots\}$ and $\gamma_\omega > 0$ and family of measures m^ω from the space $\mathscr{P}(X^\omega)$ of probability measures on X^ω such that for P-almost all (a.a.) $\omega \in \Omega$, any $x \in X^{\theta^{-N}\omega}$, and each Borel $\Gamma \subset X^\omega$,

$$P^{\theta^{-N}\omega}(N, x, \Gamma) \geqq \gamma_\omega m^\omega(\Gamma) . \tag{1.1}$$

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Clearly, (1.1) is a randomized version of the classic Doeblin condition and I will derive here a randomized version of Doeblin's conclusion. Namely, (1.1) implies that there exists, for P-a.a.ω, a unique $\mu^\omega \in \mathscr{P}(X^\omega)$ such that

$$\mu^\omega P^\omega = \mu^{\theta\omega}, \tag{1.2}$$

where $P^\omega g(x) = \int P^\omega(x,dy)g(y)$ and $\mu^\omega P^\omega(\Gamma) = \int d\mu^\omega(x)P^\omega(x,\Gamma)$. Moreover, for P-a.a.ω and each Borel $\Gamma \subset X^\omega$, $P^{\theta^{-n}\omega}(n,x,\Gamma)$ converges to $\mu^\omega(\Gamma)$ as $n \to \infty$ at an exponential rate. Finally, if, in addition to (1.1), we have the upper bound,

$$P^{\theta^{-N}\omega}(N,x,\Gamma) \leqq \gamma_\omega^{-1} m^\omega(\Gamma), \quad N = N_\omega, \tag{1.3}$$

the preceeding convergence results transfer, in the obvious way, to corresponding statements about densities.

The above results can be generalized to give a Perron–Frobenius type theorem for random operators. Let R^ω, $\omega \in \Omega$, be a family of bounded, nonnegative operators mapping a bounded Borel function g on $X^{\theta\omega}$ to a function $R^\omega g$ on X^ω by the formula

$$R^\omega g(x) = \int R^\omega(x,dy)g(y). \tag{1.4}$$

I assume that there exist random variables $c_\omega > 0$ and $N = N_\omega \in \mathbb{Z}_+$ such that for P-a.a.ω, any $x \in X^\omega$, and each Borel $\Gamma \subset X^\omega$,

$$R^{\theta^{-N}\omega}(N,x,\Gamma) \geqq c_\omega m^\omega(\Gamma) \tag{1.5}$$

and

$$\operatorname{supp} R^{\theta^{-N}\omega}(N,x,\cdot) = \operatorname{supp} m^\omega. \tag{1.6}$$

Here $R^\omega(n,x,\Gamma) = \int \cdots \int R^\omega(x,dy_1)\cdots R^{\theta^{n-1}\omega}(y_{n-1},\Gamma)$. Then, under a mild compactness assumption, I show that for P-a.a.ω there exist uniquely defined numbers $\lambda_\omega > 0$, a family of positive functions h_ω on X^ω, and a family of measures $\nu^\omega \in \mathscr{P}(X^\omega)$ (depending measurably on ω if R^ω and other objects in the assumptions above depend measurably on ω) such that for any $x \in X^\omega$ and $\Gamma \subset X^{\theta\omega}$,

$$\int d\nu^\omega(x)R^\omega(x,\Gamma) = \lambda_\omega \nu^{\theta\omega}(\Gamma) \tag{1.7}$$

and

$$\int h_{\theta\omega}(y)R^\omega(x,dy) = \lambda_\omega h_\omega(x). \tag{1.8}$$

When in addition to (1.5), we assume

$$R^{\theta^{-N_\omega}\omega}(N_\omega,x,\Gamma) \leqq c_\omega^{-1} m^\omega(\Gamma) \tag{1.9}$$

then (1.6) is automatically satisfied and these results transfer to statements in terms of densities.

The above results enable me to obtain a Donsker–Varadhan type formula in a random environment. Namely, for any measurable in ω family of

continuous functions $\varphi = \{\varphi_\omega\}$ on X^ω let $R^\omega = R^\omega_\varphi$ be the operator given by the formula

$$R^\omega g(x) = \int g(y) e^{\varphi_{\theta\omega}(y)} P^\omega(x, dy) ; \tag{1.10}$$

and let λ^φ_ω be the corresponding random variable appearing in (1.7) and (1.8). In addition to (1.5), assume that

$$\int \sup_{x \in X^\omega} |\varphi_\omega(x)| dP(\omega) < \infty . \tag{1.11}$$

Then $\log \lambda^\varphi_\omega$ is integrable and

$$Q(\varphi) \overset{\text{def}}{=} \int \log \lambda^\varphi_\omega dP(\omega) = \sup_\eta \left(\int \varphi d\eta - I(\eta) \right) , \tag{1.12}$$

where the supremum is taken over the space $\mathscr{P}_P(X \times \Omega)$ of probability measures on $X \times \Omega$ with the desintegration $d\eta(x, \omega) = d\eta^\omega(x) dP(\omega)$ and

$$I(\eta) \equiv - \inf_u \int \log \left(\frac{P^\omega u_{\theta\omega}}{u_\omega} \right) d\eta^\omega dP(\omega) , \tag{1.13}$$

the infinum being taken over measurable families $u = \{u_\omega\}$ of positive continuous functions on X^ω such that $\int \sup_x |\log u_\omega(x)| \, dP(\omega) < \infty$. It turns out that for a wide class of families $\varphi = \{\varphi_\omega\}$ of continuous functions φ_ω on X^ω there exists a unique $\mu = \mu_\varphi \in \mathscr{P}_P(X \times \Omega)$ such that

$$Q(\varphi) = \int \varphi d\mu - I(\mu) . \tag{1.14}$$

At the same time, it is not difficult to see that if θ is ergodic then for P-a.a.ω,

$$\lim_{n \to \infty} \frac{1}{n} \log R^\omega_\varphi(n) 1 = Q(\varphi) \tag{1.15}$$

where $R^\omega_\varphi(n) = R^\omega_\varphi \circ \cdots \circ R^{\theta^{n-1}\omega}_\varphi$. Assume that Ω is a locally compact metric space and endow $X \times \Omega$ with the product topology and $\mathscr{P}(X \times \Omega)$ with the weak topology. Then in view of [Ki2], these yield the "relativized" large deviations bounds for occupational measures

$$\zeta^n_\omega = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{(Z^\omega_i, \theta^i \omega)} \in \mathscr{P}(X \times \Omega) \tag{1.16}$$

(where δ_z denotes the unit mass at z) which means that with probability one for any closed $K \subset \mathscr{P}(X \times \Omega)$,

$$\limsup_{n \to \infty} \frac{1}{n} \log P^\omega \{ \zeta^n_\omega \in K \} \leqq - \inf_{\nu \in K} I_P(\nu) \tag{1.17}$$

and for any open $G \subset \mathscr{P}(X \times \Omega)$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P^\omega \{ \zeta^n_\omega \in G \} \geqq - \inf_{\nu \in G} I_P(\nu) \tag{1.18}$$

where $I_P(\nu) = I(\nu)$ if $\nu \in \mathscr{P}_P(X \times \Omega)$ and $I_P(\nu) = \infty$, otherwise.

Next, let X^ω's be measurably depending on ω smooth compact manifolds imbedded into a compact set X and $F(\omega)$ be a measurably depending on

ω family of C^2 onto maps $F(\omega)\colon X^\omega \to X^{\theta\omega}$ with nondegenerate differentials which expand distances by a factor bigger than 1. I shall consider Markov chains $Z_n^{\omega,\varepsilon}$ such that the probability that $Z_{n+1}^{\omega,\varepsilon} \in \Gamma \subset X^{\theta^{n+1}\omega}$ conditioned to $Z_n^{\omega,\varepsilon} = x \in X^{\theta^n\omega}$ equals $P_\varepsilon^{\theta^n\omega}(x,\Gamma)$ where $P_\varepsilon^\omega(x,\cdot)$ tends to $\delta_{F(\omega)x}$ as $\varepsilon \to 0$. The Markov chains $Z_n^{\omega,\varepsilon}$ will be called random perturbations of maps $F(\omega)$. Under certain conditions similar to [Ki1] and [Ki3] I shall show that the measures maximizing (1.12) for the Markov chains $Z_n^{\omega,\varepsilon}$ (in particular, their invariant measures) weakly converge as $\varepsilon \to 0$ to the corresponding equilibrium states of random transformations $F(\omega)$ (see [Ki5]). Similarly to [Ki2] this will follow from the convergence as $\varepsilon \to 0$ of the corresponding left hand side of (1.12) to the relativized topological pressure (see [Ki5] or [Bo]) of random transformations $F(\omega)$.

Some of the results of this paper can be proved under more general circumstances, but I have chosen to show how the thermodynamical formalism methods applied in [Ki5] and [KK] to random transformations work in the case of Markov chains in random environments. Many results are obtained here for each ω separately (i.e., for a fixed inhomogeneous Markov chain Z_n^ω) and they need only that the transition probabilities $P^{\theta^i\omega}(x,\cdot)$, $i \in \mathbb{Z}$ satisfy certain conditions which holds true in our circumstances for P-a.a.ω. Note that combining methods of [Co3] and this paper one can obtain a central limit theorem for Markov chains in random environments in my set up, as well. One can obtain similar results for diffusions given by stochastic differential equations with coefficients dependent on a stationary process independent of the driving Brownian motion. A related set up was considered in [Ki7].

2 Convergence of transition probabilities

The main result of this section is the following.

2.1 Theorem *Suppose that* (1.1) *holds true but no topological assumptions on X and X^ω are made. Then for P-a.a.ω there exists a unique family of probability measures μ^ω on X^ω satisfying* (1.2) *(and depending measurably on ω if the spaces X^ω, the measures m^ω, and the operators P^ω depend measurably on ω) and such that for some $\kappa = \kappa_\omega \in (0,1)$, $C = C(\omega) \in (0,\infty)$, any $x \in X^{\theta^{-n}\omega}$, and a Borel set $\Gamma \subset X^\omega$,*

$$|P^{\theta^{-n}\omega}(n,x,\Gamma) - \mu^\omega(\Gamma)| \leqq C(\omega)(1-\kappa)^n\,. \tag{2.1}$$

Furthermore, if (1.3) *is also satisfied then for any $n \geqq N_\omega$ there exist densities*

$$p^{\theta^{-n}\omega}(n,x,y) = \frac{P^{\theta^{-n}\omega}(n,x,dy)}{m^\omega(dy)}\,, \tag{2.2}$$

$$p^\omega(y) = \frac{d\mu^\omega(y)}{dm^\omega(y)} \quad \forall x \in X^{\theta^{-n}\omega},\ \forall y \in X^\omega$$

and one has

$$\sup_{x,y} |p^{\theta^{-n}\omega}(n,x,y) - p^{\omega}(y)| \leqq C(\omega)(1-\kappa)^n . \tag{2.3}$$

Proof. For each sufficiently large $M > 0$ set $A_M = \{\omega : N_\omega \leqq M \text{ and } \gamma_\omega \geqq M^{-1}\}$. Let $n_i = n_i(\omega, M) \in \mathbb{Z}_+$ be the subsequent integers so that $\theta^{-n_i}\omega \in A_M$, $n_{i+1} - n_i \geqq M$, and $n_1 \geqq M$. If $\theta^n\omega \notin A_M$ for all $n > n_{i-1}$ then I set $n_i = \infty$. Assume that $n_i < \infty$. Since $N_{\theta^{-n_i}\omega} \leqq M$ then by (1.1) and the Chapman–Kolmogorov formula for any Borel $\Gamma \subset X^{\theta^{-n_i}\omega}$,

$$\begin{aligned} P^{\theta^{-(n_i+M)}\omega}(M,x,\Gamma) &= \int P^{\theta^{-(n_i+M)}\omega}(M - N_{\theta^{-n_i}\omega}, x, dz) P^{\theta^{-(n_i+N_{\theta^{-n_i}\omega})}\omega}(N_{\theta^{-n_i}\omega}, z, \Gamma) \\ &\geqq \gamma_{\theta^{-n_i}\omega} m^{\theta^{-n_i}\omega}(\Gamma) , \end{aligned} \tag{2.4}$$

and so

$$Q^{\theta^{-(n_i+M)}\omega}(M,x,\Gamma) = P^{\theta^{-(n_i+M)}\omega}(M,x,\Gamma) - \gamma_{\theta^{-n_i}\omega} m^{\theta^{-n_i}\omega}(\Gamma)$$

is a positive measure such that

$$Q^{\theta^{-(n_i+M)}\omega}(M,x,X^{\theta^{-n_i}\omega}) = 1 - \gamma_{\theta^{-n_i}\omega} \leqq (1 - M^{-1}) . \tag{2.5}$$

By the Chapman–Kolmogorov formula

$$P^{\theta^{-(n_k+M)}\omega}(n_k + M, x, \Gamma) = \sum_{i=1}^{k} \gamma_{\theta^{-n_i}\omega} \int m^{\theta^{-n_i}\omega}(dz_i) P^{\theta^{-n_i}\omega}(n_i, z_i, G) \prod_{j=i+1}^{k} (1 - \gamma_{\theta^{-n_j}\omega}) + I^{\omega}_{k,M}(x,y) \tag{2.6}$$

where

$$\begin{aligned} I^{\omega}_{k,M}(x,\Gamma) = \int \cdots \int & Q^{\theta^{-(n_k+M)}\omega}(M,x,dz_k) P^{\theta^{-n_k}\omega}(n_k - n_{k-1} - M, z_k, d\tilde{z}_k) \\ & \times \cdots \times Q^{\theta^{-(n_2+M)}\omega}(M,\tilde{z}_3,dz_2) P^{\theta^{-n_2}\omega}(n_2 - n_1 - M, z_2, d\tilde{z}_2) \\ & \times Q^{\theta^{-(n_1+M)}\omega}(M,\tilde{z}_2,dz_1) P^{\theta^{-n_1}\omega}(n_1, z_1, \Gamma) . \end{aligned}$$

By (2.5) for any Borel $\Gamma \in X^{\omega}$,

$$\begin{aligned} I^{\omega}_{k,M}(x,\Gamma) &\leqq \prod_{j=1}^{k} (1 - \gamma_{\theta^{-n_i}\omega}) \sup_z P^{\theta^{-n_1}\omega}(n_1, z, \Gamma) \\ &\leqq (1 - M^{-1})^k \sup_z P^{\theta^{-n_1}\omega}(n_1, z, \Gamma) , \end{aligned} \tag{2.7}$$

and so if $n_k < \infty$,

$$\begin{aligned} \sup_{x,z} |P^{\theta^{-(n_k+M)}\omega}(n_k + M, x, \Gamma) - P^{\theta^{-(n_k+M)}\omega}(n_k + M, z, \Gamma)| \\ \leqq 2(1 - M^{-1})^k \sup_v P^{\theta^{-n_1}\omega}(n_1, v, \Gamma) , \end{aligned} \tag{2.8}$$

where the supremum is taken over $x,z \in X^{\theta^{-(n_k+M)}\omega}$ and $v \in X^{\theta^{-n_1}\omega}$. Next, for each $n \in \mathbb{Z}_+$ and a Borel $\Gamma \subset X^\omega$ set $\overline{P}^{\theta^{-n}\omega}(n,\Gamma) = \sup_{x \in X^{\theta^{-n}\omega}} P^{\theta^{-n}\omega}(n,x,\Gamma)$ and $\underline{P}^{\theta^{-n}\omega}(n,\Gamma) = \inf_{x \in X^{\theta^{-n}\omega}} P^{\theta^{-n}\omega}(n,x,\Gamma)$. Then

(2.9)

$$\overline{P}^{\theta^{-(n+1)}\omega}(n+1,\Gamma) = \sup_x \int P^{\theta^{-(n+1)}\omega}(x,dz) P^{\theta^{-n}\omega}(n,z,\Gamma) \leqq \overline{P}^{\theta^{-n}\omega}(n,\Gamma)\,,$$

and similarly,

$$\underline{P}^{\theta^{-(n+1)}\omega}(n+1,\Gamma) \geqq \underline{P}^{\theta^{-n}\omega}(n,\Gamma)\,. \tag{2.10}$$

These together with (2.8) yield that there exists a limit

$$\mu^\omega(\Gamma) = \lim_{n\to\infty} \overline{P}^{\theta^{-n}\omega}(n,\Gamma) = \lim_{n\to\infty} \underline{P}^{\theta^{-n}\omega}(n,\Gamma) \tag{2.11}$$

and for any $n \geqq n_1(\omega), x \in X^{\theta^{-n}\omega}$, and a Borel $\Gamma \subset X^\omega$,

(2.12)

$$|P^{\theta^{-n}\omega}(n,x,\Gamma) - \mu^\omega(\Gamma)| \leqq 2(1-M^{-1})^{L^\omega(n)} \sup_z P^{\theta^{-n_1}\omega}(n_1,z,\Gamma)$$

where $L^\omega(n) = \max\{i : n_i \leqq n\}$. By the ergodic theorem with probability one

$$\liminf_{n\to\infty} \frac{1}{n} L^\omega(n) \geqq M^{-1} E(\mathbf{1}_{A_M} | \mathscr{I})(\omega) \stackrel{\text{def}}{=} \rho_M(\omega) \tag{2.13}$$

where $E(\cdot\,|\,\cdot)$ denotes the conditional expectation, $\mathscr{I}$ is the σ-algebra of θ-invariant sets in Ω, and $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $= 0$, otherwise. If $M \uparrow \infty$ then $M\rho_M(\omega) \uparrow 1$ for P-a.a.ω. Thus for P-a.a.ω there exists M so that $\rho_M(\omega) \geqq \frac{1}{2}M^{-1} > 0$, and so I can estimate the right hand side of (2.12) by the right hand side of (2.1) with $\kappa(\omega) = 1 - (1-M^{-1})^{\rho_M(\omega)/2}$ and $C(\omega) = 2\max_{n \geqq 0}((1-M^{-1})^{L^\omega(n) - n\rho_M(\omega)/2})$ which gives (2.1). The relation (1.2) follows by letting $n \to \infty$ in the equality

$$P^{\theta^{-n}\omega}(n+1,x,\Gamma) = \int P^{\theta^{-n}\omega}(n,x,dy) P^\omega(y,\Gamma)$$

which is possible since we can approximate $P^\omega(y,\Gamma)$ by a piecewise constant in y function.

Next, by the Chapman–Kolmogorov formula for any $n \geqq N_\omega$,

$$P^{\theta^{-n}\omega}(n,x,\Gamma) = \int P^{\theta^{-n}\omega}(n-N_\omega,x,dy) P^{\theta^{-N_\omega}\omega}(N_\omega,y,\Gamma)\,, \tag{2.14}$$

and so if (1.3) is also satisfied then

$$\gamma_\omega m^\omega(\Gamma) \leqq P^{\theta^{-n}\omega}(n,x,\Gamma) \leqq \gamma_\omega^{-1} m^\omega(\Gamma)\,. \tag{2.15}$$

Thus the Radon–Nikodim derivatives $p^{\theta^{-n}\omega}(n,x,y)$ introduced in (2.2) exist for any $n \geqq N_\omega$ and

$$\gamma_\omega \leqq p^{\theta^{-n}\omega}(n,x,y) \leqq \gamma_\omega^{-1}\,. \tag{2.16}$$

By (1.3) for any $n \in \mathbb{Z}_+$ and a Borel $\Gamma \subset X^\omega$,

$$\int \mu^{\theta^{-n}\omega}(dz) P^{\theta^{-n}\omega}(n, z, \Gamma) = \mu^\omega(\Gamma)\,, \tag{2.17}$$

and so by (2.15) the Radon–Nikodim derivatives $p^\omega(y)$ defined in (2.2) exist and

$$\gamma_\omega \leqq p^\omega(y) \leqq \gamma_\omega^{-1}\,. \tag{2.18}$$

These together with (2.12) and (2.13) yield (2.3). Furthermore, (2.1) and (2.17) imply that (1.2) determines μ^ω uniquely since any other measure $\tilde{\mu}^\omega$ satisfying (1.2) will satisfy also (2.17) and by letting $n \to \infty$ in (2.17) one obtains by (2.12) that $\tilde{\mu}^\omega = \mu^\omega$ for P-a.a.ω. If the operators P^ω and the measures m^ω are measurable in ω then both the transition probabilities $P^{\theta^{-n}\omega}(n, x, \Gamma)$ and their densities $p^{\theta^{-n}\omega}(n, x, y)$ are also measurable in ω, and so their limits μ^ω and p^ω are measurable in ω, as well. □

Observe that if μ is a measure on $X \times \Omega$ defined by $d\mu(x, \omega) = d\mu^\omega(x) dp(\omega)$ with μ^ω satisfying (2.17) then μ is an invariant measure of the skew product Markov chain $\mathscr{Z}_n = (Z_n^\omega, \theta^n\omega)$ on $X \times \Omega$.

As an application of Theorem 2.1 I shall exhibit next a direct proof of ergodicity of Markov measures for random subshifts of finite type though a general proof of ergodicity of Gibbs measures can be found in [KK]. In this case $X^\omega = \{1, \ldots, k(\omega)\}$ where $k(\omega) \in \mathbb{Z}_+$ is a random variable satisfying $\int \log k dP < \infty$. Now X^ω's are compact subsets of $\bar{\mathbb{Z}}_+$ which is the one point compactification of $\mathbb{Z}_+ = \{1, 2, \ldots\}$. The transition probabilities are given here by $k(\omega) \times k(\theta\omega)$ matrices $P^\omega = (P_{ij}^\omega)$, $i = 1, \ldots, k(\omega)$, $j = 1, \ldots, k(\theta\omega)$. Consider the products $P^\omega(n) = P^\omega P^{\theta\omega} \cdots P^{\theta^{n-1}\omega}$ and assume that there exists a random variable $N = N_\omega$ such that for P-a.a.ω the matrix $P^{\theta^{-N}\omega}(N)$ has positive entries. Then (1.1) holds true where m^ω is the uniform measure on X^ω. By Theorem 2.1 there exist unique probability vectors $p^\omega = (p_i^\omega)$, $i = 1, \ldots, k(\omega)$ such that

$$p^\omega P^\omega = p^{\theta\omega}\,. \tag{2.19}$$

Next consider the space of sequences $\mathbf{X}^\omega = X^\omega \times X^{\theta\omega} \times \cdots = \{\xi = (\xi_0, \xi_1, \ldots), \xi_i \in X^{\theta^i\omega}\}$ together with the shift $\sigma : \mathbf{X}^\omega \to \mathbf{X}^{\theta\omega}$ so that $(\sigma\xi)_i = \xi_{i+1}$. Set $C_{x_0, \ldots, x_n}^\omega = \{\xi \in \mathbf{X}^\omega : \xi_i = x_i \forall i = 0, 1, \ldots, n\}$ which is called a cylinder set. Define the probability measures $\nu_{p,P}^\omega$ on $\mathbf{X}^\omega$ setting

$$\nu_{p,P}^\omega(C_{x_0, \ldots, x_n}^\omega) = p_{x_0}^\omega P_{x_0 x_1}^\omega \cdots P_{x_{n-1} x_n}^{\theta^{n-1}\omega}$$

which is legitimate in view of Kolmogorov's extension theorem. By (2.19) it follows easily that

$$\sigma \nu_{p,P}^\omega = \nu_{p,P}^{\theta\omega} \tag{2.20}$$

which yields that the measure $\nu_{p,P}$ on $\mathbf{X} = \{(x, \omega) : x \in \mathbf{X}^\omega\}$ defined by $d\nu_{p,P}(x, \omega) = d\nu_{p,P}^\omega(x) dP(\omega)$ is invariant with respect to the skew product transformation $\tau = (\sigma, \theta)$. Such measures are called Markov measures for the shift σ.

2.2 Proposition *Suppose that the probability P on Ω is ergodic for θ then under the above conditions the measure $\nu_{p,P}$ is ergodic for τ.*

Proof. Since for $n > k$,

$$\nu^{\omega}_{p,P}(C^{\omega}_{x_0,\dots,x_k} \cap \sigma^{-n} C^{\theta^n\omega}_{y_0,\dots,y_l})$$

$$= p^{\omega}_{x_0} p^{\omega}_{x_0x_1} \cdots p^{\theta^{k-1}\omega}_{x_{k-1}x_k} \left(\sum_{z_1,\dots,z_{n-k-1}} p^{\theta^k\omega}_{x_k z_1} \cdots p^{\theta^{n-1}\omega}_{z_{n-k-1}y_0} \right) p^{\theta^n\omega}_{y_0y_1} \cdots p^{\theta^{n+l-1}\omega}_{y_{l-1}y_l}$$

and by Theorem 2.1,

$$\sup_{z_0,y_0} \left| \sum_{z_1,\dots,z_{n-k-1}} p^{\theta^k\omega}_{z_0z_1} \cdots p^{\theta^{n-1}\omega}_{z_{n-k-1}y_0} - p^{\theta^n\omega}_{y_0} \right| \leqq C(\theta^n\omega)(1-\kappa)^n$$

then

$$|\nu^{\omega}_{p,P}(C^{\omega}_{x_0,\dots,x_k} \cap \sigma^{-n} C^{\theta^n\omega}_{y_0,\dots,y_l}) - \nu^{\omega}_{p,P}(C^{\omega}_{x_0,\dots,x_k}) \nu^{\theta^n\omega}_{p,P}(C^{\theta^n\omega}_{y_0,\dots,y_l})| \leqq C(\theta^n\omega)(1-\kappa)^n .$$

Approximating sets by disjoint unions of cylinder sets I conclude that there exists a sequence $n_i = n_i(\omega) \to \infty$ such that for any measurable collection of measurable sets $\Gamma^{\omega} \subset \mathbf{X}^{\omega}$,

$$\nu^{\omega}_{p,P}(\Gamma^{\omega} \cap \sigma^{-n_i}\Gamma^{\theta^{n_i}\omega}) - \nu^{\omega}_{p,P}(\Gamma^{\omega}) \nu^{\theta^{n_i}\omega}_{p,P}(\Gamma^{\theta^{n_i}\omega}) \to 0 \quad \text{as } i \to \infty . \tag{2.21}$$

Let now Γ be a measurable subset of $\mathbf{X} = \{(x,\omega) : x \in \mathbf{X}^{\omega}\}$ satisfying $\tau^{-1}\Gamma = \Gamma$ then for $\Gamma^{\omega} = \{x : (x,\omega) \in \Gamma\}$ it follows that

$$\sigma^{-n}\Gamma^{\theta^n\omega} = \Gamma^{\omega} . \tag{2.22}$$

Since $\nu_{p,P}$ is τ-invariant then by (2.20) and (2.22) for P-a.a.ω,

$$\nu^{\theta^n\omega}_{p,P}(\Gamma^{\theta^n\omega}) = \sigma^n \nu^{\omega}_{p,P}(\Gamma^{\theta^n\omega}) = \nu^{\omega}_{p,P}(\sigma^{-n}\Gamma^{\theta^n\omega}) = \nu^{\omega}_{p,P}(\Gamma^{\omega}) , \tag{2.23}$$

and so by (2.21) and (2.22) for P-a.a.ω,

$$\nu^{\omega}_{p,P}(\Gamma^{\omega}) = (\nu^{\omega}_{p,P}(\Gamma^{\omega}))^2 .$$

Hence, $\nu^{\omega}_{p,P}(\Gamma^{\omega}) = 0$ or $= 1$ for P-a.a.ω which together with (2.23) and ergodicity of P with respect to θ yield that either $\nu^{\omega}_{p,P}(\Gamma^{\omega}) = 0$ for P-a.a.ω or $\nu^{\omega}_{p,P}(\Gamma^{\omega}) = 1$ for P-a.a.ω, and so $\nu_{p,P}(\Gamma) = 0$ or $= 1$ completing the proof of ergodicity of $\nu_{p,P}$. □

3 Random Perron–Frobenius theorem

Let R^ω, $\omega \in \Omega$ be a family of positive operators satisfying (1.4)–(1.6). Assume, in addition, that there exist $\beta(\omega) > 0$ and $\alpha_\rho(\omega) \geqq 1$ with $\lim_{\rho \to 0} \alpha_\rho(\omega) = 1$ such that

$$R^\omega \tilde{\Lambda}^{\theta\omega} \subset \tilde{\Lambda}^\omega \stackrel{\text{def}}{=} \{g \in \mathscr{C}(X^\omega) : g \geqq 0 \text{ and } g(x) \leqq \alpha_\rho(\omega) g(z) \text{ if } \operatorname{dist}(x,z) = \rho \leqq \beta(\omega)\} \tag{3.1}$$

where $\mathscr{C}(Y)$ denotes the space of continuous functions on Y. In this section I shall obtain

3.1 Theorem *There exist uniquely defined numbers $\lambda_\omega > 0$, positive functions h_ω on X^ω, and measures $\nu^\omega \in \mathscr{P}(X^\omega)$ (measurably dependent on ω if the spaces X^ω, the measures m^ω, and the operators R^ω, depend measurably on ω and $\mathscr{F}$ is countably generated) such that* (1.7) *and* (1.8) *hold true. Furthermore, for some $C(\omega) > 0$, $\kappa = \kappa_\omega \in (0,1)$, any $x \in X^{\theta^{-n}\omega}$, and a Borel $\Gamma \subset X^\omega$,*

(3.2)

$$|(\lambda_{\theta^{-n}\omega} \cdots \lambda_{\theta^{-1}\omega})^{-1} (h_{\theta^{-n}\omega})^{-1} R^{\theta^{-n}\omega}(n, x, \Gamma) - \nu^\omega(\Gamma)| \leqq C(\omega)(1-\kappa)^n .$$

If also the upper bound (1.9) *holds true then for any $n \geqq N_\omega$ there are densities*

$$r^{\theta^{-n}\omega}(n,x,y) = \frac{R^{\theta^{-n}\omega}(n,x,dy)}{m^\omega(dy)}, \qquad r^\omega(y) = \frac{d\nu^\omega(y)}{dm^\omega(y)} \quad \forall x \in X^{\theta^{-n}\omega},\ y \in X^\omega$$

and

$$\sup_{x,y} |(\lambda_{\theta^{-(n-1)}\omega} \cdots \lambda_\omega)^{-1} (h_{\theta^{-n}\omega}(x))^{-1} r^{\theta^{-n}\omega}(n,x,y) h_\omega(y) - r^\omega(y)| \leqq C(\omega)(1-\kappa)^n . \tag{3.3}$$

Proof. The argument will follow the lines of [Ki5]. Let Π^ω denotes the product space $\prod_{i=-\infty}^{\infty} \mathscr{P}(X^{\theta^i\omega})$ taken with the product topology while each $\mathscr{P}(X^\omega)$ is taken with the topology of the weak convergence. Define the map $\Phi^\omega : \Pi^\omega \to \Pi^\omega$ so that for each $v = (\ldots, v_{-1}, v_0, v_1, \ldots)$,

$$(\Phi^\omega v)_{i+1} = \frac{(R^{\theta^i\omega})^* v_i}{(R^{\theta^i\omega})^* v_i (X^{\theta^{i+1}\omega})} \tag{3.4}$$

where $(R^\omega)^*$ maps $\mathscr{P}(X^\omega)$ to $\mathscr{P}(X^{\theta\omega})$ so that $\int g d(R^\omega)^* \nu = \int R^\omega g d\nu$ for any bounded Borel function on $X^{\theta\omega}$. The denominator in (3.4) is positive since R^ω were assumed to be positive operators. Clearly, Φ^ω is a continuous map of Π^ω into itself, Π^ω is compact and convex, and so by the Schauder–Tichonoff fixed point theorem Φ^ω has a fixed point. This means that there exists a sequence $v_\omega = (\ldots, v_{-1,\omega}, v_{0,\omega}, v_{1,\omega}, \ldots)$ such that $(\Phi^\omega v_\omega)_i = v_{i,\omega}$. Put $\nu^{\theta^i\omega} = v_{i,\omega}$ and $\lambda^{\theta^i\omega} = (R^{\theta^i\omega})^* v_{i,\omega}(X^{\theta^{i+1}\omega})$. This definition is legitimate if ω is not a periodic

point of θ. If $\theta^j\omega = \omega$ then it is clear that the operator Φ^ω maps the space of periodic sequences of measures with period j to itself, and so in this case we can choose a fixed point ν_ω of Φ^ω satisfying $\nu_{i+j,\omega} = \nu_{i,\omega}$. Using such ν_ω the definition of $\lambda^{\theta^i\omega}$ and $\nu^{\theta^i\omega}$ becomes legitimate for periodic points of θ, as well, and I obtain $(R^{\theta^i\omega})^*\nu^{\theta^i\omega} = \lambda^{\theta^i\omega}\nu^{\theta^{i+1}\omega}$. Taking $i = 0$ I obtain (1.7) which implies also that

$$\int d\nu^{\theta^{-n}\omega}(x)R^{\theta^{-n}\omega}(n,x,\Gamma) = \lambda_{\theta^{-n}\omega}\cdots\lambda_{\theta^{-2}\omega}\lambda_{\theta^{-1}\omega}\nu^\omega(\Gamma)\,. \tag{3.5}$$

Taking here $n = N_\omega$ I obtain from (1.5) that for any Borel $\Gamma \subset X^\omega$,

$$\nu^\omega(\Gamma) \geqq c_\omega(\lambda_{\theta^{-N_\omega}\omega}\cdots\lambda_{\theta^{-2}\omega}\lambda_{\theta^{-1}\omega})^{-1}m^\omega(\Gamma)\,. \tag{3.6}$$

On this stage I do not claim either uniqueness of λ_ω and ν^ω or their measurable dependence on ω. Next, define

$$\Lambda^\omega = \{g \in \mathscr{C}(X^\omega)\colon g \geqq 0,\ \int g d\nu^\omega = 1,\ \text{and } g(x) \leqq \alpha_\rho(\omega)g(\tilde{x}) \text{ if } \operatorname{dist}(x,\tilde{x}) = \rho \leqq \beta(\omega)\}$$

where $\alpha_\rho(\omega)$ and $\beta(\omega)$ are the same as in (3.1). First, Λ^ω is nonempty since $1 \in \Lambda^\omega$. Furthermore, Λ^ω is, clearly, convex and closed. It is also bounded (since X^ω is compact) and equicontinuous, and so it is compact. Take the infinite product $\Lambda^\omega_{\mathbb{Z}} = \cdots \times \Lambda^{\theta^{-1}\omega} \times \Lambda^\omega \times \Lambda^{\theta\omega} \times \cdots$ with the product topology. Then $\Lambda^\omega_{\mathbb{Z}}$ is a nonempty, convex, compact subset of the product $\prod_{i=-\infty}^{\infty} \mathscr{C}(X^{\theta^i\omega})$. Define the map $\Phi^\omega\colon \Lambda^\omega_{\mathbb{Z}} \to \Lambda^\omega_{\mathbb{Z}}$ which sends a sequence $q = \{q_i\}$ into the sequence $\Phi^\omega q = \{(\Phi^\omega q)_i\}$ by the formula

$$(\Phi^\omega q)_i = (\lambda^{\theta^i\omega})^{-1}R^{\theta^i\omega}q_{i+1}\,. \tag{3.7}$$

First, show that Φ^ω maps $\Lambda^\omega_{\mathbb{Z}}$ into itself. Suppose that $q = (\ldots, q_{-1}, q_0, q_1, \ldots) \in \Lambda^\omega_{\mathbb{Z}}$ so that $q_i \in \Lambda^{\theta^i\omega}$. I have to show that $(\Phi^\omega q)_i \in \Lambda^{\theta^i\omega}$ which follows because

$$\begin{aligned}\int (\Phi^\omega q)_i d\nu^{\theta^i\omega} &= (\lambda_{\theta^i\omega})^{-1}\int R^{\theta^i\omega}q_{i+1}d\nu^{\theta^i\omega} \\ &= (\lambda_{\theta^i\omega})^{-1}\int q_{i+1}d(R^{\theta^i\omega})^*\nu^{\theta^i\omega} = \nu^{\theta^{i+1}\omega}(q_{i+1}) = 1\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}(\Phi^\omega q)_i(x) &= (\lambda_{\theta^i\omega})^{-1}R^{\theta^i\omega}q_{i+1}(x) \leqq \alpha_\rho(\omega)(\lambda_{\theta^i\omega})^{-1}R^{\theta^i\omega}q_{i+1}(\tilde{x}) \\ &= (\Phi^\omega q)_i(\tilde{x})\,,\end{aligned} \tag{3.9}$$

provided $\operatorname{dist}(x,\tilde{x}) = \rho \leqq \beta(\omega)$ since by (3.1), $R^{\theta^i\omega}q_{i+1} \in \tilde{\Lambda}^\omega$. Now I can apply the Schauder–Tichonoff fixed point theorem to obtain a fixed point of Φ^ω denoted by $h_\omega = (h_{\omega,i})$ and put $h_{\theta^i\omega} = h_{\omega,i}$. Again, we have to take special care about ω which are periodic points of θ. For such ω we have to choose fixed points h_ω being periodic sequences which is possible since Φ^ω

preserves corresponding spaces of periodic sequences with period j if $\theta^j\omega = \omega$. Now we have that $(\lambda_{\theta^i\omega})^{-1}R^{\theta^i\omega}h_{\theta^i\omega} = h_{\theta^{i+1}\omega}$ and, in particular, we obtain (1.8). Moreover, since $h_\omega \in \Lambda^\omega$ it follows that $\int h_\omega dv^\omega = 1$, $h_\omega \geqq 0$, and $h_\omega(x) \leqq \alpha_r(\omega)h_\omega(\tilde{x})$ provided $\operatorname{dist}(x,\tilde{x}) \leqq \beta(\omega)$. The relation (1.8) implies that

$$\int h_{\theta^n\omega}(y)R^\omega(n,x,dy) = \lambda_\omega \cdots \lambda_{\theta^{n-1}\omega}h_\omega(x)\,.$$

Taking here $n = k = k_\omega = \min\{l \geqq 0 : N_{\theta^l\omega} \leqq l\}$, I derive from (1.5) that

$$h_\omega(x) \geqq c_{\theta^k\omega}(\lambda_\omega \cdots \lambda_{\theta^{k-1}\omega})^{-1}\int h_{\theta^k\omega}(y)dm^{\theta^k\omega}(y)\,. \tag{3.10}$$

By (1.6) and (3.5) for P-a.a.ω,

$$\operatorname{supp} v^\omega = \operatorname{supp} m^\omega \tag{3.11}$$

and since $h_\omega \in \mathscr{C}(X^\omega)$ and $\int h_\omega dv^\omega = 1$, I derive from (3.10) and (3.11) that $h_\omega > 0$.

Define the new operator P^ω mapping functions on $X^{\theta\omega}$ to functions on X^ω by the formula

$$P^\omega g(x) = (\lambda_\omega h_\omega(x))^{-1}R^\omega(h_{\theta\omega}g)(x)\,. \tag{3.12}$$

Then $P^\omega 1 = 1$, i.e., P^ω is a Markov operator. For any $x \in X^\omega$ and $\Gamma \subset X^{\theta\omega}$ set $P^\omega(x,\Gamma) = P^\omega(\mathbf{1}_\Gamma)(x)$ where $\mathbf{1}_\Gamma(y) = 1$ if $y \in \Gamma$ and $\mathbf{1}_\Gamma(y) = 0$, otherwise. If $P^\omega(n,x,\Gamma)$ is defined by means of $P^{\theta^i\omega}(y_i,dy_{i+1})$ as in Introduction then from (1.5), (3.10), and (3.11) it follows that (1.1) holds true with the same N_ω as in (1.5). Thus I can apply Theorem 2.1 and obtain the unique measure μ^ω satisfying (1.2) and (2.1). The latter can be rewritten in the following way

$$\left|(\lambda_{\theta^{-n}\omega}\cdots\lambda_{\theta^{-1}\omega})^{-1}(h_{\theta^{-n}\omega})^{-1}\int g(y)R^{\theta^{-n}\omega}(n,x,dy) - \int \frac{g}{h_\omega}d\mu^\omega\right| \tag{3.13}$$
$$\leqq C(\omega)(1-\kappa_\omega)^n \sup|g|$$

for any bounded Borel function g on X^ω. Now, I shall show that

$$h_\omega(y) = \frac{d\mu^\omega(y)}{dv^\omega(y)}\,. \tag{3.14}$$

Indeed, let $\eta^\omega \in \mathscr{P}(X^\omega)$ is defined by $\eta^\omega(\Gamma) = \int_\Gamma h_\omega dv^\omega$ for any Borel $\Gamma \subset X^\omega$. Then for any bounded Borel function g on $X^{\theta\omega}$,

$$\begin{aligned}\int g d\eta^{\theta\omega}(\Gamma) &= \int g h_{\theta\omega}dv^{\theta\omega} = (\lambda_\omega)^{-1}\int g h_{\theta\omega}d(R^\omega)^*v^\omega \\ &= (\lambda_\omega)^{-1}\int R^\omega(gh_{\theta\omega})dv^\omega = \int h_\omega P^\omega g dv^\omega = \int P^\omega g d\eta^\omega\end{aligned}$$

implying that $(P^\omega)^*\eta^\omega = \eta^{\theta\omega}$ and by the uniqueness of a measure satisfying (1.2), $\eta^\omega = \mu^\omega$. Next, applying (3.13) with g and 1 I derive

$$\lim_{n\to\infty}\frac{\int g(y)R^{\theta^{-n}\omega}(n,x,dy)}{R^{\theta^{-n}\omega}(n,x,X^\omega)} = \int g dv^\omega\,, \tag{3.15}$$

and so ν^ω is determined uniquely. Now

$$\lambda_\omega = \lambda_\omega \int 1 d\nu^{\theta\omega} = \int 1 d(R^\omega \nu^\omega) = \int R^\omega 1 d\nu^\omega \tag{3.16}$$

is determined uniquely, as well. Since both μ^ω and ν^ω are determined uniquely then by (3.14) h_ω is also determined uniquely.

It remains to establish the measurability in ω provided the operators R^ω depend measurably on ω. Denote by $L^0(\Omega, \mathscr{C}(X))$ the space of measurable maps $g\colon \Omega \to \mathscr{C}(X)$ so that $g = \{g_\omega\}$, $g_\omega \in \mathscr{C}(X^\omega)$ and by $L^1(\Omega, \mathscr{C}(X))$ the subset of $g \in L^0(\Omega, \mathscr{C}(X))$ such that $\int \sup_x |g_\omega(x)| dP(\omega) < \infty$. Observe that $L^1(\Omega, \mathscr{C}(X))$ is dense in $L^0(\Omega, \mathscr{C}(X))$ with respect to the topology where $g^{(n)} \to g$ if $\sup_{x \in X^\omega} |g_\omega^{(n)}(x) - g_\omega(x)| \to 0$ as $n \to \infty$ for P-a.a.ω. Since (3.15) is satisfied for any $g_\omega \in \mathscr{C}(X)$ and R^ω depends measurably on ω then $\int g_\omega d\nu^\omega$ depends measurably on ω. If $\mathscr{F}$ is countably generated then $L^1(\Omega, \mathscr{C}(X))$ is separable (see [BG2]) so I can choose a countable dense set of g's there and I conclude that ν^ω depends measurably on ω (and so does its density r^ω). By (3.16) λ_ω is also measurable in ω. Next, let Λ denotes the subset of $L^0(\Omega, \mathscr{C}(X))$ such that if $g \in \Lambda$ then $g_\omega \in \Lambda^\omega$. In the same way as above it follows that the operator $\Phi : L^0(\Omega, \mathscr{C}(X)) \to L^0(\Omega, \mathscr{C}(X))$ defined by $(\Phi g)_\omega = (\lambda_\omega)^{-1} R^\omega g_{\theta\omega}$ maps Λ into itself and preserves measurable dependence on ω. We know that $h = \{h_\omega\}$ is the unique fixed point of Φ. Note that $1 \in \Lambda$ and by convexity $g^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} \Phi^i 1 \in \Lambda$. Since Φ is linear it follows that any limit point of $g^{(n)}$ as $n \to \infty$ is a fixed point of Φ, and so $g^{(n)}$ must converge to h as $n \to \infty$. Since $L^0(\Omega, \mathscr{C}(X))$ is closed then $h \in L^0(\Omega, \mathscr{C}(X))$, i.e., h_ω depends measurably on ω. Finally, $\mu^\omega = h_\omega \nu^\omega$ depends measurably on ω, as well. Observe also that (3.16) gives the estimate

$$\inf_{x \in X^\omega} R^\omega(x, X^{\theta\omega}) \leqq \lambda_\omega \leqq \sup_{x \in X^\omega} R^\omega(x, X^{\theta\omega}) \tag{3.17}$$

which implies that

$$\int |\log \lambda_\omega| dP(\omega) < \infty \tag{3.18}$$

if logarithms of both the left and the right hand side of (3.17) are integrable in ω which will be used in the next section.

Assume now that (1.9) is also satisfied. Then in the same way as in Sect. 2 there exist densities $r^{\theta^{-n}\omega}(n, x, y)$, $n \geqq N_\omega$ and $r^\omega(y)$ and (3.3) follows from (2.3) in the same way as (3.13) follows from (2.1). Furthermore, taking $n = k$ and $\theta^k\omega$ in place of ω in (3.5) I obtain that for any $y \in X^{\theta^k\omega}$,

$$c_{\theta^k\omega} \leqq \lambda_{\theta^{k-1}\omega} \cdots \lambda_{\theta\omega} \lambda_\omega r^{\theta^k\omega}(y) \leqq (c_{\theta^k\omega})^{-1} \tag{3.19}$$

which together with (3.10) yield

$$\begin{aligned} h_\omega(x) &\geqq c_{\theta^k\omega} (\lambda_\omega \cdots \lambda_{\theta^{k-1}\omega})^{-1} \int h_{\theta^k\omega}(y)\, dm^{\theta^k\omega}(y) \\ &\geqq c_{\theta^k\omega}^2 \int h_{\theta^k\omega}(y)\, d\nu^{\theta^k\omega}(y) = c_{\theta^k\omega}^2 \end{aligned} \tag{3.20}$$

and similarly,

$$h_\omega(x) \leqq c^2_{\theta^k\omega}\,. \qquad \square \tag{3.21}$$

3.2 Remark One obtains an important particular case when (3.1) holds true assuming that $R^\omega(x,dy) = r^\omega(x,y)m^\omega(dy)$ with densities $r^\omega(x,y)$ satisfying

$$r^\omega(x,y) \leqq \alpha_\rho(\omega) r^\omega(z,y) \quad \text{provided dist}(x,z) = \rho \leqq \beta(\omega). \tag{3.22}$$

This condition is always satisfied if the number of elements of each X^ω is finite but may depend on ω as in [BG2]. If any two points $x,\tilde{x} \in X^\omega$ can be connected by a chain $x_i \in X^\omega$, $i = 0,\dots,n$ with $x_0 = x$, $x_n = \tilde{x}$, and $\text{dist}(x_i,x_{i+1}) \leqq \beta(\omega)$ then (3.22) implies (1.5).

3.3 Remark Observe that if I replace in the above assumptions and assertions θ by θ^{-1} and denote the corresponding quantities by the same letter with the tilde above I shall obtain that $\tilde{P}^\omega(n,x,\cdot)$ converges in variation uniformly in $x \in X^{\theta^n\omega}$ exponentially fast in n to a $\tilde{\mu}^\omega \in \mathscr{P}(X^\omega)$ satisfying

$$\tilde{\mu}^{\theta\omega}\tilde{P}^\omega = \tilde{\mu}^\omega \tag{3.23}$$

and (1.7)–(1.8) will turn into

$$\int d\tilde{v}^{\theta\omega}(x)\tilde{R}^\omega(x,\Gamma) = \tilde{\lambda}^\omega\tilde{v}^\omega(\Gamma) \tag{3.24}$$

and

$$\int \tilde{h}_\omega(y)\tilde{R}^\omega(x,dy) = \tilde{\lambda}^\omega\tilde{h}_{\theta\omega}(x)\,. \tag{3.25}$$

3.4 Remark In a particular case when X is a countable set (with a metric making it compact) Theorem 3.1 was derived in [BG] under the quite restrictive condition that N_ω is constant which was also assumed in [Se]. In order to realize how restrictive this condition is, consider the case when X is a finite set and $P^\omega, P^{\theta\omega},\dots$ and $R^\omega, R^{\theta\omega},\dots$ are matrices chosen by independent trials from finite sets of matrices $\mathscr{P}$ and $\mathscr{R}$. Then my condition follows if only some finite product of matrices from $\mathscr{P}$ (correspondingly, from $\mathscr{R}$) is a positive matrix though if $N_\omega = N$ is constant then any product of at least N matrices from $\mathscr{P}$ (correspondingly, from $\mathscr{R}$) must be a positive matrix.

4 Donsker–Varadhan formula and large deviations

Given $\varphi = \{\varphi_\omega\} \in L^1(\Omega, \mathscr{C}(X))$ and operators P^ω satisfying (1.1) and let $R^\omega = R^\omega_\varphi$ be defined by (1.10). Then (1.5) will be satisfied and the condition (1.6) is equivalent to

$$\text{supp}\, P^{\theta^{-N_\omega}\omega}(N_\omega,x,\cdot) = \text{supp}\, m^\omega \tag{4.1}$$

which will follow if, for instance, (1.3) holds true. The condition (3.1) will be satisfied if, in particular, $P^\omega(x,dy) = p^\omega(x,y)m^\omega(dy)$ and

$$p^\omega(x,y) \leqq \alpha_\rho(\omega)p^\omega(z,y) \quad \text{provided dist}(x,z) = \rho \leqq \beta(\omega)\,. \tag{4.2}$$

So assume that (1.5), (1.6), and (3.1) hold true and denote by $\lambda_\omega = \lambda_\omega^\varphi$, $h_\omega = h_\omega^\varphi$, and $\nu^\omega = \nu_\varphi^\omega$ the corresponding quantities given by Theorem 3.1. Since $\varphi = \{\varphi_\omega\} \in L^1(\Omega, \mathscr{C}(X))$, i.e., (1.11) holds true, then

$$\int \sup_{x \in X^\omega} |\log R^\omega(x, X^{\theta\omega})| \, dP(\omega) < \infty , \tag{4.3}$$

and so by (3.17) one has (3.18). I shall need also that

$$\int \sup_{x \in X^\omega} |\log h_\omega(x)| \, dP(\omega) < \infty \tag{4.4}$$

which in view of (3.20) and (3.21) will follow if

$$\int |\log c_{\theta^{k_\omega}\omega}| dP(\omega) < \infty . \tag{4.5}$$

The last condition is not good enough since it is not given in terms of φ and P^ω. In order to obtain a condition on φ and P^ω which gives (4.5) observe that for any $x \in X^{\theta^{-n}\omega}$ and $\Gamma \subset X^\omega$,

$$\begin{aligned} \exp\left(\sum_{i=0}^{n-1} \inf_{y_i \in X^{\theta^{-i}\omega}} \varphi_{\theta^{-i}\omega}(y_i)\right) P^{\theta^{-n}\omega}(n, x, \Gamma) &\leqq R^{\theta^{-n}\omega}(n, x, \Gamma) \\ &\leqq \exp\left(\sum_{i=0}^{n-1} \sup_{y_i \in X^{\theta^{-i}\omega}} \varphi_{\theta^{-i}\omega}(y_i)\right) P^{\theta^{-n}\omega}(n, x, \Gamma) . \end{aligned} \tag{4.6}$$

Thus if (1.1) and (1.3) are satisfied then (1.5) and (1.9) hold true for $R^\omega = R_\varphi^\omega$ with

$$c_\omega = \gamma_\omega \exp\left(-\sum_{i=0}^{N_\omega - 1} \sup_{y_i \in X^{\theta^{-i}\omega}} |\varphi_{\theta^{-i}\omega}(y_i)|\right). \tag{4.7}$$

Now the condition (4.5) is expressed in terms of φ and P^ω and if $N_\omega = N$ is nonrandom, (4.5) will be satisfied whenever $|\log \gamma_{\theta^k\omega}|$ and $\sup |\varphi_{\theta^k\omega}|$ are integrable and (1.11) holds true.

4.1 Theorem *Suppose that $R^\omega = R_\varphi^\omega$ given by* (1.10) *satisfies* (1.5), (1.6), *and* (3.1), *and the corresponding measures $\nu^\omega = \nu_\varphi^\omega$, numbers $\lambda_\omega = \lambda_\omega^\varphi$ and functions $h_\omega = h_\omega^\varphi$ satisfying* (1.7) *and* (1.8) *have the integrability properties* (3.18) *and* (4.4) *which follow from the conditions discussed above. Then* (1.12) *holds true. Moreover,* (1.14) *is satisfied if and only if a measure μ satisfies* (3.14).

Proof. First, observe that if $\sup_x |\log u_\omega(x)| \in L^1(\Omega, P)$ then $\sup_x |\log P^\omega u_{\theta\omega}(x)| \in L^1(\Omega, P)$, and so the right hand side of (1.13) makes sense. Now the proof proceeds in the same way as in the deterministic environment (see [Ki8]). For each $\eta \in \mathscr{P}_P(X \times \Omega)$ set $\Psi(\eta) = \int \varphi d\eta - I(\eta)$ then for $v_\omega = h_\omega e^{\varphi_\omega}$ I have

$$\begin{aligned} \Psi(\eta) &= \int \varphi d\eta + \inf_u \int \log\left(\frac{P^\omega u_{\theta\omega}}{u_\omega}\right) d\eta^\omega dP(\omega) \\ &\leqq \int \varphi d\eta + \int \log\left(\frac{P^\omega v_{\theta\omega}}{v_\omega}\right) d\eta^\omega dP(\omega) \\ &= \int \log\left(\frac{R^\omega h_{\theta\omega}}{h_\omega}\right) d\eta dP(\omega) = Q(\varphi) . \end{aligned} \tag{4.8}$$

Next, let $\mathscr{R}^\omega$ denotes the Markov operator defined by the right hand side of (3.12) and $\mu \in \mathscr{P}_P(X \times \Omega)$ be the unique measure whose desintegrations μ^ω satisfy (3.14) and

$$(\mathscr{R}^\omega)^* \mu^\omega = \mu^{\theta\omega} \ . \tag{4.9}$$

Set $u = e^w$ then by Jensen's inequality $\log \mathscr{R}^\omega u_{\theta\omega} \geqq \mathscr{R}^\omega \log u_{\theta\omega} = \mathscr{R}^\omega w_{\theta\omega}$, and so by (4.9),

$$\int \log \mathscr{R}^\omega u_{\theta\omega}\, d\mu^\omega \geqq \int \mathscr{R}^\omega w_{\theta\omega}\, d\mu^\omega = \int w_{\theta\omega}\, d\mu^{\theta\omega} = \int \log u_{\theta\omega}\, d\mu^{\theta\omega} \ . \tag{4.10}$$

Since $\mathscr{R}^\omega u_{\theta\omega} = -\log \lambda_\omega - \log h_\omega + \log P^\omega(e^{\varphi_{\theta\omega}} h_{\theta\omega} u_{\theta\omega})$ then integrating against μ^ω I obtain

$$\log \lambda_\omega \leqq \int \log P^\omega(e^{\varphi_{\theta\omega}} h_{\theta\omega} u_{\theta\omega}) d\mu^\omega - \int \log h_\omega d\mu^\omega - \int \log u_{\theta\omega} d\mu^{\theta\omega}.$$

Integrating this against P and taking into account that P is θ-invariant I derive

$$\begin{aligned} Q(\varphi) \ &\leqq \ \int \log P^\omega(e^{\varphi_{\theta\omega}} h_{\theta\omega} u_{\theta\omega}) d\mu^\omega dP(\omega) \\ &\quad - \int \log h_\omega d\mu^\omega dP(\omega) - \int \log u_\omega d\mu^\omega dP(\omega) \\ &= \ \int \varphi_\omega d\mu^\omega dP(\omega) + \int \log \frac{P^\omega v_{\theta\omega}}{v_\omega} d\mu^\omega dP(\omega) \end{aligned} \tag{4.11}$$

where $v_\omega = u_\omega h_\omega e^{\varphi_\omega}$. Taking the infinum in v I obtain $Q(\varphi) \leqq \int \varphi d\mu - I(\mu)$ which together with (4.8) yield (1.12) and (1.14).

It remains to show that μ is the unique measure from $\mathscr{P}_P(X \times \Omega)$ satisfying (1.14). For each $\eta \in \mathscr{P}_P(X \times \Omega)$ set

$$\begin{aligned} J(\eta) \ &= \ -\inf_v \int \log \frac{\mathscr{R}^\omega v_{\theta\omega}}{v_\omega} d\eta^\omega dP(\omega) \\ &= \ -\inf_u \int \log P^\omega(u_{\theta\omega} e^{\varphi_{\theta\omega}}) \lambda_\omega u_\omega d\eta^\omega dP(\omega) \\ &= \ Q(\varphi) + I(\eta) - \int \varphi d\eta \ . \end{aligned} \tag{4.12}$$

By (1.14) and (4.8), $J(\mu) = 0$ and $J(\eta) \geqq 0$ for all $\eta \in \mathscr{P}_P(X \times \Omega)$. Thus for any positive $v = \{v_\omega\}$ such that $\sup_x |\log v_\omega| \in L^1(\Omega, P)$ I have

$$\int \log \frac{\mathscr{R}^\omega v_{\theta\omega}}{v_\omega} d\mu^\omega dP(\omega) \geqq 0 \ . \tag{4.13}$$

Suppose that $Q(\tilde{\mu}) = \int \varphi d\tilde{\mu} - I(\tilde{\mu})$. For a bounded measurable $w = \{w_\omega\}$ take $v = 1 + \varepsilon w$ with ε small enough then

$$\psi(\varepsilon) \overset{\text{def}}{=} \int \log \frac{\mathscr{R}^\omega(1 + \varepsilon w_{\theta\omega})}{1 + \varepsilon w_\omega} d\tilde{\mu}^\omega dP(\omega) \geqq 0 \tag{4.14}$$

and $\psi(0) = 0$, and so $\frac{d\psi(\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = 0$ which gives

$$\int (\mathscr{R}^\omega w_{\theta\omega} - w_\omega) d\tilde{\mu}^\omega dP(\omega) = 0 \ . \tag{4.15}$$

I can take w in the form $w_\omega = \zeta(\omega)\tilde{w}_\omega$ for an arbitrary bounded measurable function ζ on Ω which enables me to derive from (4.15) that $\int \tilde{w}_\omega d((R^{\theta^{-1}\omega})^* \tilde{\mu}^{\theta^{-1}\omega}) = \int \tilde{w}_\omega d\mu^\omega$ for any bounded Borel function $\tilde{w}_\omega$ on X^ω. Thus $(R^{\theta^{-1}\omega})^* \tilde{\mu}^{\theta^{-1}\omega} = \tilde{\mu}^\omega$, and so $\tilde{\mu} = \mu$ by the uniqueness of measures satisfying (4.9) proved in Theorem 2.1. □

Observe that similar results can be obtained for operators $\tilde{R}^\omega = e^{\varphi_\omega} P^\omega$. If $\varphi \equiv 0$ then $R_\varphi^\omega = \tilde{R}_\varphi^\omega = P^\omega$, $h_\omega \equiv 1 = \lambda_\omega$, and $\mu^\omega = \nu^\omega$ satisfies (1.2).

Similarly to the deterministic environment Theorem 4.1 has applications to large deviations. Let $\mathbf{P}_x^\omega$ be the measures in path spaces of the Markov chains Z_n^ω defined in Introduction. Then the n-th iterate $R_\varphi^\omega(n)$ of the operator R_φ^ω acts on functions $g \in X^{\theta^n\omega}$ by the formula

$$R_\varphi^\omega(n)g(x) = \int g(Z_n^\omega) \exp\left(\sum_{i=1}^n \varphi_{\theta^i\omega}(Z_i^\omega)\right) d\mathbf{P}_x^\omega \,, \tag{4.16}$$

and so

$$\begin{aligned} \lambda_\omega^\varphi \lambda_{\theta\omega}^\varphi \cdots \lambda_{\theta^{n-1}\omega}^\varphi h_\omega^\varphi \left(\sup_{x\in X^{\theta^n\omega}} h_{\theta^n\omega}^\varphi(x)\right)^{-1} &= \left(\sup_{x\in X^{\theta^n\omega}} h_{\theta^n\omega}^\varphi(x)\right)^{-1} R_\varphi^\omega(n) h_{\theta^n\omega}^\varphi \\ \leqq R_\varphi^\omega(n)1 &\leqq \left(\inf_{x\in X^{\theta^n\omega}} h_{\theta^n\omega}^\varphi(x)\right)^{-1} R_\varphi^\omega(n) h_{\theta^n\omega}^\varphi \\ = \lambda_\omega^\varphi \lambda_{\theta\omega}^\varphi \cdots \lambda_{\theta^{n-1}\omega}^\varphi h_\omega^\varphi &\left(\inf_{x\in X^{\theta^n\omega}} h_{\theta^n\omega}^\varphi(x)\right)^{-1} . \end{aligned} \tag{4.17}$$

If (4.4) is satisfied txen

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in X^{\theta^n\omega}} |\log h_{\theta^n\omega}^\varphi(x)| = 0 \,. \tag{4.18}$$

Assumung that θ is an ergodic transformation of (Ω, P) I derive from (4.17) and (4.18) that with probability one

$$\lim_{n\to\infty} \frac{1}{n} \log R_\varphi^\omega(n)1 = Q(\varphi) \,. \tag{4.19}$$

Assume that Ω is a compact metric space. Then by [Ki2] (4.19) together with Theorem 4.1 yield the large deviations bounds (1.17) and (1.18) for the occupational measures ζ_ω^n given by (1.16). The compactness of Ω is needed only for a formal application of the main result from [Ki2]. In fact, it suffices to require that Ω is locally compact as in [Se] since the behaviour on Ω is controlled by one measure P which implies certain tightness property of measures under consideration.

5 Random perturbations

Let $F(\omega) : X^\omega \to X^{\theta\omega}$ be a measurable in ω family of continuous maps. Consider a family of Markov chains $Z_n^{\omega,\varepsilon}$ in the random environment such

that their transition probabilities $P_\varepsilon^\omega(x, \cdot\,)$ satisfy (1.1) and (1.3) and for any $g \in \mathscr{C}(X^{\theta\omega})$ uniformly in $x \in X^\omega$,

$$\int g(y)P_\varepsilon^\omega(x, dy) \to g(F(\omega)x) \quad \text{as } \varepsilon \to 0 \,. \tag{5.1}$$

I call Markov chains $Z_n^{\omega,\varepsilon}$ satisfying (5.1) random perturbations (in a random environment) of random transformations $F(\omega)$. Assume that the conditions of the previous section are satisfied so that for any $\varepsilon > 0$ and $\varphi = \{\varphi_\omega\} \in L^1(\Omega, \mathscr{C}(X))$ there exist unique $\lambda_\omega = \lambda_\omega^{\varphi,\varepsilon}$, $h_\omega = h_\omega^{\varphi,\varepsilon}$, $\nu^\omega = \nu_{\varphi,\varepsilon}^\omega$, and $\mu^\omega = \mu_{\varphi,\varepsilon}^\omega = h_\omega \nu^\omega$ satisfying (1.7) and (1.8) for $R^\omega = R_{\varphi,\varepsilon}^\omega = P_\varepsilon^\omega e^{\varphi_\omega}$ and such that $Q(\varphi) = Q_\varepsilon(\varphi)$ satisfies (1.12) with P_ε^ω in place of P^ω. On the other hand, one has the following variational formula for random transformations, called the relativized variational principle (see [Bo]),

$$Q_0(\varphi) = \sup_\eta \left(\int \varphi d\eta + H^{(r)}(\eta)\right) \tag{5.2}$$

where the supremum is taken over all $\eta \in \mathscr{P}_P(X)$ such that $(F(\omega))^*\eta^\omega = \eta^{\theta\omega}$ and $H^{(r)}(\eta)$ is the relativized entropy of random transformations $F(\omega)$ given the measure η. If the entropy is upper semicontinuous in η and $Q_\varepsilon(\varphi) \to Q(\varphi + \phi)$ as $\varepsilon \to 0$ for some $\phi \in L^1(\Omega, \mathscr{C}(X))$ and any $\varphi \in L^1(\Omega, \mathscr{C}(X))$ then it follows by convex analysis arguments (see [Ki3] or [Ki8]) that any weak limit $\mu_{\varphi+\phi,0}$ as $\varepsilon \to 0$ of measures $\mu_{\varphi,\varepsilon}$ is a measure from $\mathscr{P}_P(X)$ satisfying

$$(F(\omega))^*\mu_{\varphi+\phi,0}^\omega = \mu_{\varphi+\phi,0}^{\theta\omega} \tag{5.3}$$

and

$$Q_0(\varphi + \phi) = \int(\varphi + \phi)d\mu_{\varphi+\phi,0} + H^{(r)}(\mu_{\varphi+\phi,0}) \,. \tag{5.4}$$

The weak convergence $\eta_\varepsilon \xrightarrow{w} \eta$ as $\varepsilon \to \infty$ for η_ε, $\eta \in \mathscr{P}_P(X \times \Omega)$ means here that $\lim_{\varepsilon\to 0} \int g d\eta_\varepsilon = \int g d\eta$ for any $g \in L^1(\Omega, \mathscr{C}(X))$ which makes $\mathscr{P}_P(X \times \Omega)$ a compact space (see [BG2]). If one can prove the uniqueness of a measure $\mu_{\varphi+\phi,0}$ satisfying (5.3) and (5.4) then we obtain $\mu_{\varphi,\varepsilon} \xrightarrow{w} \mu_{\varphi+\phi,0}$ as $\varepsilon \to 0$.

Suppose now that X^ω's are compact smooth d-dimensional manifolds imbedded into a compact set X and $F(\omega)$ is a measurably depending on ω family of C^2 onto maps with nondegenerate differentials $DF(\omega)$ and such that there exist $0 < \kappa_1, \kappa_2 < 1$ with

$$\kappa_1 \leqq \|(DF(\omega))^{-1}\| \leqq \kappa_2 \,.$$

In fact, one can take κ_1 and κ_2 even depending on ω assuming that $-\infty < \int \log \kappa_1 dP \leqq \int \log \kappa_2 dP < 0$. I shall assume that the transition probabilities $P_\varepsilon^\omega(x, \cdot\,)$ have densities $p_\varepsilon^\omega(x, y)$ with respect to the Riemannian volume $m^{\theta\omega}$ on $X^{\theta\omega}$ such that there exist non-negative functions q_x^ω on the tangent spaces $T_x X^\omega$ satisfying

$$\int q_x^\omega(\xi)dm_x^\omega(\xi) = 1, \qquad q_x^\omega(\xi) \leqq Ce^{-\alpha\|\xi\|}, \quad C, \alpha > 0 \,, \tag{5.5}$$

where m_x^ω is the Riemannian volume on $T_x X^\omega$, and

$$\begin{aligned}(1 - \varepsilon^\alpha)\varepsilon^{-d} q_{F(\omega)x}^\omega(\varepsilon^{-1}\mathrm{Exp}_{F(\omega)x}^{-1} y) \\ \leqq p_\varepsilon^\omega(x, y) \leqq (1 + \varepsilon^\alpha)\varepsilon^{-d} q_{F(\omega)x}^\omega(\varepsilon^{-1}\mathrm{Exp}_{F(\omega)x}^{-1} y)\end{aligned} \tag{5.6}$$

provided $\operatorname{dist}(F(\omega)x, y) \leqq \varepsilon^{1-\alpha}$ where $0<\alpha<1$ and Exp denotes the exponential map. I assume also that

$$p_\varepsilon^\omega(x, y) \leqq C\varepsilon^{-d} e^{-\frac{2}{\varepsilon}\operatorname{dist}(F(\omega)x, y)} \tag{5.7}$$

and

$$\int \log \|F(\omega)\|_{C^2} dP(\omega) < \infty \tag{5.8}$$

where $\|\cdot\|_{C^2}$ denotes the C^2-norm. If $\varphi = \{\varphi_\omega\} \in L^1(\Omega, \mathscr{C}(X))$ satisfies

$$|\varphi_\omega(x) - \varphi_\omega(y)| \leqq C_\varphi(\omega)|x - y|^\beta \ , \tag{5.9}$$

for some $\beta > 0$ independent of ω, and

$$\log C_\varphi \in L^1(\Omega, P) \quad \text{and} \quad \sup_x |\varphi_\omega(x)| \in L^1(\Omega, P) \ , \tag{5.10}$$

and ϕ also satisfies the same conditions then by [KK] there exists a unique $\mu_{\varphi+\phi,0} \in \mathscr{P}_P(X)$ satisfying (5.3) and (5.4).

5.1 Theorem *Under the above conditions for any* $\varphi \in L^1(\Omega, \mathscr{C}(X))$,

$$\lim_{\varepsilon\to 0} Q_\varepsilon(\varphi) = Q_0(\varphi + \phi) \tag{5.11}$$

where $\phi_\omega(x) = -\log|\det \operatorname{Jac}_x F(\omega)|$ *and* Jac *means the Jacobian. If* (5.9) *and* (5.10) *hold true then*

$$\mu_{\varphi,\varepsilon} \xrightarrow{w} \mu_{\varphi+\phi,0} \quad \text{as } \varepsilon \to 0 \ . \tag{5.12}$$

In particular, if $\varphi \equiv 0$ *then by Theorems* 2.1, 3.1, *and* 4.1 $\mu_{\varphi,\varepsilon}$ *is the unique invariant measure of the skew product Markov chain and as* $\varepsilon \to 0$ *it converges to* $\mu_{\phi,0}$ *which is the unique invariant measure of the skew product transformation* $\tau(x,\omega) = (F(\omega)x, \theta\omega)$ *such that for P-a.a.*ω *the desintegration* μ^ω *is equivalent to the Riemannian volume* m^ω *on* X^ω (*see* [Ki5]).

Proof. The argument proceeds essentially in the same way as in [Ki3, Ki6 and Ki8]. I take $n = n(\varepsilon) \approx (\log\varepsilon)^2$ and using (5.7) I represent $R_{\varphi,\varepsilon}^\omega(n)1(x)$ in the form

$$\int \cdots \int \prod_{i=1}^n (\mathbf{1}_{\operatorname{dist}(Z_i^{\omega,\varepsilon}, F(\theta^{i-1}\omega)Z_{i-1}^{\omega,\varepsilon}) \leqq \delta} \exp(\varphi_{\theta^i\omega}(Z_i^{\omega,\varepsilon}))) dP_{\varepsilon,x}^\omega \tag{5.13}$$

plus an exponentially small in ε term where $P_{\varepsilon,x}^\omega$ is the path space probability of the Markov chain $Z^{\omega,\varepsilon}$. Employing the shadowing lemma for random expanding transformations (see [Ki6 and BG1]) I conclude that the expression (5.13) can be written essentially as the sum

$$\sum_{z\in\mathscr{E}(\delta,n,\omega)} \int \cdots \int \prod_{i=1}^n (\mathbf{1}_{\operatorname{dist}(Z_i^\omega, F^i(\omega)z) \leqq \text{const}\delta} \exp(\varphi_{\theta^i\omega}(Z_i^{\omega,\varepsilon}))) dP_{\varepsilon,x}^\omega \tag{5.14}$$

where $\mathscr{E}(\delta, n, \omega)$ is a maximal (δ, n, ω)-separated set (see [Ki5, Ki6, KK and BG1]) and $F^i(\omega) = F(\theta^{i-1}\omega)\cdots F(\theta\omega)F(\omega)$. Using (5.6) and employing the same linearization procedure as in [Ki8] I conclude that each term in the sum

(5.14) is essentially of order $\exp\sum_{i=1}^{n}(\varphi+\phi)_{\theta^i\omega}(F^i(\omega)z)$. By the definition of the relativized topological pressure I derive from here similarly to [Ki3, Ki6 and Ki8] that $Q_\varepsilon(\varphi)=\lim_{n\to\infty}\frac{1}{n}\log R^\omega_{\varphi,\varepsilon}(n)1$ is close for small ε to $Q_0(\varphi+\phi)$. Now (5.12) follows, as explained above, from the uniqueness of the maximizing measure (equilibrium state) in (5.2) proved in [KK]. □

The result of this section can be extended to the case when X^ω's are random repellers, i.e., compact subsets of a compact domain X in a manifold such that $F(\omega)$'s expand distances in some neighborhoods of X^ω's and we condition the Markov chains $Z^{\omega,\varepsilon}$ to stay in X.

Observe also that assuming

$$\lim_{\varepsilon\to 0}\varepsilon\log P^\omega_\varepsilon(x,U)=-\inf_{y\in U}\rho^\omega(x,y)$$

for any open $U\subset X^{\theta\omega}$ where $\rho^\omega(x,y)\geqq 0$ is measurable in ω and continuous in $x\in X^\omega$ and $y\in X^{\theta\omega}$ one can obtain also a discrete-time version of the Wentzel–Freidlin theory in random environments extending some of the results from Sect. 1.5 of [Ki1 and Ki4] though this does not require methods of the present paper.

6 Continuous time case

Let now θ^t, $t\in\mathbb{R}$ be a continuous parameter group of transformations acting on a probability space $(\Omega,\mathscr{F},P)$ and preserving the measure P and let X be a compact smooth manifold. I shall consider on X a diffusion Z^ω_t in a random environment given by a stochastic differential equation

$$dZ^\omega_t=\sigma_{\theta^t\omega}(Z^\omega_t)dw_t+b_{\theta^t\omega}(Z^\omega_t)dt \tag{6.1}$$

with bounded coefficients which are smooth in the first argument and measurable in the second and where the driving noise is independent of the stationary process $\theta^t\omega$ on Ω. Let $\mathbf{P}^\omega_x$ denotes the path space probability for the nonhomogeneous in time diffusion Z^ω_t starting at x and set $P^\omega(t,x,\Gamma)=\mathbf{P}^\omega_x\{Z^\omega_t\in\Gamma\}$, $\Gamma\subset X$. We have also the operators P^ω_t and $R^\omega_t=R^\omega_{t,\varphi}$ defined by

$$P^\omega_t g(x)=\int g(y)P^\omega(t,x,dy) \tag{6.2}$$

and

$$R^\omega_t g(x)=\int g(Z^\omega_t)\exp\left(\int_0^t\varphi_{\theta^s\omega}(Z^\omega_s)\right)d\mathbf{P}^\omega_x \tag{6.3}$$

for any $\varphi=\{\varphi_\omega\}\in L^1(\Omega,\mathscr{C}(X))$. Set also $L^\omega=\frac{1}{2}(\sigma_\omega,\sigma_\omega\nabla)+(b_\omega,\nabla)$ which is the elliptic operator corresponding to the process Z^ω_t. The following result follows easily from the discrete time assertions of Sects. 2–4.

6.1 Theorem *Suppose that $\sigma_\omega\sigma^*_\omega$ is uniformly positive definite and R^ω_t is given by* (6.3). *Then there exist unique measurable in ω families of $\mu^\omega\in\mathscr{P}(X)$, $\nu^\omega=\nu^\omega_\varphi\in\mathscr{P}(X)$, $h_\omega=h^\varphi_\omega\in\mathscr{C}(X)$, and $\lambda_\omega=\lambda^\varphi_\omega>0$ such that*

$$\mu^\omega P^\omega_t=\mu^{\theta^t\omega}\,, \tag{6.4}$$

$P^{\theta^{-1}\omega}(t,x,\Gamma)$ converges exponentially fast to $\mu^\omega(\Gamma)$ for any Borel $\Gamma \subset X$,

$$v^\omega R_t^\omega = \exp\left(\int_0^t \log \lambda_{\theta^s\omega} ds\right) v^{\theta^t\omega}\ , \tag{6.5}$$

and

$$R_t^\omega h_{\theta^t\omega} = \exp\left(\int_0^t \log \lambda_{\theta^s\omega} ds\right) h_\omega\ . \tag{6.6}$$

Furthermore,

$$\int \log \lambda_\omega dP(\omega) = \sup_\eta (\int \varphi d\eta - I(\eta)) \tag{6.7}$$

where the supremum is taken over the space $\mathscr{P}_P(X \times \Omega)$ and

$$I(\eta) = -\inf_u \int \left(\frac{L^\omega u_\omega}{u_\omega}\right) d\eta^\omega dP(\omega) \tag{6.8}$$

where the infinum is taken over measurable families $u = \{u_\omega\}$ of positive C^2 functions on X such that $\int \|u_\omega\|_{C^2} dP(\omega) < \infty$.

7 Invariant measures for random IFS

In this section I shall consider random iterated function systems (IFS) which also generate Markov chains in random environments. Let $(\Omega, \mathscr{F}, P)$, θ, and X^ω be as in Introduction and $\Phi(\omega) = \{F_1(\omega), \ldots, F_{m(\omega)}(\omega)\}$ be a measurable in $\omega \in \Omega$ family of continuous maps from X^ω to $X^{\theta\omega}$ where $m = m(\omega)$ is a $\mathbb{Z}_+ = \{1,2,\ldots\}$-valued random variable. Let $p^\omega = \{p_1^\omega, \ldots, p_{m(\omega)}^\omega\}$ be a measurable in ω family of probability vectors. For each $x \in X^\omega$ and $\Gamma \subset X^{\theta\omega}$ set

$$P^\omega(x,\Gamma) = \sum_{i=1}^{m(\omega)} p_i^\omega \mathbb{I}_\Gamma(F_i(\omega)x) \tag{7.1}$$

where $\mathbb{I}_\Gamma(z) = 1$ if $z \in \Gamma$ and $= 0$, otherwise. The random transition probabilities $P^\omega(x,\Gamma)$ generate a Markov chain Z_n^ω in a random environment. Though the conditions of Sect. 2 are not satisfied in this set up nevertheless if the maps $F_i(\omega)$ are contractions there exist measures μ^ω satisfying (1.2).

7.1 Theorem *Suppose that θ is ergodic and*

$$\max_{1 \leqq i \leqq m(\omega)} \operatorname{dist}(F_i(\omega)v, F_i(\omega)w) \leqq s(\omega)\operatorname{dist}(v,w) \tag{7.2}$$

for all $v, w \in X^\omega$ with

$$\int \log s\, dP < 0\ . \tag{7.3}$$

Then there exists a unique measurable in ω family of probability measures μ^ω on X^ω satisfying (1.2).

Proof. The argument goes in the same way as for usual contracting iterated function systems. Recall, that the Hutchinson metric d_H^ω on $\mathscr{P}(X^\omega)$ is given by

$$d_H^\omega(\nu,\eta) = \sup\left\{\int_{X^\omega} f d\nu - \int_{X^\omega} f d\eta : f \in L^\omega\right\} \tag{7.4}$$

where $L^\omega = \{f : |f(x) - f(y)| \leqq \operatorname{dist}(x,y), \forall x, y \in X^\omega\}$. Let $\tilde{L}\omega = \{\tilde{f} : \tilde{f} = s^{-1}(\omega)\sum_{i=1}^{m(\omega)} p_i^\omega f \circ F_i(\omega), f \in L^{\theta\omega}\}$ then $\tilde{L}^\omega \subset L^\omega$, and so

$$\begin{aligned} d_H^{\theta\omega}(\nu P^\omega, \eta P^\omega) &= \sup\left\{s(\omega)\int \tilde{f} d\nu - s(\omega)\int \tilde{f} d\eta : \tilde{f} \in \tilde{L}^\omega\right\} \\ &\leqq s(\omega) d_H^\omega(\nu,\eta)\,. \end{aligned} \tag{7.5}$$

Take $\nu, \eta \in X^{\theta^{-n}\omega}$ then

$$d_H^\omega(\nu P^{\theta^{-n}\omega}(n), \eta P^{\theta^{-n}\omega}(n)) \leqq s(\theta^{-n})\cdots s(\theta^{-1}\omega) d_H^{\theta^{-n}\omega}(\nu,\eta) \tag{7.6}$$

where $P^\omega(n) = P^\omega P^{\theta\omega}\cdots P^{\theta^{n-1}\omega}$. Since $X^\omega \subset X$ for all ω and both X^ω and X are compact I derive easily from (7.3) that for any sequence $\nu_{n_i} \in \mathscr{P}(X^{\theta^{-n_i}\omega})$ the sequence $\nu_{n_i} P^{n^{-n_i}\omega}(n_i)$ converges in d_H^ω to a measure μ^ω satisfying (1.2) and such a measure is unique. □

It is possible to study the continuity properties of measures μ^ω similarly to the deterministic case considered in [DF]. For applications of random IFS to constructions of random fractals see [Ki9].

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