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Fredholm theory of holomorphic discs under the perturbation of boundary conditions

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1 Introduction

In 1985, Gromov [G] introduced the notion of *pseudo-holomorphic curves*, proved existence theorems of such curves in various situations and developed various ways of using them to prove many nontrivial results in symplectic geometry which cannot have been proven in other methods. One of such results in [G] is the existence of holomorphic discs with boundary on any given (compact) Lagrangian submanifold in $\mathbb{C}^n$ with respect to the standard complex structure. This, as a corollary, proves the non-existence of any compact *exact* (embedded) Lagrangian submanifold in $\mathbb{C}^n$, which in turn gives rise to the existence of an exotic symplectic structure on $\mathbb{C}^n$ (for $n \geqq 2$). The non-existence of any compact exact Lagrangian submanifold L into $\mathbb{C}^n$ can be rephrased as follows: For the standard symplectic structure ω on $\mathbb{C}^n$ the homomorphism $I_\omega : \pi_2(\mathbb{C}^n, L) \to \mathbb{R}$ defined by

$$I_\omega(w) = \int_{D^2} w^*\omega$$

is not trivial for any embedded compact Lagrangian submanifold. Later Polterovich [P1] carefully analyzed Gromov's proof of the above existence result to control the Maslov index of Gromov's J-holomorphic disc and proved that one can choose a generic almost complex structure J so that there exists a J-holomorphic disc w with Maslov index bound

$$3 - n \leqq \mu_L(w) \leqq n + 1 . \tag{1.1}$$

In particular, when $n = 2$, this implies the non-triviality of the Maslov class of any compact Lagrangian embedding.

The purpose of the present paper is two-fold: One is to introduce a new approach to Gromov techniques in the study of Lagrangian embeddings in Kähler manifolds (e.g. $\mathbb{C}^n$ or $\mathbb{C}P^n$) which uses the perturbation of boundary conditions (with complex structure fixed) instead of almost complex structures as used in the traditional Gromov theory, and the other, which has been possible by this new approach, is to address this important existence result in the purely complex analytic context which would benefit those from the area of several complex variables. One of the advantages of keeping the integrable complex structure is that one can do precise local calculations, which have enabled us to prove an optimal criterion for the Fredholm-regularity of holomorphic discs in terms of the so-called *partial indices* (see [O3]), which resembles the well-known regularity criterion for holomorphic spheres in terms of the partial Chern numbers in the complex geometry.

There are three levels of perturbing boundary conditions:

i) *Hamiltonian isotopies of Lagrangian boundary conditions,*
ii) *Lagrangian isotopies of Lagrangian boundary conditions,*
iii) *Totally real isotopies of totally real boundary conditions.*

The proofs of the perturbation results below are entirely similar for all of these cases, but i) requires the most careful treatment. Therefore we give the complete proof for the case i) and indicate needed modifications for the other cases.

Now, we start with the case i). We denote by ϕ_H^1 the time one-map of Hamilton's equation

$$\dot{x} = X_{H_t}(x)$$

for $H : M \times I \to \mathbb{R}$ a smooth function and by $\mathscr{D}_\omega(M)$ the set of such ϕ_M^1's, i.e., the set of *exact* (or *Hamiltonian*) diffeomorphism of (M, ω):

$$\mathscr{D}_\omega = \mathscr{D}_\omega(M) = \{\phi \in \mathrm{Diff}(M) \mid \phi = \phi_H^1 \text{ for some } H\} .$$

And let J be an almost complex structure on (M, ω) such that the triple (M, ω, J) defines a Kähler structure, i.e., such that J is integrable and the bilinear form $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M. Indeed, we do not need the integrability of J for this theorem to hold (see Remark 3.5) but we assume it for the coherence of our exposition.

Theorem I. *Let (M, ω, J) as above and let $L \subset (M, \omega)$ be a given compact Lagrangian submanifold. Then there exists a dense subset $(\mathscr{D}_\omega)^L_{\mathrm{reg}} \subset \mathscr{D}_\omega$ such that for $\phi \in (\mathscr{D}_\omega)^L_{\mathrm{reg}}$, any (not multiply covered) holomorphic disc*

$$w : (D^2, \partial D^2) \to (M, \phi(L))$$

is Fredholm-regular. Furthermore, the same kind of genericity result as above also holds under the Lagrangian isotopies.

We refer to Sect. 2 for the precise definition of the Fredholm-regularity. This theorem makes unnecessary introducing almost complex structures in the study, via Gromov's techniques of pseudo-holomorphic discs, of Lagrangian embeddings on Kähler manifolds (e.g., $\mathbb{C}^n$ or $\mathbb{C}P^n$) and enables us to work only with the standard integrable complex structure. Because of this, together with the optimal regularity criterion we prove in [O3], we hope that one might be able to exploit the classical theory of analytic discs in several complex variables in the study of Lagrangian embeddings.

In this respect, it would be useful to know the existence of Fredholm-regular *holomorphic* discs. For *any compact* Lagrangian submanifold in $\mathbb{C}^n$, Gromov [G] proved the existence of a non-trivial holomorphic disc $w : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$ which is a limit of J-holomorphic discs as J converges to the standard complex structure on $\mathbb{C}^n$. Because of the way how the construction goes, the disc found may not be regular in general. There exists a Lagrangian submanifold for which we have a non-regular holomorphic disc (see Example 3.1). After we prove Theorem I, it is straightforward to modify Polterovich's argument [P1] to prove the following complex analytic analogue of Polterovich's result in [P1], replacing the perturbation of almost complex structures by the one of Lagrangian submanifolds (with the standard complex structure on $\mathbb{C}^n$ fixed).

Theorem II (Compare with [P1]). *Let $L \in (\Lambda_{L_0}(\mathbb{C}^n))_{\mathrm{reg}}$. Then there exists a non-trivial regular holomorphic disc $w : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$ i.e., a non-trivial solution of the equation*

$$\begin{cases} \bar{\partial} w = 0 & \text{in } D^2 \\ w|_{\partial D^2} \subset L \end{cases}$$

such that

$$3 - n \leqq \mu_L(w) \leqq n + 1 .$$

Finally we study a Fredholm theory similar to Theorem I for holomorphic discs with totally real boundary conditions in (almost) complex manifolds (M,J). We denote by $\mathscr{T}(M)$ the set of totally real embeddings in (M,J) and for each given totally real embedding R_0, we define

$$\mathscr{T}_{R_0}(M) = \{R \in \mathscr{T}(M) \,|\, R \text{ isotopic to } R_0 \text{ through totally real embeddings}\} .$$

Then we also prove the following theorem.

Theorem III. *Let (M,J) be a (almost) complex manifold and R_0 be a given totally real submanifold. Then there exists a dense subset $\mathscr{T}_{R_0,\mathrm{reg}}(M) \subset \mathscr{T}_{R_0}(M)$ such that if $R \in \mathscr{T}_{R_0,\mathrm{reg}}(M)$, any (not multiply-covered) J-holomorphic disc $w : (D^2, \partial D^2) \to (M, R)$ is Fredholm-regular.*

Now comes the organization of this paper. In Sect. 2, we briefly summarize basic facts on the Maslov index and its relation to the Fredholm index of holomorphic discs. Section 3 starts with an example of non-regular holomorphic disc showed to us by Polterovich and then develops a new Fredholm theory of holomorphic discs under Hamiltonian (also under Lagrangian) pertubations of Lagrangian embeddings and proves Theorem I. We also take this chance to point out some nontrivial technical points in relation to the structure of the image of J-holomorphic discs, which has not been addressed in the literature [P1,2], [S] and [L]. Finally in Sect. 4, we study the case of totally real boundary conditions and prove Theorem III.

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2 Maslov index and Fredholm index

In this section, we summarize the well-known facts on the Masolv index and its relation to the Fredholm index of (pseudo)-holomorphic discs in a calibration (or an *almost Kähler structure*) (M,ω,J): We call a triple (M,ω,J) a calibration or an almost Kähler structure if the bilinear form $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M.

First, we recall the definition of the Maslov index $\mu_L(w)$ for *any smooth* disc $w : (D^2, \partial D^2) \to (M,L)$ that is not necessarily (pseudo)-holomorphic: For a smooth disc $w : (D^2, \partial D^2) \to (M,L)$, we can find a unique trivialization (up to homotopy) of the pull-back bundle $w^*TM \simeq D^2 \times \mathbb{C}^n$ as a symplectic vector bundle. This trivialization defines a map α_w from $S^1 = \partial D^2$ to $\Lambda(n) = \Lambda(\mathbb{C}^n) :=$ *the set of Lagrangian planes in* $\mathbb{C}^n$ on which there is the well-known universal Maslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z})$ (see [A] for a detailed description). We define the Maslov index $\mu_L(w)$ of the disc $w : (D^2, \partial D^2) \to (M,L)$ by

$$\mu_L(w) := \mu(\alpha_w) \in \mathbb{Z}\,.$$

This definition depends only on the homotopy class in $\pi_2(M,L)$. If we have another disc w' with

$$w|_{\partial D^2} = w'|_{\partial D^2} \subset L\,,$$

then we have (see [V] for the proof)

$$\mu_L(w) - \mu_L(w') = 2c_1([w - w']) \tag{2.1}$$

where c_1 is the first Chern class of (M,J) and the map $w - w' : S^2 \to M$ is defined by

$$(w - w')(z) = \begin{cases} w(z) & \text{if } z \in D^2 \\ w'(\bar z) & \text{if } z \in \overline{D}^2 \end{cases}.$$

Here we identify S^2 with $D^2 \cup \overline{D}^2$ where $\overline{D}^2$ is the disc with the opposite orientation. In particular when c_1 vanishes (e.g., in $\mathbb{C}^n$ or more generally in Ricci-flat Kähler manifolds), the Maslov index $\mu_L(w)$ depends only on the homotopy class in $\pi_1(L)$ of the boundary map of the disc, and so defines a cohomology class in $H^1(L;\mathbb{Z})$. If one restricts to Lagrangian submanifolds in $\mathbb{C}^n$, this cohomology class is exactly the one defined by Maslov (see [A]), which is called the *Maslov class* of the Lagrangian submanifold $L \subset \mathbb{C}^n$.

Now if we are given a J-holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$, the linearization operator $E_w := D\bar{\partial}_J(w)$ of the map $\bar{\partial}_J = \frac{1}{2}(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y})$ at w is a Fredholm operator of Fredholm index given by

$$\operatorname{Index} E_w = \mu_L(w) + n \tag{2.2}$$

(see [G] for more details). Note that if w is non-trivial, the automorphism group of the disc $(D^2, \partial D^2)$ gives 3-dimensional family of J-holomorphic disc and so generates at least 3 dimensional kernel of the linearization E_w. This implies

$$3 \leqq \dim \operatorname{Ker} E_\omega = \operatorname{Index} E_w = \mu(w) + n$$

and so

$$\mu(w) \geqq 3 - n \tag{2.3}$$

provided w is Fredholm-regular, i.e. provided Coker $E_w = \{0\}$.

3 Proof of Theorem I

One might wonder if there may not exist any non-regular holomorphic disc $w : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$, but this is not the case and the following example, provided by Polterovich, shows that there really exists a non-regular (not multiply covered) holomorphic disc w unless we perturb the given Lagrangian embedding.

Example 3.1 [*Polterovich*]. Consider the standard torus $T^2 = T^2(1, r) = S^1(1) \times S^1(r) \subset \mathbb{C}^2$, where $S^1(r)$ is the circle with radius r and with its center at the origin in $\mathbb{C}$. The loop $S^1(1) \times \{r\} \subset T^2$ obviously bounds the standard holomorphic disc w_0 defined by

$$w_0(z) = (z, r) \in \mathbb{C}^2 .$$

We will construct a Lagrangian submanifold $L \subset \mathbb{C}^2$ that is a Hamiltonian deformation of T^2 which still bounds the holomorphic disc but its Maslov index with respect L can be arbitrarily large non-positive even number, if we vary the radius r. Note that the above standard disc has Maslov index $\mu_{T_2}(\partial w_0) = 2$ and the symplectic period

$$\omega[w_0] = \theta(\partial w_0) = \pi . \tag{3.1}$$

We now recall a fact, which seems to be known to some experts for some times, that any two embedded smooth loops in $\mathbb{C}^n$ with the same symplectic periods are isotopic through an ambient Hamiltonian isotopy (see e.g. [LO; Appendix 1] for the proof of this fact). Then we consider the radii r given by

$$r = \sqrt{k}, \quad k \in \mathbb{Z}_+ .$$

For a fixed $k \in \mathbb{Z}_+$, the loop $\lambda_1 \subset T^2$ defined by

$$\lambda_1(\theta) = (e^{-i(k-1)\theta}, \sqrt{k}e^{i\theta})$$

has its Maslov index and symplectic period given by

$$\mu_{T^2}(\lambda_1) = -2(k-2), \qquad \theta(\lambda_1) = \pi \tag{3.2}$$

Since $\lambda_0 = \partial w_0$ and λ_1 have the same symplectic period, there exists an exact symplectic diffeomorphism ϕ_0 of $\mathbb{C}^n$ that maps λ_0 to λ_1. Now, we choose

$$L = \phi_0(T^2)$$

and then L still bounds the same holomorphic disc w_0. However the boundary of the disc lying on $L = \phi_0(T^2)$ has Maslov index

$$\mu_L(\lambda_0) = -2(k-2)\,.$$

This is because the Maslov index is invariant under Hamiltonian isotopies in the sense that

$$\mu_{\phi(L)}(\phi(\lambda)) = \mu_L(\lambda)$$

for any given L and $\lambda \subset L$. In particular, $\mu_L(w_0) \leqq 0$ if $k \geqq 2$, and so w_0 cannot be regular by (2.3). Hence the theorem in this section will show that this kind of holomorphic discs are non-generic under the perturbation of Lagrangian boundary conditions.

In the rest of this section, we develop a Fredholm theory that takes a different point of view from the traditional one in the Gromov theory of pseudoholomorphic discs: *we fix the* (*integrable*) *complex structure, or rather the Kähler structure* (M, ω, J) *and vary the given compact Lagrangian submanifold* L *in* (M, ω) *through either ambient Hamiltonian isotopies or Lagrangian isotopies while the traditional approach in the Gromov theory is to change the complex structure into* (*non-integrable*) *almost complex structures as parameters.* This approach is motivated by our attempt to exploit the standard Kähler structure of $\mathbb{C}^n$ in the study of the Maslov class of Lagrangian embeddings in $\mathbb{C}^n$.

We first start with the problem of Hamiltonian perturbations and then indicate later how to modify this for the perturbation by Lagrangian isotopies. We first recall the description by Weinstein [W2] of the space of Hamiltonian (or exact) deformations of a given Lagrangian submanifold $L \subset (M, \omega)$, which he calls the *isodrast* of L. Following [W2], we denote by $\Lambda = \Lambda(M, \omega)$ the space of closed, embedded Lagrangian submanifolds of (M, ω) and the isodrast of L by $\Lambda_L = \Lambda_L(M, \omega)$. Using the cotangent coordinates [W1], we see that the tangent space to Λ (as well as the local model) at a given point L is naturally isomorphic to the space $Z^1(L)$ of closed one-forms on L. Then the tangent space of the isodrast $\Lambda_L = \Lambda_L(M, \omega) \subset \Lambda(M, \omega)$ of L is naturally isomorphic to the subspace $B^1(L)$ in $Z^1(L)$ consisting of exact one-forms. Since the tangent space $T_L\Lambda_L$ to Λ_L of L is isomorphic to the space $B^1(L)$, we may also identify $T_L\Lambda_L$ with the space of the smooth functions of f on L with

$$\int_L f = 0\,.$$

In other words, the map $X : f \mapsto X_f$ defines an isomorphism between

$$C_0^\infty(L) := \left\{ f \in C^\infty(L) \,\middle|\, \int_L f = 0 \right\}$$

and $T_L \Lambda_L$ where X_f is defined by

$$X_f(x) := J \operatorname{grad}_L f(x), \quad x \in L .$$

Here we used the measure on L to integrate that is induced from the ambient metric on M which is in turn induced from the Kähler structure (M, ω, J). The induced metric on L is also used to define $\operatorname{grad}_L f(x)$. Similarly, the metric on M induces a function on $C^\infty(L)$ defined by

$$\| f \|_{\bar{\varepsilon}} = \sum_{k \in \mathbb{N}} \varepsilon_k \max_{x \in L} | D^k f(x) | \tag{3.3}$$

which defines a norm on the space

$$C^{\bar{\varepsilon}}(L) = \{ f \in C_0^\infty(L) | \, \| f \|_{\bar{\varepsilon}} < \infty \} .$$

This norm has been introduced by Floer [F] in a different context. Furthermore, one can prove in the same way as in [F: Lemma 5.1] that $C^{\bar{\varepsilon}}(L)$ is a Banach space and can choose $\bar{\varepsilon}$ so that $C^{\bar{\varepsilon}}(L)$ is dense in $C_0^\infty(L)$ with respect to L^2-norm on $C_0^\infty(L)$. We choose $r > 0$ small enough so that for $\| f \|_{\bar{\varepsilon}} < r$, the cotangent coordinate chart map of Λ_L restricted to

$$C_r^{\bar{\varepsilon}}(L) := \{ f \in C^{\bar{\varepsilon}}(L) | \, \| f \|_{\bar{\varepsilon}} < r \}$$

is injective, which is possible because the topology given by $\| \cdot \|_{\bar{\varepsilon}}$ is stronger than C^∞-topology. More precisely, we note that the cotangent coordinate chart map gives an isomorphism between a neighborhood V_L of L in M and a neighborhood U_L of the zero section in T^*L. We denote this cotangent chart map by

$$\Phi_L : V_L \subset M \to U_L \subset T^*L .$$

We note that each Lagrangian submanifold in T^*L C^∞-close to the zero section can be represented by the graph of the exact one-form df for some smooth function f on L. If we impose the condition $\int_L f = 0$, this correspondence is one to one.

Furthermore one can also give a canonical parametrization of the graphs

$$\operatorname{Graph}(df) = \{ (x, df(x)) \in T^*L \, | \, x \in L \} ,$$

using the unique representative f, by the map

$$\phi_f : x \mapsto (x, df(x)) .$$

In this way, once we fix the cotangent coordinate chart

$$\Phi_{L_0} : V_{L_0} \to U_{L_0} ,$$

we can canonically represent each Lagrangian submanifold L C^∞-close to L_0 as

$$L = \phi_L(L_0)$$

where $\phi_L = \Phi_{L_0}^{-1} \circ \phi_{f_L} \circ \Phi_{L_0}$. Here f_L is the unique function on L_0 with $\int_{L_0} f = 0$ that represents the Lagrangian submanifold $\Phi_{L_0}(L) \subset T^*L_0$. Moreover, it is also well-known (see e.g. [W1]) that this ϕ_L can be extended to an ambient Hamiltonian isotopy $\phi_t : M \to M$ so that

$$\phi_L \equiv \phi_1|_{L_0} .$$

In particular,

$$\phi_1 \in \mathscr{D}_\omega(M) .$$

We will abuse our notation so that ϕ_L also denotes any such Hamiltonian diffeomorphism. We denote by $X_{L_0} : C_r^{\bar{\varepsilon}}(L_0) \to \Lambda_{L_0}(M,\omega)$ the map defined by

$$X_{L_0}(f) = \Phi_{L_0}^{-1} \text{ (Graph } df)$$

and by $\mathscr{N}(L_0)$ the image of $C_r^{\bar{\varepsilon}}(L_0)$ under the map X_{L_0}. Then $\mathscr{N}(L)$ is certainly a (open) Banach manifold since $C_r^{\bar{\varepsilon}}(L_0)$ is an open set of Banach space.

Now, we fix $s > 1$ and define

$$\mathscr{F} = \mathscr{F}^s = H^{s+1}(D^2, M)$$

which is the Sobolev space of all maps $\omega : D^2 \to M$ whose $(s+1)^{\text{th}}$ derivative is in L^2. For a given compact Lagrangian submanifold $L_0 \subset (M,\omega)$, we define

$$\begin{aligned}\mathscr{M} &= \mathscr{M}(\mathscr{N}(L_0)) \\ &= \left\{ (w,L) \in \mathscr{F} \times \mathscr{N}(L_0) \,\middle|\, \begin{array}{l} \bar{\partial}_J w = 0,\ w|_{\partial D^2} \subset L \text{ and } w^{-1}(w(z)) \cap \partial D^2 = \{z\} \\ \text{and } Dw(z) \neq 0 \text{ for some } z \in \partial D^2 \end{array} \right\}\end{aligned} \tag{3.4}$$

where

$$\bar{\partial}_J = \frac{1}{2}\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}\right) .$$

Although we are not going to need it in this paper, we state the following theorem to justify the terminology "multiply-covered" below, which were implicitly assumed in the previous literature [P1,2], [S] and [L]. This will be the boundary analogoue of Lemma 4.4 [Mc]. It turns out that the proof for the boundary case requires a proof which is different from and more complicated than for the interior case even in $\mathbb{C}^n$ with respect to the standard complex structure (see [O2] for details).

Theorem 3.2 [O2]. *Let (M,J) be an almost complex manifold and R be a totally real submanifold in (M,J). We assume that (M,J) does not contain any nontrivial J-holomorphic closed surface. Suppose that a J-holomorphic disc $w : (D^2, \partial D^2) \to (M,R)$ does not satisfy the condition*

$$w^{-1}(w(z)) \cap \partial D^2 = \{z\} \quad \textit{and} \quad Dw(z) \neq 0 \quad \textit{for some } z \in \partial D^2 . \tag{3.5}$$

Then there exists some J-holomorphic disc $\tilde{w}$ satisfying the condition (3.5) such that

$$w = \tilde{w} \circ b \quad \text{in } D^2$$

where $b : D^2 \to D^2$ is a finite Blaschke product whose multiplicity is greater than or equal to 2. In other words, b is of the form, with $s \geqq 2$,

$$b(z) = e^{ic} \prod_{k=1}^{s} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}$$

where c is real and α_k are complex constants.

In terms of this theorem, we will call a (J)-holomorphic disc *multiply-covered* if it does not satisfy (3.5).

Remark 3.3. (i) We remark that without the condition that (M,J) does not contain any nontrivial J-holomorphic closed surface, the above theorem is not true in general illustrated by the following example (a similar remark was made before about the proof of Lemma 5.3 [F] in the appendix of our paper [O1]): Consider the unit sphere $S^2(1)$ with the standard complex structure and let $C \subset S^2$ be a closed line segment (that is not a point) inside an equator $R \subset S^2$. Now consider a Riemann map from D^2 to $S^2 \backslash C$ such that 1 and -1 in ∂D^2 maps to each of the ends of the line segment C. One can easily check that this is a holomorphic disc with boundary in R which does not satisfy the condition (3.5) but is not of form as in the theorem, i.e, "not multiply-covered".

(ii) More generally, consider the holomorphic map

$$w : \mathbb{H} \to \mathbb{C} \qquad w(z) = z^k$$

and identify $\mathbb{C} \cup \{\infty\} = S^2$ and $\mathbb{H} \cup \{\infty\} = D^2 \subset \mathbb{C}$. The example (i) corresponds to the case $k = 2$. If k is odd at least 5, then this map satisfies the condition (3.5) but neither injective on an open dense subset of D^2 nor multiply-covered in the sense of Theorem 3.2. These examples show that the structure of the images of J-holomorphic discs is more complex than that of J-holomorphic spheres (see [O2] for further discussions on this matter).

Now, we note that since each $L \in \mathcal{N}(L_0)$ is Hamiltonian isotopic to L_0 and $L = \phi_L(L_0)$ as above, the pair $(w,L) \in \mathcal{M}$ uniquely defines the homotopy class $[\phi_L^{-1} \circ w] \in \pi_2(M,L_0)$ where we mean by ϕ_L any ambient Hamiltonian diffeomorphism that extends the map $\phi_L : L_0 \to L$. However this class does not depend on the choice of the extensions. Hence we can define, for each $A \in \pi_2(M,L_0)$,

$$\mathcal{M}_A = \mathcal{M}_A(\mathcal{N}(L_0)) = \{(w,L) \in \mathcal{M} \,|\, [\phi_L^{-1} \circ w] = A \text{ in } \pi_2(M,L_0)\}\,.$$

Proposition 3.4. *$\mathcal{M}_A$ is a Banach submanifold of $\mathcal{F}^s \times \mathcal{N}(L_0)$, which does not depend on $s > 1$ if s is sufficiently big. Indeed, if we denote $\mathcal{F}^\infty = C^\infty(D^2, M)$, then we have*

$$\mathcal{M}_A \subset \mathcal{F}^\infty \times \mathcal{N}(L_0)\,.$$

Proof. The second statement follows from the elliptic regularity and so we prove only the first statement. The general scheme of the proof is quite standard by now but our set-up has never been used before and so we give a complete proof of it.

We define for each $w \in \mathscr{F}$,

$$\mathscr{H}_w = H^s(w^*TM)$$

the space of H^s-sections of w^*TM on D^2 and

$$\mathscr{H} = \bigcup_{w\in\mathscr{F}} \mathscr{H}_w = \bigcup_{w\in\mathscr{F}} H^s(W^*TM),$$

which becomes a smooth vector bundle over $\mathscr{F}$. We also define $\Omega(M) = \Omega^{s+\frac{1}{2}}(M) = H^{s+\frac{1}{2}}(S^1;M)$ and

$$\Omega(L_0) = \Omega^{s+\frac{1}{2}}(L_0) = H^{s+\frac{1}{2}}(S^1,L_0) := H^{s+\frac{1}{2}}(S^1,M)\cap C^0(S^1,L_0)$$

which is the space of $H^{(s+\frac{1}{2})}$-maps for S^1 to L_0. Note that by the trace theorem (see e.g. [LM]), for each map $w \in H^{s+1}(D^2,M)$, its boundary map $w|_{\partial D^2}$ lies in $H^{s+\frac{1}{2}}(S^1,M)$. Now, we define a smooth map

$$\Delta : \mathscr{F} \times \mathscr{N}(L_0) \to \mathscr{H} \times \Omega(M)$$

by

$$\Delta(w,L) = (\bar\partial_J w, \phi_L^{-1}\circ w|_{\partial D^2}).$$

Then we have

$$\mathscr{M}_A = \Delta^{-1}(\{0\}\times\Omega_A(L_0))$$

where

$$\Omega_A(L_0) = \{\alpha\in\Omega^{s+\frac{1}{2}}(L_0)\,|\,\alpha = w|_{\partial D^2} \text{ for some } w\in\mathscr{F} \text{ with } [w]=A\}.$$

Therefore to prove the proposition, it is enough to prove that the map $\Delta : \mathscr{F}\times\mathscr{N}(L_0)\to\mathscr{H}\times\Omega(M)$ is transverse to the submanifold

$$\{0\}\times\Omega_A(L_0)\subset\mathscr{H}\times\Omega(M).$$

Let $(w,L)\in\mathscr{M}_A$, i.e., it satisfy

$$\begin{cases} \bar\partial_J w = 0 & \text{in } D^2 \\ \phi_L^{-1}\circ w|_{\partial D^2}\subset L_0 & \\ [\phi_L^{-1}\circ w] = A & \text{in } \pi_2(M,L). \end{cases} \tag{3.6}$$

To prove the transversality, we need to show

$$\begin{aligned} \operatorname{Im}(T_{(w,L)}\Delta) + \{0\}\oplus T_{\phi^{-1}\circ w|_{\partial D^2}}(\Omega_A(L_0)) &= T_{(0,\phi_L^{-1}\circ w|_{\partial D^2})}(\mathscr{H}\times\Omega(M)) \\ &= \mathscr{H}_w\oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(M). \end{aligned}\tag{3.7}$$

Now, we note that each function $f \in C_0^\infty(L_0)$ gives rise to a function $f \circ \phi_L^{-1} \in C^\infty(L)$ and so an element $X_{f\circ\phi_L^{-1}} \in T_L\mathcal{N}(L_0)$ and vice versa (since L is contained in the cotangent coordinate neighborhood of L_0). Because of this, we will represent an element in $T_L\mathcal{N}(L_0)$ with $X_{f\circ\phi_L^{-1}}$ for a function f in $C_0^\infty(L_0)$. If $(\xi, X_{f\circ\phi_L^{-1}}) \in T_{(w,L)}(\mathcal{F} \times \mathcal{N}(L_0))$, then a straightforward computation shows

$$T_{(w,L_0)}\Delta(\xi, X_{f\circ\phi_L^{-1}}) = (\overline{\nabla}_J\xi, X_f(\phi_L^{-1} \circ w|_{\partial D^2}) - \xi(\phi_L^{-1} \circ w|_{\partial D^2}))$$
$$\in \mathcal{H}_w \oplus T_{(0,\phi_L^{-1}\circ w|_{\partial D^2})}\Omega(M) \tag{3.8}$$

where ∇ is the Hermitian connection on TM and

$$\overline{\nabla}_J := \frac{1}{2}\left(\frac{D}{dx} + J\frac{D}{dy}\right) .$$

Note that since we assume (M, ω, J) is Kähler, the Hermitian connection coincides with the Levi-Civita connection. We denote $E = E_{(w,L)} := T_{(w,L)}\Delta$ and its adjoint by

$$E^* = (\mathcal{H}_w \oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(M))^* \to (T_{(w,L)}(\mathcal{F} \times \mathcal{N}(L_0)))^* .$$

Since the L^2-inner product gives an isomorphism between $\mathcal{H}_w^* = (H^s(w^*TM))^*$ and $H^{-s}(w^*TM)$ and between $(T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega^{s+\frac{1}{2}}(M))^*$ and $T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega^{-(s+\frac{1}{2})}(M)$, we define the L^2-adjoint of E, denoted by E^+,

$$E^+ : H^{-s}(w^*TM) \times T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega^{-(s+\frac{1}{2})}(M) \to (T_{(w,L)}(\mathcal{F} \times \mathcal{N}(L_0)))^*$$

by the composition of E^* and the above isomorphism. Now, to show (3.7), it is enough to prove

$$(\mathrm{Im}\, E_{(w,L)} + \{0\} \oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp = \{0\} , \tag{3.9}$$

where $(\cdot)^\perp$ is the L^2-orthogonal complement of $(\cdot)$ in $H^{-s}(w^*TM) \oplus T_{\phi_L^{-1}\circ|_{\partial D^2}}\Omega^{-(s+\frac{1}{2})}(M)$. On the other hand, we note that

$$(\mathrm{Im}\, E_{(w,L)} + \{0\} \oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp = (\mathrm{Im}\, E_{(w,L)})^\perp \cap (\{0\} \oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp$$
$$= (\mathrm{Im}\, E_{(w,L)})^\perp \cap \{H^{-s}(w^*TM) \oplus (T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp\} . \tag{3.10}$$

Therefore, if $(\gamma, \alpha) \in (\mathrm{Im}\, E_{(w,L)} + \{0\} \oplus T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp$, then we in particular have $\alpha \in (T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp$ i.e.

$$\alpha(\theta) = N_{\phi_L^{-1}\circ w(\theta)}L_0, \quad \theta \in \partial D^2 \tag{3.11}$$

where NL_0 is the normal bundle of L_0 in M. Since we will not lose any generality for the following discussions, we will assume from now on that $\phi_L \equiv \mathrm{id}$, i.e. $L = L_0$ for the simplicity of the exposition. Now, we characterize the element $E^+(\gamma,\alpha)$, for each $(\gamma,\alpha) \in H^s(w^*TM) \times T_{w|_{\partial D^2}}\Omega^{-(s+\frac{1}{2})}(M)$, by the equation

$$\langle\langle E^+(\gamma,\alpha),(\xi,X_f)\rangle\rangle = \int_{D^2}(\overline{\nabla}_J\xi,\gamma) + \int_{\partial D^2}(X_f(w|_{\partial D^2}) - \xi|_{\partial D^2},\alpha)$$

where $\langle\langle\cdot\,,\cdot\rangle\rangle$ is the natural pairing between $T_{(w,L)}(\mathscr{F}\times\mathscr{N}(L_0))$ and its dual. If $(\gamma,\alpha) \in (\mathrm{Im}\,E_{(w,L)} + \{0\} \oplus (T_{\phi_L^{-1}\circ w|_{\partial D^2}}\Omega(L_0))^\perp$, then we have in particular $E^+(\gamma,\alpha) = 0$ from (3.10), and so we are given

$$\int_{D^2}(\overline{\nabla}_J\xi,\gamma) + \int_{\partial D^2}(X_f(w|_{\partial D^2}) - \xi|_{\partial D^2},\alpha) = 0 \tag{3.12}$$

for all $(\xi,f) \in H^{s+1}(w^*TM) \times C^{\bar\varepsilon}(L_0)$. Since γ is smooth by elliptic regularity, we can integrate by parts to get

$$\begin{aligned} 0 &= -\int_{D^2}(\xi,\nabla_J\gamma) + \int_{\partial D^2}(\xi,e^{-J\theta}\gamma)\,d\theta + \int_{\partial D^2}(X_f(w|_{\partial D^2}) - \xi|_{\partial D^2},\alpha) \\ &= -\int_{D^2}(\xi,\nabla_J\gamma) + \int_{\partial D^2}(\xi,-\alpha + e^{-J\theta}\gamma)\,d\theta + \int_{\partial D^2}(X_f\circ w|_{\partial D^2},\alpha)\,d\theta \end{aligned}$$

for all ξ and f where $\nabla_J := \frac{1}{2}(\frac{D}{dx} - J\frac{D}{dy})$. Therefore, we have proven that (γ,α) satisfies

$$\begin{cases} \nabla_J\gamma = 0 & \text{in } D^2 \\ -\alpha + e^{-J\theta}\gamma = 0 & \text{on } \partial D^2 \\ \alpha^\perp = 0 & \text{on some open subset } \Lambda \subset \partial D^2 \end{cases}$$

where $\alpha^\perp$ is the normal component of α to L_0. The third equation follows because of set of $X_f(x)$'s span N_xL_0 for each $x = w(z) \in L_0$ at which w satisfies the condition (3.5) (Recall the definition of $\mathscr{M}$ in (3.4)). However from (3.11), this implies

$$\alpha = 0 \quad \text{on } \Lambda \subset \partial D^2\,. \tag{3.13}$$

Then the second equation and (3.13) imply that $\gamma|_{\partial D^2} = 0$ on $\Lambda \subset \partial D^2$. This together with the equation $\nabla_J\gamma = 0$ implies

$$\gamma \equiv 0 \quad \text{in } D^2 \tag{3.14}$$

by the unique continuation theorem [Ar] adapted for J-holomorphic maps (see Theorem 2.1) [O2]). Then back from the second equation and (3.14), we have

$$\alpha = 0 \quad \text{on } \partial D^2\,. \tag{3.15}$$

Combining (3.14) and (3.15), we have proved $(\gamma, \alpha) = 0$ which proves (3.9) and so (3.7). Hence the proof. □

Now, it is straightforward to check from the ellipticity of the $\bar{\partial}_J$-equation that the projection map

$$\Pi_A = \Pi_{A,L_0} : \mathscr{M}_A(\mathscr{N}(L_0)) \to \mathscr{N}(L_0)$$

is Fredholm. Then by the Sard–Smale theorem, there is a subset of the second category $\mathscr{N}(L_0)^A_{\text{reg}} \subset \mathscr{N}(L_0)$ which consists of regular elements. For these $L \in \mathscr{N}(L_0)$, the inverse image $\Pi_A^{-1}(L)$ becomes a smooth manifold. Now let $\Lambda^A_{L_0}(M,\omega)_{\text{reg}} \subset \Lambda_{L_0}(M,\omega)$ be the union of all $\mathscr{N}(L)^A_{\text{reg}}$ for $L \in \Lambda_{L_0}(M,\omega)$. Then the intersection $\Lambda_{L_0}(M,\omega)_{\text{reg}} := \bigcap_{A\in\pi_2(M,L_0)}\{\Lambda^A_{L_0}(M,\omega)_{\text{reg}}\}$ is a dense subset of $\Lambda_{L_0}(M,\omega)$ as well as each $\Lambda^A_{L_0}(M,\omega)$ is dense in $\Lambda_{L_0}(M,\omega)$. Now the standard index formula [G] implies

$$\text{Index}\, \Pi_A = \mu_{L_0}(A) + n\,.$$

Hence we have finally finished the proof of Theorem I.

Theorem 3.5. *Let (M,ω,J) be a Kähler manifold and let $L \subset (M,\omega)$ be a compact Lagrangian submanifold. Then there exists a dense subset of $\phi \in (\mathscr{D}_w)^L_{\text{reg}} \subset \mathscr{D}_w$ such that any (not multiply-covered) holomorphic disc*

$$w : (D^2, \partial D^2) \to (M, \phi(L))$$

is Fredholm regular.

Proof. We define

$$(\mathscr{D}_\omega)^L_{\text{reg}} := \{\phi \in \mathscr{D}_\omega | \phi(L) \in \Lambda_L(M,\omega)_{\text{reg}}\}$$

and then the denseness of $(\mathscr{D}_\omega)^L_{\text{reg}}$ in $\mathscr{D}_\omega$ follows from that $\Lambda_L(M,\omega)_{\text{reg}}$ is dense in $\Lambda_L(M,\omega)$. □

Remark 3.6. (i) One can check, by carefully looking at the above proof, that Theorem 3.5 holds even for general calibrations (M,ω,J) for which J is not necessarily integrable. The only difference from the integrable case is that the linearization operator $E_w = D\bar{\partial}_J(w)$ which is the first component of $T_{(w,L_0)}\Delta$, will contain a torsion term in addition to $\overline{\nabla}_J$ and so its L^2-adjoint will be a sum of ∇_J and some zero order operator. But this does not keep us from applying the unique continuation theorem [O2] in the last step like in (3.13). We emphasize here that the boundary condition does not change at all from the integrable case.

(ii) One can also check that the above proof can be easily modified to deal with the Fredholm theory under the Lagrangian isotopies. For this case, we replace the isodrast $\Lambda_L = \Lambda_L(M,\omega)$ by $\Lambda = \Lambda(M,\omega)$, $B^1(L)$ by $Z^1(L)$ and give $Z^1(L)$ a Floer norm that is similar to the one used n (3.3) for $C^\infty(L)$ and so for $B^1(L)$. The remaining argument goes through with obvious modifications.

By applying the reasoning in Remark 3.6 (ii) to each connected component of $\Lambda(M,\omega)$, we have the following theorem.

Theorem 3.7. *Let (M,ω,J) be a Kähler manifold. Then there exists a dense subset $\Lambda_{\text{reg}}(M,\omega)$ of $\Lambda(M,\omega)$ such that for $L \in \Lambda_{\text{reg}}(M,\omega)$, any (not multiply covered) holomorphic disc*

$$w : (D^2, \partial D^2) \to (M,L)$$

is Fredholm-regular.

4 Fredholm theory of holomorphic discs with totally real boundary conditions

The proof of Theorem I in Sect. 3 can be equally carried out for the totally real boundary condition once we set up the framework and modify the proof appropriately, which turns out to be easier to deal with than the case in Sect. 3. In this section, we explain the necessary modifications to deal with totally real boundary conditions.

Let (M,J) be a complex manifold. We note that the set of totally real embeddings is an open subset of all smooth embeddings and so the local structure of the set of totally real embeddings, denoted by $\mathscr{T}(M)$, is the same as the set of all smooth embeddings. For a given totally real embedding $R_0 \subset (M,J)$, we denote

$$\mathscr{T}_{R_0}(M) = \{R \in \mathscr{T}(M) \,|\, R \text{ isotopic to } R_0 \text{ through totally real embeddings}\}\,.$$

We denote by NR the normal bundle of R in M with respect any fixed metric. Then the tangent space to $\mathscr{T}_{R_0}(M)$ (as well as the local model) at a given point $R \in \mathscr{T}_{R_0}(M)$ is isomorphic to the space $\Lambda(NR)$ of all smooth sections of NR. As in Sect. 3, we define a Floer norm on $\Lambda(NR)$ by

$$\|X\|_{\bar{\varepsilon}} = \sum_{k\in\mathbb{N}} \varepsilon_k \max_{x\in R} |D^k X(x)|$$

and

$$\Lambda^{\bar{\varepsilon}}(NR) = \{x \in \Lambda(NR) \,|\, \|\dot{X}\|_{\bar{\varepsilon}} < \infty\}$$

$$\Lambda^{\bar{\varepsilon}}_r(NR) = \{x \in \Lambda(NR) \,|\, \|X\|_{\bar{\varepsilon}} < r\}\,,$$

We denote by $\exp_R : NR \to M$ the exponential map on NR and then it defines a diffeomorphism

$$\exp_R : U_R \subset NR \to V_R \subset M$$

where U_R and V_R are open neighborhoods of the zero section of NR and R in M respectively. As before, $\Lambda^{\bar{\varepsilon}}(NR)$ becomes a Banach space and one can choose $\bar{\varepsilon}$ so that $\Lambda^{\bar{\varepsilon}}(NR)$ is dense in $\Lambda(NR)$ with respect to L^2-norm on $\Lambda(NR)$. We choose $r > 0$ small enough so that for $\|X\|_{\bar{\varepsilon}} < r$, the induced map $\widetilde{\exp}_R : \Lambda^{\bar{\varepsilon}}_r(NR) \to \mathscr{T}_{R_0}(M)$ defined by

$$\widetilde{\exp}_R(X) = (\exp_R \circ X)(R)$$

is injective. Now we denote by $\mathcal{N}(R)$ the image of $\Lambda_r^{\bar{\varepsilon}}$ under the map $\widetilde{\exp}_R$, which is certainly a Banach manifold.

Now we fix sufficiently large $s > 1$ and define

$$\mathcal{F} = \mathcal{F}^s = H^{s+1}(D^2, M)$$

and

$$\mathcal{M} = \mathcal{M}(\mathcal{N}(R_0)) = \left\{ (w, R) \in \mathcal{F} \times \mathcal{N}(R_0) \;\middle|\; \begin{array}{l} \bar{\partial}_J w = 0,\ w|_{\partial D^2} \subset R \\ \text{and } w \text{ satisfies } (3.5) \end{array} \right\}.$$

Since by definition $R \in \mathcal{N}(R_0)$ is isotopic to R_0 through totally real embeddings, the pair $(w, R) \in \mathcal{M}$ uniquely defines the homotopy class $[(w, R)]$ in $\pi_2(M, R_0)$. Now we define as before

$$\mathcal{M}_A = \mathcal{M}_A(\mathcal{N}(R_0)) = \{(w, R) \in \mathcal{M} \mid [(w, R)] = A \text{ in } \pi_2(M, R_0)\} .$$

Once we have set up these framework, the proof of Proposition 3.4 with minor modifications gives the following proposition. (See Remark 3.6 (i)).

Proposition 4.1. *$\mathcal{M}_A$ is a Banach submanifold of $\mathcal{F}^s \times \mathcal{N}(R_0)$, which does not depend on $s > 1$ if s is sufficiently big. Indeed, we have*

$$\mathcal{M}_A \subset \mathcal{F}^\infty \times \mathcal{N}(R_0) .$$

Again by applying the Sard–Smale theorem, we immediately get the following theorem as in Sect. 3.

Theorem 4.2. *Let (M, J) be a complex manifold and R_0 be a given totally real submanifold. Then there exists a dense subset $\mathcal{T}_{R_0,\mathrm{reg}}(M) \subset \mathcal{T}_{R_0}(M)$ such that if $R \in \mathcal{T}_{R_0,\mathrm{reg}}(M)$, any J-holomorphic disc (satisfying (3.5)) $w : (D^2, \partial D^2) \to (M, R)$ is Fredholm-regular.*

Again, the integrability of the complex structure can be dropped in the hypotheses as in Remark 3.6 (i).

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