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On the singular continuous spectrum of self-adjoint extensions

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1 Introduction

In the famous papers [7] and [9] by Friedrichs and Krein it has been shown that every closed symmetric operator H in a Hilbert space $\mathscr{H}$ with gap J has a self-adjoint extension $\hat{H}$ such that J is contained in the resolvent set of $\hat{H}$; an open interval (a,b) is called a *gap* of H if

$$\left\| \left(H - \frac{a+b}{2} \right) f \right\| \geqq \frac{b-a}{2} \|f\|, \quad f \in D(H), \quad \text{if } -\infty < a < b < \infty\,,$$

$$(Hf,f) \geqq b\|f\|^2, \quad f \in D(H), \quad \text{if } -\infty = a < b < \infty\,.$$

Moreover Krein has found that if in addition H has finite deficiency indices (n,n), then within the gap J the spectrum of every self-adjoint extension consists of a finite number of eigenvalues such that the sum of their multiplicities does not exceed n, cf. [9], Theorem 22. Conversely, if $\{\lambda_j\}_{j=1}^s$, $1 \leqq s < \infty$, is an arbitrary sequence of points of J and $\{p_j\}_{j=1}^s$ is an arbitrary sequence of positive integers obeying $\sum_{j=1}^s p_j \leqq n$, then there exists a self-adjoint extension $\tilde{H}$ of H such that within the gap J the spectrum of $\tilde{H}$ coincides with the points λ_j which are eigenvalues of multiplicity p_j, $1 \leqq j \leqq s$ ([9], Theorem 23). So the problem which spectrum can the self-adjoint extensions have within the gap is completely solved for finite deficiency indices.

In [3,4,5] and [10] an attempt was made to extend these results to the case of infinite deficiency indices. It turned out that Theorem 23 of [9] has a straightforward generalization. Let $\mathscr{E}$ be a countable set within the gap J and let $p : \mathscr{E} \to \mathbf{N} \cup \{\aleph_0\}$ be an arbitrary function. Then there exists a self-adjoint extension $\tilde{H}$ of H such that $\sigma_p(\tilde{H}) \cap J = \mathscr{E}$, the multiplicity of each eigenvalue $\lambda \in \mathscr{E}$ equals $p(\lambda)$ and no point of the gap J belongs to the continuous spectrum of $\tilde{H}$. In other words, any pure point spectrum can be generated within the gap J by choosing an appropriate extension. Here $\sigma_p(\cdot)$ denotes the set of eigenvalues of an operator.

However, provided the deficiency indices of H are infinite it seems naturally to believe that other kinds of spectra (singular and absolutely continuous spectra) can arise within the gap J. In fact, for a large class of operators H, including all symmetric operators with infinite deficiency indices and compact resolvent, we have shown that every kind of absolutely continuous spectrum within a gap J of H can be generated by a self-adjoint extension $\tilde{H}$ of H, cf. [2]. In this paper we shall show that a symmetric operator with infinite deficiency indices and some gap has self-adjoint extensions with non-empty singular continuous spectrum. Actually we shall prove the following stronger.

Theorem 1 *Let H be a symmetric operator in some Hilbert space $\mathscr{H}$. Suppose that the operator H has some gap J and infinite deficiency indices. Let J_0 be any open subset of J. Then H has a self-adjoint extension $\tilde{H}$ with the following properties*:

(i) $\sigma_{sc}(\tilde{H}) \cap J = \sigma_{ess}(\tilde{H}) \cap J = \overline{J_0} \cap J$.
(ii) $\sigma_{ac}(\tilde{H}) \cap J = \emptyset$.
(iii) *$\tilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$.*

Here $\sigma, \sigma_{ac}, \sigma_{sc}$ and σ_{ess} denote the spectrum, the absolutely continuous, the singular continuous and the essential spectrum, respectively. $\bar{S}$ denotes the closure of the set S.

The method of proof is as follows. Without loss of generality we assume $0 \in J$. First one constructs an auxiliary invertible self-adjoint extension H_{aux} of H such that H_{aux} has pure point spectrum within the gap J of H, the eigenvalues of H_{aux} within J are simple and form a dense subset of J_0. Then one chooses a vector $g \in \operatorname{ran}(H)^{\perp}$ such that $(g,e) \neq 0$ for every eigenvector e of H_{aux} corresponding to an eigenvalue in J and shows that the operator $H_{aux}^{-1} + \alpha(g,\cdot)g$ is invertible and its inverse $\tilde{H}_\alpha$ is a self-adjoint extension of H for every real number α. Finally one proves that for every α in some dense G_δ-subset of $\mathbf{R}$ the operator $\tilde{H}_\alpha$ has the required spectral properties. This easily follows from the following recent result by A. Gordon resp. by R. del Rio, N. Makarov and B. Simon.

Theorem 2 (A. Gordon [8]; R. del Rio, N. Makarov, B. Simon [6], Theorem 3) *Let A be a self-adjoint operator and g a cyclic vector of A. Then the set*

$$\{\alpha \in \mathbf{R} : A + \alpha(g,\cdot)g \quad \textit{has no eigenvalue in } \sigma(A)\}$$

is a dense G_δ subset of $\mathbf{R}$.

For convenience of the reader we shall give a proof of the existence of the auxiliary operator H_{aux} which is more simple and much shorter than our original proof in [3]. Moreover we shall need the mentioned result by A. Gordon and by R. del Rio, N. Makarov and B. Simon only in a very special case. Instead to show that this result can be used in our situation we shall give a short direct proof that the operator $\tilde{H}_\alpha$ has the required spectral properties.

In our very special case we get absence of eigenvalues in $\overline{J_0} \cap J$ even for *every* $\alpha \in \mathbf{R}$, $\alpha \neq 0$.

Finally we mention that Theorem 1 allows only to generate so-called "fat" singular continuous spectrum by extensions, i.e., singular continuous spectrum which coincides with the closure of its inner points. For spectrum which does not have this property (so-called "thin" spectrum) we cannot make any conclusions. In particular, we cannot generate singular continuous spectrum which is a Cantor set. The problem is that for thin sets the used proof technique does not allow to decide whether the generated spectrum is really singular continuous or results from the closure of the discrete spectrum which is outside the thin set. So the problem remains open for thin sets.

2 The singular continuous spectrum of self-adjoint extensions

In this section we shall prove Theorem 1. We shall use the following

Lemma 3 *Let H be a symmetric operator in some separable Hilbert space $\mathscr{H}$. Let b be a strictly positive real number and $J = (-b,b)$ or $J = (-\infty,b)$. Suppose that J is a gap of H. For every $\lambda \in J$ let $P_\lambda : \ker(H^*) \to \ker(H^* - \lambda)$ be the mapping given by*

$$P_\lambda f := P_{\ker(H^* - \lambda)} f, \quad f \in \ker(H^*), \tag{1}$$

where $P_{\mathscr{L}}$ denotes the orthogonal projection in $\mathscr{H}$ onto the subspace $\mathscr{L}$. Then for every $\lambda \in J$ the mapping P_λ is bijective and

$$\|P_\lambda^{-1} g\| \leqq \frac{b + |\lambda|}{b - |\lambda|} \|g\|, \quad g \in \operatorname{ran}(P_\lambda), \tag{2}$$

when $J = (-b,b)$ and

$$\|P_\lambda^{-1} g\| \leqq \max\left\{\frac{b}{b-\lambda}, \frac{b-\lambda}{b}\right\}, \quad g \in \operatorname{ran}(P_\lambda), \tag{3}$$

when $J = (-\infty,b)$.

Proof. Since J is a gap of H the symmetric operator H has a self-adjoint extension $\hat{H}$ such that $J \cap \sigma(\hat{H}) = \emptyset$, e.g., the Friedrichs and the Krein extension of H in the case when $J = (-\infty,b)$ and $J = (-b,b)$, respectively. Note that

$$\int_J F(t) d(E(t)f,g) = 0$$

for all $f,g \in \mathscr{H}$ and every Borel-measurable function F where $\{E(t)\}_{t\in\mathbf{R}}$ denotes the spectral family of the self-adjoint operator $\hat{H}$.

Let $\lambda \in J$. Let $f \in \ker(H^*) = \operatorname{ran}(H)^\perp$, $f \neq 0$ and $g \in D(H)$. We have

$$\begin{aligned}(\hat{H}(\hat{H} - \lambda)^{-1} f, (H - \lambda)g) &= \int \frac{t}{t-\lambda}(t-\lambda) d(E(t)f,g) \\ &= \int t\, d(E(t)f,g) = (f, Hg) = 0\,.\end{aligned}$$

Thus $\tilde{f} := \hat{H}(\hat{H} - \lambda)^{-1} f \in \operatorname{ran}(H - \lambda)^{\perp} = \ker(H^* - \lambda)$ and consequently we have

$$\|P_\lambda f\| \geqq \left(\frac{\tilde{f}}{\|\tilde{f}\|}, f \right) = \frac{\int_{\mathbf{R}\setminus J} t/(t-\lambda) d\|E(t)f\|^2}{\{\int_{\mathbf{R}\setminus J} (t/(t-\lambda))^2 d\|E(t)f\|^2\}^{1/2}} . \tag{4}$$

Since

$$\frac{b}{b + |\lambda|} \leqq \frac{t}{t-\lambda} \leqq \frac{b}{b - |\lambda|}, \quad t \in \mathbf{R}\setminus J ,$$

when $J = (-b, b)$ and

$$\min\left\{1, \frac{b}{b-\lambda}\right\} \leqq \frac{t}{t-\lambda} \leqq \max\left\{1, \frac{b}{b-\lambda}\right\}, \quad t \in \mathbf{R}\setminus J ,$$

when $J = (-\infty, b)$ this implies that

$$\|P_\lambda f\| \geqq \frac{b - |\lambda|}{b + |\lambda|} \|f\| \tag{5}$$

and

$$\|P_\lambda f\| \geqq \frac{\min\{1, b/(b-\lambda)\}}{\max\{1, b/(b-\lambda)\}} \|f\| \tag{6}$$

when $J = (-b, b)$ and $J = (-\infty, b)$, respectively. Thus P_λ is invertible and (2) and (3) hold.

By (5) and (6) the operator P_λ has a trivial kernel and a closed range. Hence it remains to show that $f \in \ker(H^* - \lambda)$ and $(f, h) = 0$ for each $h \in \ker(H^*)$ yields $f = 0$. Since

$$D(H^*) = D(\hat{H}) \dotplus \ker(H^*) ,$$

we obtain elements $g \in D(\hat{H})$ and $k \in \ker(H^*)$ such that $f = g + k$. By $H^* f = \lambda f$ and $(f, h) = 0$, $h \in \ker(H^*)$, we find $H^* f \in \operatorname{ran}(H)$. Hence one gets $H^* f = \hat{H} g \in \operatorname{ran}(H)$. However, this yields $g \in D(H)$. Using that we obtain

$$(H - \lambda) g = \lambda k .$$

Since $k \in \ker(H^*)$ we have

$$(Hg, (H - \lambda) g) = \|Hg\|^2 - \lambda (Hg, g) = 0$$

which implies

$$\|Hg\| \leqq |\lambda| \|g\| .$$

Let $|\lambda| < b$. Since $b\|g\| \leqq \|Hg\|$ we immediately find

$$b\|g\| \leqq \|Hg\| \leqq |\lambda| \|g\|$$

which proves $g = 0$. If $\lambda \leqq -b$, then the result is obvious. Therefore $k = 0$ and $f = 0$. $\square$

Proof of Theorem 1. Since H has a self-adjoint extension $\hat{H}$ such that the gap J is contained in the resolvent set of $\hat{H}$ the theorem is true (with $\tilde{H} = \hat{H}$) in the special case when $J_0 = \emptyset$. Thus we may assume that $J_0 \neq \emptyset$. Moreover we may assume that $J = (-b, b)$ or $J = (-\infty, b)$ for some strictly positive real number b.

It suffices to show that there exists a self-adjoint extension $\tilde{H}$ of H such that $\sigma_{\text{ess}}(\tilde{H}) \cap J = \overline{J_0} \cap J$, $\sigma_{\text{ac}}(\tilde{H}) \cap J = \emptyset$ and $\tilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$. In fact, then on the one hand every $\lambda \in J_0$ belongs to the singular continuous spectrum of $\tilde{H}$ and consequently we have $\overline{J_0} \subset \sigma_{\text{sc}}(\tilde{H})$, on the other hand we have $\sigma_{\text{sc}}(\tilde{H}) \subset \sigma_{\text{ess}}(\tilde{H})$ and consequently $\sigma_{\text{sc}}(\tilde{H}) \cap J \subset \overline{J_0}$.

We choose any square summable sequence $\{\alpha_n\}_{n \in \mathbf{N}}$ of numbers such that $\alpha_n \neq 0$ for every $n \in \mathbf{N}$ and any sequence $\{\eta_n\}_{n \in \mathbf{N}}$ in $J_0^{-1} := \{1/t : t \in J_0, t \neq 0\}$ such that $\eta_n \neq \eta_m$ for $n \neq m$ and for every $\eta \in \overline{J_0^{-1}}$

$$|\eta_n - \eta| < |\alpha_n| \tag{7}$$

for infinitely many $n \in \mathbf{N}$.

Such sequences always exist. For instance we start with a partion Γ_1 of the real axis into intervals $[k, k+1)$, $k \in \mathbf{Z}$. Dividing the intervals $[k, k+1)$ into two intervals $[k, k+\frac{1}{2})$ and $[k+\frac{1}{2}, k+1)$ we get a new partion Γ_2. Dividing again the intervals $[k, k+\frac{1}{2})$ and $[k+\frac{1}{2}, k+1)$ into two subintervals of half length we get a further partion Γ_3. Repeating this procedure again and again we obtain a sequence of partions $\{\Gamma_l\}_{l \in \mathbf{N}}$. Choosing now from the intersection of J_0^{-1} with the intervals of the partion Γ_l, provided this intersection is not empty, points we get for each $l \in \mathbf{N}$ a sequence of points $\{\eta_{lm}\}_{m \in \mathbf{Z}}$. Obviously all those points η_{lm} can be chosen different from each other. Making a suitable renumeration of the sequence $\{\eta_{lm}\}_{l \in \mathbf{N}, m \in \mathbf{Z}}$ we find the desired sequence $\{\eta_n\}_{n \in \mathbf{N}}$ of J_0^{-1}.

For notational brevity we put $\lambda_n := 1/\eta_n$ and $p_n := P_{\lambda_n}$ for every $n \in \mathbf{N}$ where for every $\lambda \in J$ the linear mapping $P_\lambda : \ker(H^*) \to \ker(H^* - \lambda)$ is given by (1).

We choose any $e_1 \in \ker(H^* - \lambda_1)$ such that $\|e_1\| = 1$. Let $n \in \mathbf{N}$ and suppose that $e_j \in \ker(H^* - \lambda_j), 1 \leqq j \leqq n$, have been chosen. Then we choose any $e_{n+1} \in \ker(H^* - \lambda_{n+1})$ such that $\|e_{n+1}\| = 1$,

$$e_{n+1} \perp e_j\,, \qquad e_{n+1} \perp p_j^{-1} e_j\,,$$

$$p_{n+1}^{-1} e_{n+1} \perp p_j^{-1} e_j\,, \qquad p_{n+1}^{-1} e_{n+1} \perp e_j\,,$$

$1 \leqq j \leqq n$. Since, by Lemma 3, for every $\lambda \in J$ the linear mapping P_λ is bijective and consequently the space $\ker(H^* - \lambda)$ is infinite dimensional each of these choices is possible. In this way we get, by induction, an orthonormal system $\{e_n\}_{n \in \mathbf{N}}$ with the following properties:

$$e_n \in \ker(H^* - \lambda_n), \quad n \in \mathbf{N}\,, \tag{8}$$

$$(g_n, g_m) = 0 = (g_n, e_m) \quad \text{for } n \neq m \tag{9}$$

where

$$g_n := p_n^{-1} e_n, \quad n \in \mathbf{N}. \tag{10}$$

Next we shall show that there exists an auxiliary self-adjoint extension H_{aux} of H with the following properties:

(i) H_{aux} has a pure point spectrum within J.

(ii) λ_n is a simple eigenvalue of H_{aux} and e_n a corresponding eigenvector for every $n \in \mathbf{N}$.

(iii) $\sigma_p(H_{\text{aux}}) \cap J = \{\lambda_n : n \in \mathbf{N}\}$.

Since $\{\lambda_n : n \in \mathbf{N}\}$ is a dense subset of $\overline{J_0}$ and $\lambda_n \neq 0$ for every $n \in \mathbf{N}$ it follows from (i) and (iii) that such an operator also satisfies

(iv) $\sigma_{\text{ess}}(H_{\text{aux}}) \cap J = \overline{J_0} \cap J$.

(v) H_{aux} is invertible.

We denote by $\mathscr{H}_0$ the closure of the span of $\{e_n : n \in \mathbf{N}\}$ and by M the self-adjoint operator in the Hilbert space $\mathscr{H}_0$ given by

$$D(M) := \left\{ \sum_{n=1}^{\infty} \beta_n e_n : \sum_{n=1}^{\infty} (1 + \lambda_n^2) |\beta_n|^2 < \infty \right\},$$

$$M \sum_{n=1}^{\infty} \beta_n e_n := \sum_{n=1}^{\infty} \lambda_n \beta_n e_n, \quad \sum_{n=1}^{\infty} (1 + \lambda_n^2) |\beta_n|^2 < \infty.$$

Obviously the operator M has a pure point spectrum, λ_n is a simple eigenvalue of M and e_n a corresponding eigenvector for every $n \in \mathbf{N}$,

$$\sigma_p(M) = \{\lambda_n : n \in \mathbf{N}\}$$

and

$$(Mf, f) \leqq b \|f\|^2, \quad f \in D(M), \tag{11}$$

in the case when $J = (-\infty, b)$ and

$$\|Mf\| \leqq b \|f\|, \quad f \in D(M),$$

in the case when $J = (-b, b)$.

M is a restriction of H^* since $e_n \in \ker(H^* - \lambda_n)$ for every $n \in \mathbf{N}$ and H^* is a closed operator. Thus we can define an extension H' of H by

$$D(H') := D(H) \dot{+} D(M), \qquad H'g := H^*g, \quad g \in D(H').$$

A short computation shows that H' is a symmetric operator.

Let $f \in D(H')$. For every $n \in \mathbf{N}$ we have

$$(H'f, e_n) - (f, Me_n) = \lambda_n (f, e_n).$$

Thus

$$\sum_{n=1}^{\infty} \lambda_n^2 |(f, e_n)|^2 = \|P_{\mathscr{H}_0} H'f\|^2 < \infty.$$

Hence $P_{\mathscr{H}_0} f \in D(M)$. For every $n \in \mathbf{N}$ we have

$$(P_{\mathscr{H}_0} H' f, e_n) = (f, Me_n) = (MP_{\mathscr{H}_0} f, e_n) .$$

Thus

$$P_{\mathscr{H}_0} H' f = MP_{\mathscr{H}_0} f, \quad f \in D(H') .$$

This implies that the operator H' can be written in the form

$$H' = M \oplus G_0 ,$$

where the symmetric operator G_0 in the Hilbert space $\mathscr{H}_0^\perp$ is given by

$$G_0 := H'_{|D(H') \cap \mathscr{H}_0^\perp} .$$

We shall show by contradiction that the gap J of H is also a gap of G_0. We shall give the proof for $J = (-\infty, b)$. The proof in the other case is virtually the same. Suppose that

$$(G_0 f, f) < b\|f\|^2 \tag{12}$$

for some $f \in D(G_0)$. We choose $g \in D(H)$ and $h \in D(M)$ such that $f = g + h$. Then we have

$$\begin{aligned}(Hg, g) = (H'(f-h), f-h) &= (G_0 f, f) + (Mh, h) \\ &< b\|f\|^2 + b\|h\|^2 = b\|f-h\|^2 = b\|g\|^2 .\end{aligned}$$

Here we have used that $H' = M \oplus G_0$, as well as our assumption (12) and (11). Thus the assumption (12) leads to a contradiction to the hypothesis that $(-\infty, b)$ is a gap of H. Thus J is also a gap of G_0.

Since J is a gap of symmetric operator G_0 in $\mathscr{H}_0^\perp$ there exists a self-adjoint operator G in $\mathscr{H}_0^\perp$ such that $G_0 \subset G$ and $\sigma(G) \cap J = \emptyset$. We put

$$H_{\text{aux}} := M \oplus G .$$

Obviously H_{aux} has the required properties.

We put

$$g := \sum_{n=1}^{\infty} \alpha_n \frac{g_n}{\|g_n\|} ,$$

where the $g_n, n \in \mathbf{N}$, are given by (10) and the $\alpha_n, n \in \mathbf{N}$, are any numbers different from zero such that the sequence $\{\alpha_n\}_{n\in\mathbf{N}}$ is square summable and (7) holds. Since, by (9), $\{g_n/\|g_n\|\}_{n\in\mathbf{N}}$ is an orthonormal system the series converges and g is well-defined. Since $g_n \in \ker(H^*)$ for every $n \in \mathbf{N}$ and $\ker(H^*)$ is closed we have that $g \in \ker(H^*)$. Obviously $g \neq 0$.

We choose any $\alpha \in \mathbf{R}, \alpha \neq 0$. Since along with H_{aux} also the inverse H_{aux}^{-1} of H_{aux} is a self-adjoint operator and $\alpha(g, \cdot)g$ is a bounded self-adjoint operator the sum $H_{\text{aux}}^{-1} + \alpha(g, \cdot)g$ is also self-adjoint. Let $h \in D(H_{\text{aux}}^{-1})$ be such that

$$H_{\text{aux}}^{-1} h + \alpha(g, h)g = 0 .$$

Then $(g,h)g \in \operatorname{ran}(H_{\text{aux}}^{-1}) = D(H_{\text{aux}})$. If g would be in $D(H_{\text{aux}})$ then we would have $H_{\text{aux}}g = H^*g = 0$ which is impossible since H_{aux} is invertible. Thus we have $(g,h) = 0$. It follows that $H_{\text{aux}}^{-1}h = 0$ which implies that $h = 0$. Thus we have shown that the operator $H_{\text{aux}}^{-1} + \alpha(g,\cdot)g$ is invertible. Along with this operator also its inverse

$$\tilde{H} := (H_{\text{aux}}^{-1} + \alpha(g,\cdot)g)^{-1}$$

is self-adjoint.

Let $h \in D(H^{-1}) = \operatorname{ran}(H)$. Since $H \subset H_{\text{aux}}$ and $g \in \ker(H^*) = \operatorname{ran}(H)^{\perp}$ we have that $H^{-1}h = H_{\text{aux}}^{-1}h = \tilde{H}^{-1}h$. Thus $\tilde{H}$ is a self-adjoint extension of H. Since the resolvent difference $\tilde{H}^{-1} - H_{\text{aux}}^{-1}$ of the self-adjoint operators $\tilde{H}$ and H_{aux} is nuclear we have that $\sigma_{\text{ac}}(\tilde{H}) = \sigma_{\text{ac}}(H_{\text{aux}})$ and $\sigma_{\text{ess}}(\tilde{H}) = \sigma_{\text{ess}}(H_{\text{aux}})$. In particular, we have

$$\sigma_{\text{ac}}(\tilde{H}) \cap J = \emptyset\,, \qquad \sigma_{\text{ess}}(\tilde{H}) \cap J = \overline{J_0} \cap J\,.$$

Thus we have only to show that $\tilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$.

The point zero is not an eigenvalue of $\tilde{H}$ since $\tilde{H}$ is invertible. Let $\lambda \in \overline{J_0} \cap J$ and $\lambda \neq 0$. We have only to show that $\eta := 1/\lambda$ is not an eigenvalue of $\tilde{H}^{-1}$. Let $h \in D(\tilde{H}^{-1}) = D(H_{\text{aux}}^{-1})$ and

$$\tilde{H}^{-1}h = H_{\text{aux}}^{-1}h + \alpha(g,h)g = \eta h\,.$$

By taking the scalar product with e_n we get from the last relation that

$$\eta_n(e_n,h) + \alpha(g,h)\frac{\alpha_n}{\|g_n\|} = \eta(e_n,h)$$

for every $n \in \mathbf{N}$. Thus we have

$$|\eta_n - \eta||(e_n,h)| = |\alpha(g,h)|\frac{|\alpha_n|}{\|g_n\|}, \quad n \in \mathbf{N}\,. \tag{13}$$

By (7), there exists a subsequence $\{\eta_{n_j}\}_{j\in\mathbf{N}}$ of $\{\eta_n\}_{n\in\mathbf{N}}$ such that

$$|\eta_{n_j} - \eta| < \alpha_{n_j}, \quad j \in \mathbf{N}\,. \tag{14}$$

By (2) resp. (3) in the Lemma 3 and (10) there exists a finite constant c such that

$$\|g_{n_j}\| < c, \quad j \in \mathbf{N}\,. \tag{15}$$

Since $\sum_{n=1}^{\infty} |(e_n,h)|^2 = \|P_{\mathscr{H}_0}h\|^2 < \infty$ it follows from (13), (14) and (15) that

$$(g,h) = 0\,.$$

Thus we have

$$H_{\text{aux}}^{-1}h = \eta h\,.$$

Since the only eigenvalues of the operator H_{aux}^{-1} in J^{-1} are the numbers η_n, $n \in \mathbf{N}$, and η_n is a simple eigenvalue of H_{aux} with corresponding eigenvector e_n

for every $n \in \mathbf{N}$ this implies that $h = ae_n$ for some constant a and some $n \in \mathbf{N}$. Since

$$0 = (g, h) = a\frac{\alpha_n}{\|g_n\|}$$

it follows that $a = 0$ and $h = 0$. Thus η is not an eigenvalue of the operator $\tilde{H}^{-1}$ and the theorem is proven. $\square$

Remark 4 The operator $\tilde{H}$ given in the proof of the Theorem 1 depends on $\{\eta_n\}_{n\in\mathbf{N}}, g$ and α. Let us write $\tilde{H}_{\{\eta_n\},g,\alpha}$ in order to indicate this dependence. We have shown that the operator $\tilde{H}_{\{\eta_n\},g,\alpha}$ has the spectral properties described in the Theorem 1 for *every* $\alpha \in \mathbf{R}$, $\alpha \neq 0$, provided $\{\eta_n\}_{n\in\mathbf{N}}$ and g are constructed as in the Proof of Theorem 1. Actually one can admit a larger class of sequences $\{\eta_n\}_{n\in\mathbf{N}}$ and vectors g. In fact, let $\{\eta_n\}_{n\in\mathbf{N}}$ be any sequence in J_0^{-1} such that $\eta_n \neq \eta_m$ for $n \neq m$ and $\{\eta_n : n \in \mathbf{N}\}$ is dense in J_0^{-1}. Let g be any vector in $\ker(H^*)$ such that $(g, e_n) \neq 0$ for every $n \in \mathbf{N}$. Then with the aid of the result by A. Gordon resp. by R. del Rio, N. Makarov and B. Simon mentioned in the introduction one can show that the operator $\tilde{H}_{\{\eta_n\},g,\alpha}$ has the spectral properties described in Theorem 1 for all α from some dense G_δ set in $\mathbf{R}$.

Example 5 Let Ω be a bounded non-empty domain in $\mathbf{R}^d, d > 1$. Then the minimal Laplacian on Ω, i.e. the operator $-\Delta^\Omega_{\min}$ in $L^2(\Omega)$ given by

$$D(-\Delta^\Omega_{\min}) := C_0^\infty(\Omega),$$

$$-\Delta^\Omega_{\min} f := -\Delta f, \quad f \in C_0^\infty(\Omega),$$

is a symmetric operator with infinite deficiency indices. Here $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. Thus, by Theorem 1, there exist self-adjoint realizations of the Laplacian on Ω, i.e. self-adjoint extensions of $-\Delta^\Omega_{\min}$, with non-empty singular continuous spectrum. Thus (the proof of) Theorem 1 enables us to construct self-adjoint realizations of the Laplacian on a bounded domain Ω in $\mathbf{R}^d, d > 1$, with spectral properties very different from the properties of the self-adjoint realizations investigated before.

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