Werk

Titel: On the partial Euler-Poincare characteristics of certain systems of parameters in...

Autor: Cuong, Nguyen Tu; Khoi, Vu The

Jahr: 1996

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0222|log35

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Z. 222, 383–390 (1996)

Mathematische Zeitschrift
© Springer-Verlag 1996

On the partial Euler–Poincare characteristics of certain systems of parameters in local rings

Nguyen Tu Cuong, Vu The Khoi

Institute of Mathematics, P.O. Box 631 Bo Ho, 10.000 Hanoi, Vietnam
e-mail: ntcuong@thevinh.ac.vn

Received 20 June 1994; in final form 1 December 1994

1 Introduction

Let $(A, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$ and M a finitely generated A-module with $\dim M = d$. Let $x = \{x_1, \ldots, x_d\}$ be a system of parameters (abbr. s.o.p) of M. Denote by $H_i(x; M)$ the i-th Koszul homology module of M with respect to the s.o.p x. Following Serre [9], Appendice II, the partial Euler–Poincare characteristic of M with respect to x is defined by

$$\chi_k(x; M) = \sum_{i \geq k} (-1)^{i-k} l(H_i(x; M))$$

for $k = 0, \ldots, d$, where $l(N)$ will be denoted for the length of the A-module N. Let $n = (n_1, \ldots, n_d)$ be a d-tuple of positive integers. We denote by $x^{(n)}$ the s.o.p $\{x_1^{n_1}, \ldots, x_d^{n_d}\}$ of M. Then we can consider $\chi_k(x^{(n)}; M)$ as a function in n. This function in general is not a polynomial (cf. [6]). But, in [5] Garcia Roig has shown that for $n_1 = \cdots = n_d = t$ the function $\chi_k(x^{(t)}; M)$ of variable t is bounded above by polynomials in t and that the least degree of all polynomials bounding above this function is independent of the choice of the s.o.p x. The first author has shown in [3] that the above statement is still true for the case of d variables $n_1, \ldots, n_d$ when $k = 1 (\chi_1(x^{(n)}; M)$ is denoted in [3] by $I_M(n; x))$. And he proved in [2] that the least degree of all polynomials bounding above $\chi_1(x^{(n)}; M)$ is exactly equal to $\dim A/(\mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M))$ if A admits a dualizing complex, where $\mathfrak{a}_i(M)$ is the annihilator of the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ of M with respect to the maximal ideal $\mathfrak{m}$.

The purpose of this paper is to generalize the above results. Namely, we will show in Sect. 3 that the least degree of all polynomials in n bounding

The first author is partially supported by the National Basic Research Program, Vietnam, and, during the completion of this paper, by the Mathematics Section of the international Centre for Theoretical Physics, Italy

above $\chi_k(x^{(n)};M)$ is independent of the choice of x for $k = 0,\dots,d$. Let us denote this invariant by $p_k(M)$. Then our main result is the following theorem.

Theorem *Let M be a finitely generated A-module of dimension d. Suppose that A admits a dualizing complex. Then*

$$p_k(M) = \dim A/(\mathfrak{a}_0(M)\cdots\mathfrak{a}_{d-k}(M))$$

for $k = 0,\dots,d$.

This theorem will be proved in Sect. 4. In Sect. 2 we will show a generalization of the formula for multiplicity of Auslander–Buchsbaum [1] which plays an important role in the proof of the theorem.

2 A formula for multiplicity

Throughout this paper we always denote by $(A,\mathfrak{m})$ a local commutative Noetherian ring with the maximal ideal $\mathfrak{m}$ and by M a finitely generated A-module of dimension d.

Let $x = (x_1,\dots,x_t)$ be a multiplicity system of M. We denote by $e(x;M)$ the Serre-multiplicity of M with respect to x. First of all, we have the following lemma.

Lemma 2.1 *Let $x = (x_1,\dots,x_t)$ be a multiplicity system of M. Let j,k be two positive integers such that $t \geqq j \geqq k > 0$. Then*

$$e(x_{j+1},\dots,x_t;H_k(x_1,\dots,x_j;M)) = \sum_{i=k}^{j} e(x_{i+1},\dots,x_t;(0:x_i)_{H_{k-1}(x_1,\dots,x_{i-1};M)}) + \sum_{i=k+1}^{j} e(x_{i+1},\dots,x_t;(0:x_i)_{H_k(x_1,\dots,x_{i-1};M)}),$$

where we set $e(x_{i+1},\dots,x_t;(0:x_i)_{H_k(x_1,\dots,x_{i-1};M)}) = l((0:x_i)_{H_k(x_1,\dots,x_{i-1};M)})$ if $i = t$.

Proof. We prove by induction on j. It is easy to see for the case $j = 1$. Suppose that $j > 1$. From the exact sequence of Koszul homology modules

$$0 \to H_k(x_1,\dots,x_{j-1};M)/x_jH_k(x_1,\dots,x_{j-1};M) \to H_k(x_1,\dots,x_j;M) \to (0:x_j)_{H_{k-1}(x_1,\dots,x_{j-1};M)} \to 0$$

and the additive property of the multiplicity, we get

$$\begin{aligned} &e(x_{j+1},\dots,x_t;H_k(x_1,\dots,x_j;M)) \\ &\quad = e(x_{j+1},\dots,x_t;(0:x_j)_{H_{k-1}(x_1,\dots,x_{j-1};M)}) \\ &\qquad + e(x_{j+1},\dots,x_t;H_k(x_1,\dots,x_{j-1};M)/x_jH_k(x_1,\dots,x_{j-1};M)) \\ &\quad = e(x_{j+1},\dots,x_t;(0:x_j)_{H_{k-1}(x_1,\dots,x_{j-1};M)}) \\ &\qquad + e(x_{j+1},\dots,x_t;(0:x_j)_{H_k(x_1,\dots,x_{j-1};M)}) \\ &\qquad + e(x_{j+1},\dots,x_t;H_k(x_1,\dots,x_{j-1};M)). \end{aligned}$$

Thus the lemma easily follows from the inductive hypothesis.

As an immediate consequence of Lemma 2.1 we have the following corollary which will be often used in next sections.

Corollary 2.2 *Let $x = (x_1,\dots,x_t)$ be a multiplicity system of M. Then*

$$\chi_k(x;M) = \sum_{i=k}^{t} e(x_{i+1},\dots,x_t;(0 : x_i)_{H_{k-1}(x_1,\dots,x_{i-1};M)}) .$$

Corollary 2.2 has some interesting consequences. First, since the multiplicity is a non-negative integer, we obtain again the following well-known result of Serre (see [9], Appendice II).

Corollary 2.3 *Let $x = (x_1,\dots,x_t)$ be a multiplicity system of M. Then $\chi_k(x;M) \geqq 0$ for all $k \geqq 0$.*

Note that $\chi_0(x;M) = e(x;M)$ (see [1] or [9]) it follows that

$$\chi_1(x;M) = l(H_0(x;M)) - \chi_0(x;M) = l(M/(x)M) - e(x;M) .$$

Therefore Corollary 2.2 leads to the following formula for the multiplicity (see [1], Corollary 4.3).

Corollary 2.4 *Let $x = (x_1,\dots,x_t)$ be a multiplicity system of M. Then*

$$e(x;M) = l(M/(x)M) - \sum_{i=1}^{t} e(x_{i+1},\dots,x_t;(x_1,\dots,x_{i-1})M : x_i/(x_1,\dots,x_{i-1})M) .$$

3 The invariant $p_k(M)$

Keep all notations in Sects. 1 and 2. We begin with the following lemmas.

Lemma 3.1 (see [5], Lemma 2). *Let $a \in A$ and let t be a positive integer. Then*

(i) $l(0 : a^t)_M \leqq tl(0 : a)_M$;
(ii) $l(0 : a^t)_M \leqq l(0 : a^{t'})_M$ *for all* $t' \geqq t$.

Proof. Straightforward.

Lemma 3.2 *Let $x = \{x_1,\dots,x_d\}$ be a s.o.p of M and $n = (n_1,\dots,n_d)$ a d-tuple of positive integers. Then, for $k = 0,\dots,d$, it holds*

(i) $\chi_k(x_1^{n_1},\dots,x_d^{n_d};M) \leqq n_1,\dots,n_d\chi_k(x;M)$;
(ii) $\chi_k(x_1^{n_1},\dots,x_d^{n_d};M) \leqq \chi_k(x_1^{t_1},\dots,x_d^{t_d};M)$ *for all* $t_1 \geqq n_1,\dots,t_d \geqq n_d$.

Proof. (i) Since $\chi_0(x;M) = e(x;M)$ the lemma is clear for $k = 0$. Suppose that $k \geqq 1$. Note that the partial Euler–Poincare characteristic is independent of the order of the sequence $x_1,\dots,x_d$ therefore it suffices to show that

$\chi_k(x_1,\ldots,x_{d-1},x_d^{n_d};M) \leqq n_d\chi_k(x;M)$. Thus by Corollary 2.2 we only have to verify that

$$l(0:x_d^{n_d})_{H_{k-1}(x_1,\ldots,x_{d-1};M)} \leqq n_d l(0:x_d)_{H_{k-1}(x_1,\ldots,x_{d-1};M)}.$$

But, this is clear by Lemma 3. We also get (ii) by the same method.

An immediate consequence of Lemma 3.2 is the following corollary.

Corollary 3.3 *Keep x and n as in Lemma* 3.2. *Then, for $k = 0,\ldots,d$, it holds*

(i) $l(H_k(x_1^{n_1},\ldots,x_d^{n_d};M)) \leqq n_1,\ldots,n_d l(H_k(x;M))$;

(ii) $l(H_k(x_1^{n_1},\ldots,x_d^{n_d};M)) \leqq l(H_k(x_1^{t_1},\ldots,x_d^{t_d};M))$

for all $t_1 \geqq n_1,\ldots,t_d \geqq n_d$.

Lemma 3.2 shows that if we consider $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$ as a function in d variables $n_1,\ldots,n_d$ then this function is bounded above by the polynomial $n_1,\ldots,n_d\chi_k(x;M)$. Moreover, we have

Proposition 3.4 *Let $x = \{x_1,\ldots,x_d\}$ be a s.o.p of M. Then the least degree of all polynomials in $n_1,\ldots,n_d$ bounding above the function $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$ is independent of the choice of x for all $k = 0,\ldots,d$. We will denote this invariant by $p_k(M)$.*

Proof. Let t be a positive integer. By [5], Theorem 6, the least degree of all polynomials in t bounding above $\chi_k(x_1^t,\ldots,x_d^t;M)$ is independent of the choice of x. Denote by $p'_k(M)$ this invariant and by $p_k(x;M)$ the least degree of all polynomials in $n_1,\ldots,n_d$ bounding above $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$. Then is clear that $p'_k(M) \leqq p_k(x;M)$. On the other hand, by Lemma 3.2 we get

$$\chi_k(x_1^t,\ldots,x_d^t;M) \geqq \chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$$

for all $t \geqq \max\{n_1,\ldots,n_d\}$; it follows that $p'_k(M) \geqq p_k(x;M)$. Thus $p_k(x;M) = p'_k(M)$. So the proposition is proved.

Applying Corollary 3.3 we also have the following corollary by the same method used in the proof of the proposition 3.4.

Corollary 3.5 *Let $x = \{x_1,\ldots,x_d\}$ be a s.o.p of M. Then the least degree of all polynomials in $n_1,\ldots,n_d$ bounding above the function $l(H_k(x_1^{n_1},\ldots,x_d^{n_d};M))$ is independent of the choice of x for $k = 0,\ldots,d$. We will denote this invariant by $q_k(M)$.*

Remark 3.6 (i) Note that $\chi_0(x;M) = e(x;M)$ and $\chi_d(x;M) = l(H_d(x;M)) = l(0:x)_M$. Hence $p_0(M) = d$ and $p_d(M) \leqq 0$.

(ii) We will take $-\infty$ for the degree of the zero-polynomial then $d-1 \geqq p_k(M) \geqq -\infty$ for $k = 1,\ldots,d-1$.

Proof. $p_1(M)$ is just the polynomial type of M defined in [3] therefore $d-1 \geqq p_1(M)$ by 2.5, (ii) of [3]. Moreover, since $\chi_k(x;M) + \chi_{k+1}(x;M) = l(H_k(x;M))$ and $q_k(M) \leqq d-k$ by Corollary 7 of [5], we can easily obtain the above inequalities by induction on k.

(iii) Let $\hat{M}$ be the m-adic completion of M. Since every s.o.p of M is also a s.o.p of $\hat{M}$ and $H_k(x;M) \cong H_k(x;\hat{M})$ it follows that $p_k(M) = p_k(\hat{M})$ for $k = 0,\ldots,d$.

4 The main result

In this section we always assume that the ring A admits a dualizing complex. The basic properties of dualizing complex used in this section are from [8].

First, we need some definitions and notions as follows.

Let k be a positive integer. A sequence of elements $x_1,\ldots,x_j$ of A is called *k-reducing sequence* of M if the following conditions hold:

Either $j < k$ or $x_i \notin P$ for all $P \in \mathrm{Ass}(H_{k-1}(x_1,\ldots,x_{i-1};M))$ with $\dim A/P \geqq d - i$ for all $i = k,\ldots,j$.

Now, we define

$r_k(M) = \min\{j$: every subset of a s.o.p. of M having $(d-j-1)$-elements is a k-reducing sequence of $M\}$.

For the case $k = 0$ we set $r_0(M) = d$.

Remark 4.1 (i) For the case $k = 1$ the notion 1-reducing sequence is just the reducing sequence defined in [2] which is a generalization of the reducing s.o.p of Auslander–Buchsbaum in [1].

(ii) It is not difficult to verify that $x_i \notin P$ for all $P \in \mathrm{Ass}(H_{k-1}(x_1, \ldots,x_{i-1};M))$ with $\dim A/P \geqq d - i$ if and only if $\dim (0:x_i)_{H_{k-1}(x_1,\ldots,x_{i-1};M)} < d - i$. Thus, in the case $j \geqq k$, a sequence $x_1,\ldots,x_j$, which is a subset of a s.o.p $x = \{x_1,\ldots,x_d\}$ of M, is a k-reducing sequence if and only if $e(x_{i+1},\ldots,x_d;(0:x_i)_{H_{k-1}(x_1,\ldots,x_{i-1};M)}) = 0$ for all $i = k,\ldots,j$.

(iii) A sequence $x_1,\ldots,x_j$ of A with $j < k$ is always a k-reducing sequence. Therefore $d - k \geqq r_k(M)$.

For short we set $\mathfrak{b}_k(M) = \mathfrak{a}_0(M)\cdots\mathfrak{a}_k(M)$, $k = 0,\ldots,d$. Recall that $\mathfrak{a}_i(M)$ is the annihilator of the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ of M. Now we are able to prove the main theorem announced in the introduction.

Theorem 4.2 *Suppose that A admits a dualizing complex then*

$$p_k(M) = r_k(M) = \dim A/\mathfrak{b}_{d-k}(M)$$

for $k = 0,\ldots,d$. Here we stipulate that the zero-module is of dimension $-\infty$.

Proof. For $k = 0$ then $p_0(M) = d$ by 3.6, (i). Since $\mathfrak{b}_d(M) \subseteq \mathrm{Ann}\, M$ by [8], Lemma 2.4.4, the theorem is true in this case. Assume now that $d \geqq k > 0$. We will prove that

$$p_k(M) \geqq r_k(M) \geqq \dim A/\mathfrak{b}_{d-k}(M) \geqq p_k(M)$$

for $k = 1,\ldots,d$.

Claim 1 $p_k(M) \geqq r_k(M)$.

Proof. Let $p_k(M) = j$. If $d - j - 1 < k$, then the inequality is true by 4.1, (iii). Therefore we may assume that $d - j - 1 \geqq k$. Let $x = \{x_1, \ldots, x_d\}$ be an arbitrary s.o.p of M. Using Corollary 2.2 we have

$$\chi_k(x; M) = \sum_{i=k}^{d} e(x_{i+1}, \ldots, x_d; (0 : x_i)_{H_{k-1}(x_1, \ldots, x_{i-1}; M)}) .$$

Since this function is bounded above by a polynomial in $n_1, \ldots, n_d$ of degree j we derive that $e(x_{i+1}, \ldots, x_d; (0 : x_i)_{H_{k-1}(x_1, \ldots, x_{i-1}; M)}) = 0$ for $i = k, \ldots, d - j - 1$. Thus $x_1, \ldots, x_j$ is a k-reducing sequence by 4.1, (ii). The claim follows.

To prove the inequality $r_k(M) \geqq \dim A/\mathfrak{b}_{d-k}(M)$ we need the following lemma.

Lemma 4.3 *Let $P \in \operatorname{Supp} M$ and let k be an integer such that $d \geqq k \geqq 0$. The following conditions are equivalent:*

(i) $\mathfrak{b}_k(M) \subseteq P$;
(ii) $\operatorname{depth} M_p + \dim A/P < k + 1$.

Proof. Denote by $D^{\cdot}$ the dualizing complex of A. Then it is easy to see that the condition (i) is equivalent to

(a) $\mathfrak{a}_i(M) \subseteq P$ for some $i \in \{0, \ldots, k\}$.

Applying Lemma 2.2.2 of [8], (a) is equivalent to

(b) $P \in \operatorname{Supp}(H^{-i}(\operatorname{Hom}_A(M; D^{\cdot})))$ for some $i \in \{0, \ldots, k\}$.

By Lemma 2.2.3 of [8], (b) is equivalent to

(c) $H^{i-\dim A/P}_{PA_P}(M_P) \neq 0$ for some $i \in \{0, \ldots, k\}$.

The last condition is clearly equivalent to (ii) as required.

Claim 2 $r_k(M) \geqq \dim A/\mathfrak{b}_{d-k}(M)$.

Proof. Set $r_k(M) = r$. By virtue of Lemma 4.3 it is enough to show that for all $P \in \operatorname{Supp} M$ with $\dim A/P > r$ then $\operatorname{depth} M_P + \dim A/P > d - k$. In fact, let $\dim A/P = d - j$. If $j < k$ then $\dim A/P = d - j > d - k$; the claim is proved. Assume that $j \geqq k > 0$. It is easy to verify that there is a sequence of elements $x_1, \ldots, x_j$ contained in P which forms a subset of a s.o.p of M. We denote again by x_i the image of x_i in A_P then $x_1, \ldots, x_j$ is a multiplicity system of M_P. Since $d - j > r$, $d - r - 1 \geqq j$. Therefore it follows from hypothesis that $x_1, \ldots, x_j$ is an k-reducing sequence of M. Thus, since $\operatorname{Ass}(H_{k-1}(x_1, \ldots, x_{i-1}; M_P)) = \operatorname{Ass}(H_{k-1}(x_1, \ldots, x_{i-1}; M)) \bigcap \operatorname{Supp} M_P$, we get $x_i \notin QA_P$ for all prime ideals $Q \subseteq P$ and $QA_P \in \operatorname{Ass}(H_{k-1}(x_1, \ldots, x_{i-1}; M_P))$ with $\dim A/Q \geqq d - i$ for $i = k, \ldots, j$. Furthermore, since A is catenar, $\dim A/Q \geqq d - i$ if and only if $\dim A_P/QA_P \geqq d - i - \dim A/P = d - i - (d - j) = j - i$. Thus $x_i \notin QA_P$ for all prime ideals $Q \subseteq P$ and $QA_P \in \operatorname{Ass}(H_{k-1}(x_1, \ldots, x_{i-1}; M_P))$ with $\dim A_P/QA_P \geqq j - i$ for $i = k, \ldots, j$; hence $\dim (0 : x_i)_{H_{k-1}(x_1, \ldots, x_{i-1}; M_P)} < j - i$ for $i = k, \ldots, j$. Especially, x_j is a non-zerodivisor of $H_{k-1}(x_1, \ldots, x_{j-1}; M_P)$. Therefore by Corollary 2.2 we get

$$\chi_k(x_1, \ldots, x_j; M_P) = \sum_{i=k}^{j} e(x_{i+1}, \ldots, x_j; (0 : x_i)_{H_{k-1}(x_1, \ldots, x_{i-1}; M_P)}) = 0 .$$

By [9], Appendice II, it follows that $H_i(x_1,\ldots,x_j;M_P)=0$ for $i=k,\ldots,j$. Therefore depth $M_P > j-k$. Thus depth $M_P+\dim A/P > j-k+(d-j)=d-k$. The claim is proved.

For the proof of the last inequality we need the following lemma.

Lemma 4.4 *Let $x=\{x_1,\ldots,x_d\}$ be a s.o.p of M and k a positive integer. Denote by $\mathfrak{v}_k(x)$ the intersection of all annihilators of modules $(0:x_i^{n_i})_{H_{k-1}(x_1^{n_1},\ldots,x_{i-1}^{n_{i-1}};M)}$ for $i=1,\ldots,d$ and for all $n_1,\ldots,n_d>0$. Then $\mathfrak{b}_{d-k}(M)\subseteq\mathfrak{v}_k(x)$.*

Proof. By Theorem 2.3.1 of [8] we have $\mathfrak{b}_{d-k}(M)H_k(x_1^{n_1},\ldots,x_d^{n_d};M)=0$ for all $n_1,\ldots,n_d>0$. Therefore, from the exact sequence

$$0\to H_k(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)/x_d^{n_d}H_k(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)$$
$$\to H_k(x_1^{n_1},\ldots,x_d^{n_d};M)\to(0:x_d^{n_d})_{H_{k-1}(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)}\to 0\,,$$

we deduce that

$$\mathfrak{b}_{d-k}(M)(0:x_d^{n_d})_{H_{k-1}(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)}=0$$

and

$$\mathfrak{b}_{d-k}(M)H_k(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)\subseteq\bigcap_{n_d\geqq 1}(x_d^{n_d}H_k(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M))=0\,.$$

Continue the above procedure, eventually we get

$$\mathfrak{b}_{d-k}(M)(0:x_i^{n_i})_{H_{k-1}(x_1^{n_1},\ldots,x_{i-1}^{n_{i-1}};M)}=0$$

for $i=1,\ldots,d$ and for all $n_1,\ldots,n_d>0$ as required.

Claim 3 $\dim A/\mathfrak{b}_{d-k}(M)\geqq p_k(M)$.

Proof. Set $\dim A/\mathfrak{b}_{d-k}(M)=j$. The claim is clear for $j=d$. Let now $d>j$. Then we can choose a s.o.p $x=\{x_1,\ldots,x_d\}$ of M such that $x_{j+1},\ldots,x_d\in\mathfrak{b}_{d-k}(M)$; therefore $x_{j+1},\ldots,x_d\in\mathfrak{v}_k(x)$ by Lemma 4.4. Thus we deduce by using Corollary 2.2 that

$$\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)=\sum_{i=k}^{d}e(x_{i+1}^{n_{i+1}},\ldots,x_d^{n_d};(0:x_i^{n_i})_{H_{k-1}(x_1^{n_1},\ldots,x_{i-1}^{n_{i-1}};M)})$$
$$=l(0:x_d^{n_d})_{H_{k-1}(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)}=l(0:x_d)_{H_{k-1}(x_1^{n_1},\ldots,x_{d-1}^{n_{d-1}};M)}\,.$$

This shows that $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$ is not dependent on n_d. Because our proof is independent of the order of the sequence, it follows that $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$ is not dependent on $n_{j+1},\ldots,n_d$. Therefore by Lemma 3.2, (i), we have

$$\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)=\chi_k(x_1^{n_1},\ldots,x_j^{n_j},x_{j+1},\ldots,x_d;M)\leqq n_1,\ldots,n_j\chi_k(x;M)\,.$$

Thus $\chi_k(x_1^{n_1},\ldots,x_d^{n_d};M)$ is bounded above by a polynomial of degree j. The claim is proved. Therefore the proof of Theorem 4.2 is now complete.

Theorem 4.2 leads to following consequences.

Corollary 4.5 *Let $x = \{x_1,\dots,x_d\}$ be s.o.p of M and k a positive integer. Then the following conditions are equivalent*:

(i) $l(H^i_{\mathfrak{m}}(M)) < \infty$ *for* $i = 0,\dots,d-k$;
(ii) $\sup_n \chi_k(x_1^{n_1},\dots,x_d^{n_d};M) < \infty$;
(iii) $\sup_n \chi_i(x_1^{n_1},\dots,x_d^{n_d};M) < \infty$ *for all* $i \geqq k$.

Proof. (i) is equivalent to the fact that $\dim A/\mathfrak{b}_{d-k} = 0$ and this is equivalent to (ii) by Theorem 4.2. Since $p_k(M) \geqq p_i(M)$ for all $i \geqq k$, (ii) is equivalent to (iii).

Corollary 4.6 *Let $q_k(M)$ be the least degree of all polynomials in n bounding above the function $l(H_k(x_1^{n_1},\dots,x_d^{n_d};M))$. Then*

$$q_k(M) = r_k(M) = \dim A/\mathfrak{b}_{d-k}(M)$$

for $k = 0,\dots,d$.

Proof. Since $p_k(M) \geqq p_{k+1}(M)$, the corollary easily follows from Theorem 4.2 and the fact that

$$\chi_k(x_1^{n_1},\dots,x_d^{n_d};M) + \chi_{k+1}(x_1^{n_1},\dots,x_d^{n_d};M) = l(H_k(x_1^{n_1},\dots,x_d^{n_d};M)) .$$

References

1. Auslander, M., Buchsbaum, D.A.: Codimension and multiplicity. Ann. Math. **68**, 625–657 (1958)
2. Cuong, N.T.: On the dimension of the non-Cohen–Macaulay locus of local ring admitting dualizing complexes. Math. Proc. Cambridge Phil. Soc. **109(2)**, 479–488 (1991)
3. Cuong, N.T.: On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain systems of parameters in local rings. Nagoya Math. J. **125**, 105–114 (1992)
4. Cuong, N.T., Minh, N.D.: On the lengths of Koszul homology modules and generalized fractions. (Preprint)
5. Garcia Roig, J-L.: On polynomial bounds for the Koszul homology of certain multiplicity systems. J. London Math. Soc. **34(2)**, 411–416 (1986) .
6. Garcia Roig, J-L., Kirby, D.: On the Koszul homology modules for the powers of a multiplicity system. Mathematika **33**, 96–101 (1986)
7. Matsumura, H.: Commutative Algebra. Second Edition, London: Benjamin 1980
8. Schenzel, P.: Dualisierende Komplex in der lokalen Algebra und Buchsbaum Ringe. Lect. Notes Math., vol. 907, Berlin Heidelberg New York: Springer 1982
9. Serre, J.P.: Algebre locale multiplicites. Lect. Notes Math., vol. 11, Berlin Heidelberg New York: Springer 1965