Math. Z. 222, 305 – 327 (1996)

Mathematische Zeitschrift

Boundary value problems for elliptic integro-differential operators

Kazuaki Taira★

Institute of Mathematics, University of Tsukuba, Tsukuba 305, Japan

Received 11 November 1994; in final form 13 February 1995

Dedicated to Professor Kiyosi Itô on his 80th birthday

0 Introduction and results

This paper is a continuation of the previous note [T2] where we studied a class of degenerate boundary value problems for second-order elliptic differential operators and proved that this class of boundary value problems generates analytic semigroups both in the L^p topology and in the topology of uniform convergence. The purpose of this paper is to extend these results to the elliptic *integro-differential operator* case.

Let D be a bounded, *convex* domain of Euclidean space $\mathbf{R}^N$, with C^∞ boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact C^∞ manifold with boundary.

Let W be a second-order, *elliptic* integro-differential operator with real coefficients such that

$$\begin{aligned} Wu(x) &= Au(x) + Su(x) \\ &:= \left(\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \right) \\ &\quad + \int_{\mathbf{R}^N \setminus \{0\}} \left(u(x+z) - u(x) - \sum_{j=1}^{N} z_j \frac{\partial u}{\partial x_j}(x) \right) s(x,z) m(dz) . \end{aligned}$$

Here:

1) $a^{ij} \in C^\infty(\overline{D})$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x) \xi_i \xi_j \geqq a_0 |\xi|^2, \quad x \in D, \xi \in \mathbf{R}^N .$$

★ *Current address*: Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739, Japan

2) $b^i \in C^\infty(\overline{D})$.

3) $c \in C^\infty(\overline{D})$, and $c \leqq 0$ in D but $c \not\equiv 0$ in D.

4) $s \in C(\overline{D} \times \mathbf{R}^N)$ and $0 \leqq s \leqq 1$ in $D \times \mathbf{R}^N$, and there exist constants $C_0 > 0$ and $0 < \theta_0 < 1$ such that

$$|s(x,z) - s(y,z)| \leqq C_0|x-y|^{\theta_0}, \quad x, y \in D, \ z \in \mathbf{R}^N ,$$

and

$$s(x,z) = 0 \quad \text{if } x+z \notin \overline{D} . \tag{0.1}$$

Condition (0.1) implies that the integral operator S may be considered as an operator acting on functions u defined on the closure $\overline{D}$ (see [G-M, Chapter II, Remark 1.19]).

5) The measure $m(dz)$ is a Radon measure on $\mathbf{R}^N \setminus \{0\}$ such that

$$\int_{\{|z| \leqq 1\}} |z|^2 m(dz) + \int_{\{|z|>1\}} |z| m(dz) < \infty . \tag{0.2}$$

The operator W is called a second-order *Waldenfels operator*. The differential operator A is called a diffusion operator which describes analytically a strong Markov process with continuous paths in the interior D. The integral operator S is called a second-order Lévy operator which is supposed to correspond to the jump phenomenon in the closure $\overline{D}$ (see [B-C-P], [T1]).

Let L be a first-order, boundary condition with real coefficients such that

$$Lu(x') = \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') + \gamma(x') u(x') .$$

Here:

1) $\mu \in C^\infty(\partial D)$ and $\mu \geqq 0$ on ∂D.

2) $\gamma \in C^\infty(\partial D)$ and $\gamma \leqq 0$ on ∂D.

3) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D.

The boundary condition L is called a first-order *Ventcel' boundary condition*. The terms $\mu \partial u / \partial \mathbf{n}$ and γu of L are supposed to correspond to the reflection phenomenon and the absorption phenomenon, respectively.

Our fundamental hypothesis is the following:

$$\mu(x') - \gamma(x') > 0 \text{ on } \partial D. \tag{H}$$

The intuitive meaning of hypothesis (H) is that either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary ∂D.

The first purpose of this paper is to prove an existence and uniqueness theorem for the following nonhomogeneous boundary value problem in the framework of *Hölder spaces*:

$$\begin{cases} Wu = f & \text{in } D , \\ Lu = \varphi & \text{on } \partial D . \end{cases} \tag{$*$}$$

The crucial point is how to define a version of Hölder spaces in which problem (∗) is uniquely solvable.

We introduce a subspace of the Hölder space $C^{1+\theta}(\partial D)$, $0 < \theta < 1$, which is associated with the boundary condition L in the following way: We let

$$C_L^{1+\theta}(\partial D) = \{\varphi = \mu\varphi_1 - \gamma\varphi_2;\ \varphi_1 \in C^{1+\theta}(\partial D),\ \varphi_2 \in C^{2+\theta}(\partial D)\},$$

and define a norm

$$|\varphi|_{C_L^{1+\theta}(\partial D)} = \inf\{|\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)};\ \varphi = \mu\varphi_1 - \gamma\varphi_2\}.$$

Then it is easy to verify that the space $C_L^{1+\theta}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C_L^{1+\theta}(\partial D)}$. We remark that the space $C_L^{1+\theta}(\partial D)$ is an "interpolation space" between $C^{2+\theta}(\partial D)$ and $C^{1+\theta}(\partial D)$. More precisely, we have

$$\begin{cases} C_L^{1+\theta}(\partial D) = C^{2+\theta}(\partial D) & \text{if } \mu \equiv 0 \text{ on } \partial D, \\ C_L^{1+\theta}(\partial D) = C^{1+\theta}(\partial D) & \text{if } \mu > 0 \text{ on } \partial D. \end{cases}$$

Now we can state our existence and uniqueness theorem for problem (∗):

Theorem 1 *If hypothesis (H) is satisfied, then the mapping*

$$(W, L) : C^{2+\theta}(\overline{D}) \to C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all* $0 < \theta < \theta_0$. *In particular, for any* $f \in C^{\theta}(\overline{D})$ *and any* $\varphi \in C_L^{1+\theta}(\partial D)$, *there exists a unique solution* $u \in C^{2+\theta}(\overline{D})$ *of problem* (∗).

As an application of Theorem 1, we consider the problem of existence of Markov processes in probability theory. To do so, we let

$$M = \{x' \in \partial D; \mu(x') = 0\}.$$

Then, in view of condition (H), it follows that the boundary condition $Lu = 0$ on ∂D includes the condition $u = 0$ on M. With this fact in mind, we let

$$C_0(\overline{D}\backslash M) = \{u \in C(\overline{D}); u = 0 \text{ on } M\}.$$

The space $C_0(\overline{D}\backslash M)$ is a closed subspace of $C(\overline{D})$; hence it is a Banach space.

A strongly continuous semigroup $\{T_t\}_{t \geqq 0}$ on the space $C_0(\overline{D}\backslash M)$ is called a *Feller semigroup* on $\overline{D}\backslash M$ if it is non-negative and contractive on $C_0(\overline{D}\backslash M)$:

$$f \in C_0(\overline{D}\backslash M),\ 0 \leqq f \leqq 1 \quad \text{on } \overline{D}\backslash M \Rightarrow 0 \leqq T_t f \leqq 1 \quad \text{on } \overline{D}\backslash M.$$

It is known (see [T1, Chapter 9]) that if T_t is a Feller semigroup on $\overline{D}\backslash M$, then there exists a unique Markov transition function p_t on $\overline{D}\backslash M$ such that

$$T_t f(x) = \int_{\overline{D}\backslash M} p_t(x, dy) f(y), \quad f \in C_0(\overline{D}\backslash M),$$

and further p_t is the transition function of some *strong Markov process*.

We define a linear operator $\mathcal{W}$ from $C_0(\overline{D}\backslash M)$ into itself as follows:

(a) The domain of definition $D(\mathcal{W})$ is the set

$$D(\mathcal{W}) = \{u \in C^2(\overline{D}) \cap C_0(\overline{D}\backslash M); Wu \in C_0(\overline{D}\backslash M), Lu = 0\} .$$

(b) $\mathcal{W}u = Wu, \ u \in D(\mathcal{W})$.

The next theorem is a generalization of Theorem 4 of [T2] to the integro-differential operator case:

Theorem 2 *If hypothesis* (H) *is satisfied, then the operator* $\mathcal{W}$ *is closable in the space* $C_0(\overline{D}\backslash M)$, *and its minimal closed extension* $\overline{\mathcal{W}}$ *is the infinitesimal generator of some Feller semigroup* $\{T_t\}_{t \geqq 0}$ *on* $\overline{D}\backslash M$.

Theorem 2 asserts that there exists a Feller semigroup on $\overline{D}\backslash M$ corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space $\overline{D}\backslash M$ until it "dies" at the time when it reaches the set M where the particle is definitely absorbed (see [K, Theorem 5.2], [S, Theorem 2.2], [G-M, Chapter VIII, Theorem 3.3]).

The second purpose of this paper is to study problem $(*)$ from the point of view of analytic semigroup theory in functional analysis. The forthcoming two theorems generalize Theorems 2 and 3 of [T2] to the integro-differential operator case.

First we state a generation theorem of analytic semigroups in the L^p topology. To do so, we associate with problem $(*)$ an unbounded linear operator W_p from $L^p(D)$ into itself as follows:

(a) The domain of definition $D(W_p)$ is the set

$$D(W_p) = \{u \in H^{2,p}(D); Lu = 0\} .$$

(b) $W_p u = Wu, \ u \in D(W_p)$.

Then we can prove the following:

Theorem 3 *Let* $1 < p < \infty$. *Assume that hypothesis* (H) *is satisfied. Then we have the following*:

(i) *For every* $\varepsilon > 0$, *there exists a constant* $r_p(\varepsilon) > 0$ *such that the resolvent set of* W_p *contains the set* $\Sigma_p(\varepsilon) = \{\lambda = r^2 e^{i\vartheta}; r \geqq r_p(\varepsilon), -\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon\}$, *and that the resolvent* $(W_p - \lambda I)^{-1}$ *satisfies the estimate*

$$\|(W_p - \lambda I)^{-1}\| \leqq \frac{c_p(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma_p(\varepsilon) , \tag{0.3}$$

where $c_p(\varepsilon) > 0$ *is a constant depending on* ε.

(ii) *The operator* W_p *generates a semigroup* e^{zW_p} *on the space* $L^p(D)$ *which is analytic in the sector* $\Delta_\varepsilon = \{z = t + is; z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ *for any* $0 < \varepsilon < \pi/2$.

Secondly, we state a generation theorem of analytic semigroups in the topology of uniform convergence. We introduce a linear operator $\mathfrak{W}$ from $C_0(\overline{D}\backslash M)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{W})$ is the set

$$D(\mathfrak{W}) = \{u \in C_0(\overline{D}\backslash M) \cap H^{2,p}(D); Wu \in C_0(\overline{D}\backslash M), Lu = 0\}.$$

(b) $\mathfrak{W}u = Wu, \; u \in D(\mathfrak{W})$.

Here we remark that the domain $D(\mathfrak{W})$ is *independent* of $N < p < \infty$ (see the proof of Lemma 4.2).

Then Theorem 3 remains valid with $L^p(D)$ and W_p replaced by $C_0(\overline{D}\backslash M)$ and $\mathfrak{W}$, respectively:

Theorem 4 *If hypothesis* (H) *is satisfied, then we have the following*:

(i) *For every* $\varepsilon > 0$, *there exists a constant* $r(\varepsilon) > 0$ *such that the resolvent set of* $\mathfrak{W}$ *contains the set* $\Sigma(\varepsilon) = \{\lambda = r^2 e^{i\vartheta}; r \geqq r(\varepsilon), -\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon\}$, *and that the resolvent* $(\mathfrak{W} - \lambda I)^{-1}$ *satisfies the estimate*

$$\|(\mathfrak{W} - \lambda I)^{-1}\| \leqq \frac{c(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma(\varepsilon), \tag{0.4}$$

where $c(\varepsilon) > 0$ *is a constant depending on* ε.

(ii) *The operator* $\mathfrak{W}$ *generates a semigroup* $e^{z\mathfrak{W}}$ *on the space* $C_0(\overline{D}\backslash M)$ *which is* analytic *in the sector* $\Delta_\varepsilon = \{z = t + is; z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ *for any* $0 < \varepsilon < \pi/2$.

Theorems 3 and 4 express a *regularizing effect* for the parabolic integro-differential operator $\partial/\partial t - W$ with homogeneous boundary condition L (see [G-M, Chapter VIII, Theorem 3.1]).

The rest of this paper is organized as follows. In Section 1 we study problem $(*)$ in the framework of Hölder spaces, and prove Theorem 1. The essential point in the proof is to estimate the integral operator S in terms of Hölder norms. We show that the operator (W, L) may be considered as a perturbation of a compact operator to the operator (A, L) in the framework of Hölder spaces. Thus the proof of Theorem 1 is reduced to the differential operator case which is studied in detail in [T2]. Section 2 is devoted to the proof of Theorem 2. The proof is based on a version of the Hille–Yosida theorem in semigroup theory in terms of the maximum principle. In Section 3 we prove Theorem 3. We estimate the integral operator S in terms of L^p norms, and show that S is an A_p-completely continuous operator in the sense of Gohberg and Kreĭn [G-K]. Section 4 is devoted to the proof of Theorem 4. Theorem 4 follows from Theorem 3 by using Sobolev's imbedding theorems and a λ-dependent localization argument, just as in [T2].

1 Proof of Theorem 1

I) First we prove Theorem 1 in the case when $S \equiv 0$:

Theorem 1.1 *If hypothesis* (H) *is satisfied, then the mapping*

$$(A,L) : C^{2+\theta}(\overline{D}) \to C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all* $0 < \theta < 1$.

Proof. The proof is divided into four steps.

i) Let (f, φ) be an arbitrary element of $C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$ with $\varphi = \mu\varphi_1 - \gamma\varphi_2$.

First we show that the boundary value problem

$$\begin{cases} Au = f & \text{in } D\,, \\ Lu = \varphi & \text{on } \partial D \end{cases} \tag{$**$}$$

can be reduced to the study of an operator on the boundary.

To do so, we consider the following Neumann problem:

$$\begin{cases} Av = f & \text{in } D\,, \\ \frac{\partial v}{\partial \mathbf{n}} = \varphi_1 & \text{on } \partial D\,. \end{cases} \tag{N}$$

Recall that the existence and uniqueness theorem for problem (N) is well established in the framework of Hölder spaces (see [G-T, Theorem 6.31]). Thus we find that a function $u \in C^{2+\theta}(\overline{D})$ is a solution of problem $(*)$ if and only if the function $w = u - v \in C^{2+\theta}(\overline{D})$ is a solution of the problem

$$\begin{cases} Aw = 0 & \text{in } D\,, \\ Lw = \varphi - Lv & \text{on } \partial D\,. \end{cases}$$

Here we remark that

$$Lv = \mu\frac{\partial v}{\partial \mathbf{n}} + \gamma v = \mu\varphi_1 + \gamma v\,;$$

so that

$$Lw = -\gamma(\varphi_2 + v) \in C^{2+\theta}(\partial D)\,.$$

But we know that every solution $w \in C^{2+\theta}(\overline{D})$ of the homogeneous equation: $Aw = 0$ in D can be expressed as follows (see [G-T, Theorem 6.14]):

$$w = \mathscr{P}\psi, \quad \psi \in C^{2+\theta}(\partial D)\,.$$

Thus one can reduce the study of problem $(**)$ to that of the equation

$$T\psi := L\mathscr{P}\psi = -\gamma(\varphi_2 + v) \quad \text{on } \partial D\,. \tag{$+$}$$

More precisely, we have the following:

Proposition 1.2 *For functions* $f \in C^{\theta}(\overline{D})$ *and* $\varphi \in C_L^{1+\theta}(\partial D)$, *there exists a solution* $u \in C^{2+\theta}(\overline{D})$ *of problem* $(**)$ *if and only if there exists a solution* $\psi \in C^{2+\theta}(\partial D)$ *of equation* $(+)$.

ii) We study the operator T in question. It is known (see [H, Chapter XX]) that the operator

$$T\psi = L\mathscr{P}\psi = \mu\frac{\partial}{\partial \mathbf{n}}(\mathscr{P}\psi) + \gamma\psi$$

is a first-order, *pseudo-differential operator* on the boundary ∂D.

The next proposition is an essential step in the proof of Theorem 1.1:

Proposition 1.3 *If hypothesis* (H) *is satisfied, then there exists a* parametrix E *in the Hörmander class* $L^0_{1,1/2}(\partial D)$ *for* T *which maps* $C^{k+\theta}(\partial D)$ *continuously into itself for any integer* $k \geqq 0$.

Proof. By making use of Theorem 22.1.3 of [H, Chapter XXII] just as in [T2, Lemma 4.2], one can construct a parametrix E in the Hörmander class $L^0_{1,1/2}(\partial D)$ for T:

$$ET \equiv TE \equiv I \bmod L^{-\infty}(\partial D)\,.$$

The boundedness of $E : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$ follows from an application of [B, Theorem 1], since $C^{k+\theta}(\partial D) = B^{k+\theta}_{\infty,\infty}(\partial D)$. □

iii) We consider problem $(**)$ in the framework of Sobolev spaces of L^p style, and prove an L^p version of Theorem 1.1.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$H^{k,p}(D)$ = the space of (equivalence classes of) functions $u \in L^p(D)$ whose derivatives $D^\alpha u$, $|\alpha| \leqq k$, in the sense of distributions are in $L^p(D)$,

and the Besov space

$B^{k-1/p,p}(\partial D)$ = the space of the boundary values φ of functions $u \in H^{k,p}(D)$.

In the space $B^{k-1/p,p}(\partial D)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf \|u\|_{H^{k,p}(D)}\,,$$

where the infimum is taken over all functions $u \in H^{k,p}(D)$ which equal φ on the boundary ∂D. The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to this norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$ (cf. [B-L]).

We introduce a subspace of $B^{1-1/p,p}(\partial D)$ which is an L^p version of $C_L^{1+\theta}(\partial D)$. We let

$$B_L^{1-1/p,p}(\partial D) = \{\varphi = \mu\varphi_1 - \gamma\varphi_2;\, \varphi_1 \in B^{1-1/p,p}(\partial D),\ \varphi_2 \in B^{2-1/p,p}(\partial D)\}\,,$$

and define a norm

$$|\varphi|_{B_L^{1-1/p,p}(\partial D)} = \inf\{|\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)}; \varphi = \mu\varphi_1 - \gamma\varphi_2\} .$$

Then it is easy to verify that the space $B_L^{1-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B_L^{1-1/p,p}(\partial D)}$.

Then, arguing just as in the proof of [T2, Theorem 1], we can obtain the following L^p version of Theorem 1.1:

Theorem 1.4 *If hypothesis* (H) *is satisfied, then the mapping*

$$(A,L) : H^{2,p}(D) \to L^p(D) \oplus B_L^{1-1/p,p}(\partial D)$$

is an algebraic and topological isomorphism.

iv) Now we remark that

$$\begin{cases} C^\theta(\overline{D}) \subset L^p(D) , \\ C_L^{1+\theta}(\partial D) \subset B_L^{1-1/p,p}(\partial D) . \end{cases}$$

Thus, we find from Theorem 1.4 that problem (**) has a unique solution $u \in H^{2,p}(D)$ for any $f \in C^\theta(\overline{D})$ and any $\varphi \in C_L^{1+\theta}(\partial D)$. Furthermore, by virtue of Proposition 1.2, it follows that the solution u can be written in the form

$$u = v + \mathscr{P}\psi, \quad v \in C^{2+\theta}(\overline{D}), \quad \psi \in B^{2-1/p,p}(\partial D) .$$

But, Proposition 1.3 tells us that

$$\psi \in C^{2+\theta}(\partial D) ,$$

since we have $\psi \equiv E(T\psi) = -E(\gamma(\varphi_2 + v)) \bmod C^\infty(\partial D)$.

Therefore, we obtain that

$$u = v + \mathscr{P}\psi \in C^{2+\theta}(\overline{D}) .$$

The proof of Theorem 1.1 is complete. □

II) Next we study the integral operator S in the framework of Hölder spaces. To do so, we need the following elementary estimates for the measure $m(dz)$:

Claim 1.5 *For* $\varepsilon > 0$, *we let*

$$\sigma(\varepsilon) = \int_{\{|z| \leqq \varepsilon\}} |z|^2 m(dz) ,$$

$$\delta(\varepsilon) = \int_{\{|z| > \varepsilon\}} |z| m(dz) ,$$

$$\tau(\varepsilon) = \int_{\{|z| > \varepsilon\}} m(dz) .$$

Then we have, as $\varepsilon \downarrow 0$,

$$\sigma(\varepsilon) \to 0\,, \tag{1.1}$$

$$\delta(\varepsilon) \leqq \frac{C_1}{\varepsilon} + C_2\,, \tag{1.2}$$

$$\tau(\varepsilon) \leqq \frac{C_1}{\varepsilon^2} + C_2\,, \tag{1.3}$$

where

$$C_1 = \int_{\{|z| \leqq 1\}} |z|^2 m(dz), \qquad C_2 = \int_{\{|z|>1\}} |z| m(dz)\,.$$

Proof. Assertion (1.1) follows immediately from condition (0.2).

The term $\delta(\varepsilon)$ can be estimated as follows:

$$\begin{aligned} \delta(\varepsilon) &= \int_{\{|z|>1\}} |z| m(dz) + \int_{\{\varepsilon<|z|\leqq 1\}} |z| m(dz) \\ &\leqq \int_{\{|z|>1\}} |z| m(dz) + \frac{1}{\varepsilon} \int_{\{\varepsilon<|z|\leqq 1\}} |z|^2 m(dz) \\ &\leqq \int_{\{|z|>1\}} |z| m(dz) + \frac{1}{\varepsilon} \int_{\{|z|\leqq 1\}} |z|^2 m(dz)\,. \end{aligned}$$

The term $\tau(\varepsilon)$ is estimated in a similar way. □

By virtue of Claim 1.5, we can estimate the term Su in terms of Hölder norms, just as in [G-M, Chapter II, Lemmas 1.2 and 1.5]:

Lemma 1.6 *For every* $\eta > 0$, *there exists a constant* $C_\eta > 0$ *such that we have, for all* $u \in C^2(\overline{D})$,

$$\|Su\|_\infty \leqq \eta \|\nabla^2 u\|_\infty + C_\eta(\|u\|_\infty + \|\nabla u\|_\infty)\,.$$

Here

$$\|u\|_\infty = \sup_{x \in D} |u(x)|\,.$$

Lemma 1.7 *For every* $\eta > 0$, *there exists a constant* $C_\eta > 0$ *such that we have, for all* $u \in C^{2+\theta_0}(\overline{D})$,

$$\|Su\|_{C^{\theta_0}(\overline{D})} \leqq \eta \|\nabla^2 u\|_{C^{\theta_0}(\overline{D})} + C_\eta(\|u\|_{C^{\theta_0}(\overline{D})} + \|\nabla u\|_{C^{\theta_0}(\overline{D})})\,.$$

Here

$$\|u\|_{C^{\theta_0}(\overline{D})} = \|u\|_\infty + [u]_{\theta_0}, \qquad [u]_{\theta_0} = \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^{\theta_0}}\,.$$

III) *End of Proof of Theorem 1.* First, Theorem 1.1 implies that

$$\operatorname{ind}(A, L) = 0 .$$

On the other hand, Lemma 1.7 tells us that the operator S maps $C^{2+\theta_0}(\overline{D})$ continuously into $C^{\theta_0}(\overline{D})$. Hence it follows from an application of [B-C-P, Théorème XXII] that S is a *compact* operator from $C^{2+\theta}(\overline{D})$ into $C^{\theta}(\overline{D})$ for all $0 < \theta < \theta_0$. This implies that the operator (W, L) is a perturbation of a compact operator to the operator (A, L).

Hence we find that

$$\operatorname{ind}(W, L) = \operatorname{ind}(A, L) = 0 .$$

Therefore, in order to show the bijectivity of (W, L), it suffices to prove its *injectivity*:

$$\begin{cases} u \in C^{2+\theta}(\overline{D}), Wu = 0 & \text{in } D, \quad Lu = 0 \quad \text{on } \partial D \\ \Rightarrow u = 0 & \text{in } D . \end{cases}$$

But, this is an immediate consequence of the following *maximum principle*:

Proposition 1.8 *If hypothesis (H) is satisfied, then we have*:

$$\begin{cases} u \in C^{2}(\overline{D}), Wu \geqq 0 & \text{in } D, \quad Lu \geqq 0 \quad \text{on } \partial D \\ \Rightarrow u \leqq 0 & \text{on } \overline{D} . \end{cases}$$

Proof. If u is a constant m, then we have $0 \leqq Wu = mc$ in D. This implies that $u \equiv m$ is non-positive, since $c \leqq 0$ and $c \not\equiv 0$ in D.

Now we consider the case when u is not a constant. Assume to the contrary that:

$$m = \max_{\overline{D}} u > 0 .$$

Then, applying the strong maximum principle (see [B-C-P, Théorème VII]) to the operator W, we obtain that there exists a point x_0' of ∂D such that

$$\begin{cases} u(x_0') = m , \\ u(x) < u(x_0') \quad \text{for all } x \in \dot{D} . \end{cases}$$

Furthermore, it follows from an application of the boundary point lemma (see [B-C-P, Théorème VIII]) that

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0 .$$

Hence we have

$$\mu(x_0') = 0, \qquad \gamma(x_0') = 0 ,$$

since $Lu(x_0') \geqq 0$. This contradicts hypothesis (H). □

The proof of Theorem 1 is now complete. □

2 Proof of Theorem 2

The proof of Theorem 2 is based on the following version of the Hille–Yosida theorem in terms of the maximum principle (see [B-C-P, Théorème de Hille–Yosida–Ray]):

Theorem 2.1 *Let $\mathscr{A}$ be a linear operator from the space $C_0(\overline{D}\backslash M)$ into itself, and assume that:*

(α) *The domain $D(\mathscr{A})$ is dense in the space $C_0(\overline{D}\backslash M)$.*

(β) *For any $u \in D(\mathscr{A})$ such that $\sup u > 0$, there exists a point $x \in \overline{D}\backslash M$ such that $u(x) = \sup u$ and $\mathscr{A}u(x) \leqq 0$.*

(γ) *For all $\alpha > 0$, the range $R(\mathscr{A} - \alpha I)$ is dense in the space $C_0(\overline{D}\backslash M)$.*

Then the operator $\mathscr{A}$ is closable in the space $C_0(\overline{D}\backslash M)$, and its minimal closed extension $\overline{\mathscr{A}}$ generates a Feller semigroup $\{T_t\}_{t \geqq 0}$ on $\overline{D}\backslash M$.

Proof of Theorem 2. We have only to verify conditions (α), (β) and (γ) in Theorem 2.1 for the operator $\mathscr{W}$.

(γ) We obtain from Theorem 1 (and its proof) that the mapping

$$(W - \alpha, L) : C^{2+\theta}(\overline{D}) \to C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism for all $\alpha > 0$. This verifies condition (γ), since the range $R(\mathscr{W} - \alpha I)$ contains the space $C^{\theta}(\overline{D}) \cap C_0(\overline{D}\backslash M)$ which is dense in $C_0(\overline{D}\backslash M)$.

(β) First let x_0 be a point of D such that $u(x_0) = \sup u$. Then it follows from an application of [B-C-P, Théorème V] that

$$\mathscr{W}u(x_0) = Wu(x_0) \leqq 0 .$$

Next let x_0' be a point of $\partial D\backslash M$ such that $u(x_0') = \sup u$. Assume to the contrary that

$$\mathscr{W}u(x_0') = Wu(x_0') > 0 .$$

We have only to consider the case when u is not a constant. Then it follows from an application of the boundary point lemma that $(\partial u/\partial \mathbf{n})(x_0') < 0$. Hence we have

$$\mu(x_0') = 0 ,$$

since $Lu(x_0') = 0$. This contradicts the hypothesis: $x_0' \in \partial D\backslash M$, that is, $\mu(x_0') > 0$.

(α) The density of the domain $D(\mathscr{W})$ can be proved just as in the proof of [T2, Theorem 8.20], by using [B-C-P, Proposition III.1.6].

The proof of Theorem 2 is complete. □

3 Proof of Theorem 3

The next theorem, which is a generalization of [T2, Theorem 6.1] to the integro-differential operator case, proves Theorem 3:

Theorem 3.1 *If hypothesis (H) is satisfied, then, for every $0 < \varepsilon < \pi/2$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of W_p contains the set $\Sigma_p(\varepsilon) = \{\lambda = r^2 e^{i\vartheta}; r \geqq r_p(\varepsilon), -\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon\}$, and that the resolvent $(W_p - \lambda I)^{-1}$ satisfies estimate* (0.3).

Proof. The proof is divided into three steps.

i) We show that there exist constants $r_p(\varepsilon)$ and $c_p(\varepsilon)$ such that we have, for all $\lambda = r^2 e^{i\vartheta}$ satisfying $r \geqq r_p(\varepsilon)$ and $-\pi + \varepsilon \leqq \vartheta \leqq \pi + \varepsilon$,

$$|u|_{2,p} + |\lambda|^{1/2}|u|_{1,p} + |\lambda| \|u\|_p \leqq c_p(\varepsilon)\|(W_p - \lambda I)u\|_p . \tag{3.1}$$

Here

$$\|u\|_p = \|u\|_{L^p(D)}, \qquad |u|_{1,p} = \|\nabla u\|_{L^p(D)}, \qquad |u|_{2,p} = \|\nabla^2 u\|_{L^p(D)} .$$

First we recall (see [T2, formula (6.2)]) that estimate (3.1) is proved for the differential operator A:

$$|u|_{2,p} + |\lambda|^{1/2}|u|_{1,p} + |\lambda| \|u\|_p \leqq c'_p(\varepsilon)\|(A_p - \lambda I)u\|_p . \tag{3.2}$$

Here the operator A_p is an unbounded linear operator from $L^p(D)$ into itself defined by the following:

(a) The domain of definition $D(A_p)$ is the set

$$D(A_p) = \{u \in H^{2,p}(D); Lu = 0\} .$$

(b) $A_p u = Au, u \in D(A_p)$.

In order to replace the last term $\|(A_p - \lambda I)u\|_p$ by the term $\|(W_p - \lambda I)u\|_p$, we need the following L^p-estimate for the operator S:

Lemma 3.2 *For every $\eta > 0$, there exists a constant $C_\eta > 0$ such that we have, for all $u \in H^{2,p}(D)$,*

$$\|Su\|_p \leqq \eta |u|_{2,p} + C_\eta(\|u\|_p + |u|_{1,p}) . \tag{3.3}$$

Proof. We decompose the term Su into the following three terms:

$$\begin{aligned} Su(x) &= \int_0^1 (1-t)dt \int_{\{|z| \leqq \varepsilon\}} z \cdot \nabla^2 u(x+tz) z s(x,z) m(dz) \\ &\quad + \int_{\{|z| > \varepsilon\}} (u(x+z) - u(x)) s(x,z) m(dz) - \int_{\{|z| > \varepsilon\}} z \cdot \nabla u(x) s(x,z) m(dz) \\ &:= S_1 u(x) + S_2 u(x) - S_3 u(x) . \end{aligned}$$

First we estimate the L^p norm of the term $S_3 u$. By using estimate (1.2), we obtain that

$$\left| \int_{\{|z|>\varepsilon\}} z \cdot \nabla u(x) s(x,z) m(dz) \right| \leqq \delta(\varepsilon) |\nabla u(x)| \leqq \left(\frac{C_1}{\varepsilon} + C_2 \right) |\nabla u(x)| .$$

Hence we have the L^p estimate of the term $S_3 u$:

$$\|S_3 u\|_p \leqq \left(\frac{C_1}{\varepsilon} + C_2 \right) \|\nabla u\|_p .$$

Secondly, we have

$$\left\| \int_{\{|z|>\varepsilon\}} u(\,\cdot\,) s(\,\cdot\,, z) m(dz) \right\|_p \leqq \left(\frac{C_1}{\varepsilon^2} + C_2 \right) \|u\|_p .$$

Furthermore, by using Hölder's inequality and Fubini's theorem, we obtain from condition (0.1) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} \left| \int_{\{|z|>\varepsilon\}} u(x+z) s(x,z) m(dz) \right|^p dx \\
&\leqq \int_{\mathbf{R}^N} \left(\int_{\{|z|>\varepsilon\}} |u(x+z)| s(x,z) m(dz) \right)^p dx \\
&\leqq \int_{\mathbf{R}^N} \left(\int_{\{|z|>\varepsilon\}} |u(x+z)|^p s(x,z)^p m(dz) \right) \left(\int_{\{|z|>\varepsilon\}} m(dz) \right)^{p/q} dx \\
&= \tau(\varepsilon)^{p/q} \int_{\mathbf{R}^N} \int_{\{|z|>\varepsilon\}} |u(x+z)|^p s(x,z)^p m(dz) dx \\
&= \tau(\varepsilon)^{p/q} \int_{\{|z|>\varepsilon\}} \left(\int_{\mathbf{R}^N} |u(x+z)|^p s(x,z)^p dx \right) m(dz) \\
&\leqq \tau(\varepsilon)^{p/q} \left(\int_D |u(y)|^p dy \right) \left(\int_{\{|z|>\varepsilon\}} m(dz) \right) = \tau(\varepsilon)^p \|u\|_p^p .
\end{aligned}$$

By estimate (1.3), we have the L^p estimate of the term $S_2 u$:

$$\|S_2 u\|_p \leqq \left(\frac{C_1}{\varepsilon^2} + C_2 \right) \|u\|_p .$$

Similarly, by using Hölders's inequality and Fubini's theorem, we find that

$$\int_{\mathbf{R}^N} \left| \int_0^1 (1-t)dt \int_{\{|z|\leqq\varepsilon\}} z \cdot \nabla^2 u(x+tz)zs(x,z)m(dz) \right|^p dx$$

$$\leqq \int_{\mathbf{R}^N} \left(\int_0^1 dt \int_{\{|z|\leqq\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)| s(x,z) m(dz) \right)^p dx$$

$$\leqq \int_{\mathbf{R}^N} \int_0^1 dt \left(\int_{\{|z|\leqq\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)|^p s(x,z)^p m(dz) \right)$$

$$\times \left(\int_{\{|z|\leqq\varepsilon\}} |z|^2 m(dz) \right)^{p/q} dx$$

$$= \sigma(\varepsilon)^{p/q} \int_{\mathbf{R}^N} \int_0^1 dt \left(\int_{\{|z|\leqq\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)|^p s(x,z)^p m(dz) \right) dx$$

$$= \sigma(\varepsilon)^{p/q} \int_0^1 dt \int_{\{|z|\leqq\varepsilon\}} |z|^2 \left(\int_{\mathbf{R}^N} |\nabla^2 u(x+tz)|^p s(x,z)^p dx \right) m(dz)$$

$$\leqq \sigma(\varepsilon)^{p/q} \left(\int_D |\nabla^2 u(y)|^p dy \right) \left(\int_{\{|z|\leqq\varepsilon\}} |z|^2 m(dz) \right)$$

$$\leqq \sigma(\varepsilon)^p \left(\int_D |\nabla^2 u(y)|^p dy \right) .$$

Hence we have the L^p estimate of the term $S_1 u$:

$$\|S_1 u\|_p \leqq \sigma(\varepsilon) \|\nabla^2 u\|_p .$$

Summing up, we have proved that

$$\begin{aligned} \|Su\|_p &\leqq \|S_1 u\|_p + \|S_2 u\|_p + \|S_3 u\|_p \\ &\leqq \sigma(\varepsilon)|u|_{2,p} + \left(\frac{C_1}{\varepsilon} + C_2 \right) |u|_{1,p} + \left(\frac{C_1}{\varepsilon^2} + C_2 \right) \|u\|_p . \end{aligned}$$

In view of assertion (1.1), this proves estimate (3.3) if we choose ε sufficiently small. □

Since we have

$$(A - \lambda)u = (W - \lambda)u - Su ,$$

it follows from estimate (3.3) that

$$\|(A_p - \lambda)u\|_p \leqq \|(W_p - \lambda)u\|_p + \eta |u|_{2,p} + C_\eta (|u|_{1,p} + \|u\|_p) .$$

Thus, carrying this estimate into estimate (3.2), we obtain that

$$|u|_{2,p} + |\lambda|^{1/2}|u|_{1,p} + |\lambda| \, \|u\|_p$$
$$\leqq c'_p(\varepsilon)\|(W_p - \lambda)u\|_p + \eta c'_p(\varepsilon)|u|_{2,p} + C_\eta c'_p(\varepsilon)(|u|_{1,p} + \|u\|_p) \,. \quad (3.4)$$

Therefore, the desired estimate (3.1) follows from estimate (3.4) if we take the constant η so small that

$$\eta c'_p(\varepsilon) < 1$$

and the parameter λ so large that

$$|\lambda|^{1/2} > C_\eta c'_p(\varepsilon) \,.$$

ii) By estimate (3.1), we find that the operator $W_p - \lambda I$ is injective and its range $R(W_p - \lambda I)$ is closed in $L^p(D)$, for all $\lambda \in \Sigma_p(\varepsilon)$.

We show that the operator $W_p - \lambda I$ is surjective for all $\lambda \in \Sigma_p(\varepsilon)$:

$$R(W_p - \lambda I) = L^p(D), \quad \lambda \in \Sigma_p(\varepsilon) \,.$$

To do so, it suffices to show that the operator $W_p - \lambda I$ is a Fredholm operator with

$$\operatorname{ind}(W_p - \lambda I) = 0, \quad \lambda \in \Sigma_p(\varepsilon) \,, \quad (3.5)$$

since $W_p - \lambda I$ is injective for all $\lambda \in \Sigma_p(\varepsilon)$.

In order to prove assertion (3.5), we need the following:

Lemma 3.3 *The operator S is A_p-*completely continuous, *that is, the operator S: $D(A_p) \to L^p(D)$ is completely continuous where the domain $D(A_p)$ is endowed with the graph norm of A_p.*

Proof. Let $\{u_j\}$ be an arbitrary bounded sequence in the domain $D(A_p)$; hence there exists a constant $K > 0$ such that

$$\|u_j\|_p \leqq K, \qquad \|A_p u_j\|_p \leqq K \,.$$

Then we have, by [T2, estimate (0.1)],

$$\|u_j\|_{2,p} \leqq C(\|A_p u_j\|_p + \|u_j\|_p) \leqq 2CK \,. \quad (3.6)$$

Therefore, by Rellich's theorem, one may assume that the sequence $\{u_j\}$ itself is a Cauchy sequence in the space $H^{1,p}(D)$. Then, applying estimate (3.3) to the sequence $\{u_j - u_k\}$ and using estimate (3.6), we obtain that

$$\|Su_j - Su_k\|_p \leqq \eta|u_j - u_k|_{2,p} + C_\eta(\|u_j - u_k\|_p + |u_j - u_k|_{1,p})$$
$$\leqq 4\eta CK + C_\eta\|u_j - u_k\|_{1,p} \,.$$

Hence we have

$$\limsup_{j,k\to\infty} \|Su_j - Su_k\|_p \leqq 4\eta CK \,.$$

This proves that the sequence $\{Su_j\}$ is a Cauchy sequence in the space $L^p(D)$, since η is arbitrary. □

In view of Lemma 3.3, assertion (3.5) follows from an application of [G-K, Theorem 2.6]. Indeed, we have, by [T2, Theorem 6.1],

$$\operatorname{ind}(W_p - \lambda I) = \operatorname{ind}(A_p - \lambda I + S) = \operatorname{ind}(A_p - \lambda I) = 0 .$$

iii) Summing up, we have proved that the operator $W_p - \lambda I$ is bijective for all $\lambda \in \Sigma_p(\varepsilon)$ and its inverse $(W_p - \lambda I)^{-1}$ satisfies estimate (0.3).

The proof of Theorem 3.1 is now complete. □

4 Proof of Theorem 4

The proof is carried out in a chain of auxilliary lemmas.

I) We begin with a version of estimate (3.1):

Lemma 4.1 *Let $N < p < \infty$. If hypothesis (H) is satisfied, then, for every $\varepsilon > 0$, there exists a constant $r_p(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\vartheta}$ with $r \geqq r_p(\varepsilon)$ and $-\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon$, we have, for all $u \in D(W_p)$,*

$$|\lambda|^{1/2} \|u\|_{C^1(\bar{D})} + |\lambda| \, \|u\|_{C(\bar{D})} \leqq C_p(\varepsilon) |\lambda|^{N/2p} \|(W - \lambda)u\|_p , \tag{4.1}$$

with a constant $C_p(\varepsilon) > 0$.

Proof. First, it follows an application of the Gagliardo–Nirenberg inequality (see [F, Part I, Theorem 10.1] that

$$\|u\|_{C(\bar{D})} \leqq C |u|_{1,p}^{N/p} \|u\|_p^{1-N/p}, \quad u \in H^{1,p}(D) . \tag{4.2}$$

Here and in the following the letter C denotes a generic positive constant depending on p and ε, but independent of u and λ.

Combining inequality (4.2) with inequality (3.1), we obtain that

$$\begin{aligned} \|u\|_{C(\bar{D})} &\leqq C \left(|\lambda|^{-1/2} \|(W - \lambda)u\|_p\right)^{N/p} \left(|\lambda|^{-1} \|(W - \lambda)u\|_p\right)^{1-N/p} \\ &= C |\lambda|^{-1+N/2p} \|(W - \lambda)u\|_p , \end{aligned}$$

so that

$$|\lambda| \, \|u\|_{C(\bar{D})} \leqq C |\lambda|^{N/2p} \|(W - \lambda)u\|_p, \quad u \in D(W_p) . \tag{4.3}$$

Similarly, applying inequality (4.2) to the functions $D_i u \in H^{1,p}(D)$ $(1 \leqq i \leqq n)$, we obtain that

$$\begin{aligned} \|\nabla u\|_{C(\bar{D})} &\leqq C |\nabla u|_{1,p}^{N/p} \|\nabla u\|_p^{1-N/p} \leqq C |u|_{2,p}^{N/p} |u|_{1,p}^{1-N/p} \\ &\leqq C (\|(W - \lambda)u\|_p)^{N/p} (|\lambda|^{-1/2} \|(W - \lambda)u\|_p)^{1-N/p} \\ &= C |\lambda|^{-1/2+N/2p} \|(W - \lambda)u\|_p . \end{aligned}$$

This proves that

$$|\lambda|^{1/2}\|u\|_{C^1(\overline{D})} \leqq C|\lambda|^{N/2p}\|(W-\lambda)u\|_p, \quad u \in D(W_p). \tag{4.4}$$

Therefore, the desired inequality (4.1) follows from inequalities (4.3) and (4.4).

II) The next lemma proves estimate (0.4):

Lemma 4.2 *Let $N < p < \infty$. If hypothesis (H) is satisfied, then, for every $\varepsilon > 0$, there exists a constant $r(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\vartheta}$ with $r \geqq r(\varepsilon)$ and $-\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon$, we have, for all $u \in D(\mathfrak{W})$,*

$$|\lambda|^{1/2}\|u\|_{C^1(\overline{D})} + |\lambda|\,\|u\|_{C(\overline{D})} \leqq c(\varepsilon)\|(\mathfrak{W}-\lambda I)u\|_{C(\overline{D})}, \tag{4.5}$$

with a constant $c(\varepsilon) > 0$.

Proof. 1) First we show that the domain

$$D(\mathfrak{W}) = \{u \in C_0(\overline{D}\backslash M) \cap H^{2,p}(D); Wu \in C_0(\overline{D}\backslash M), Lu = 0\}$$

is independent of $N < p < \infty$.

We let

$$\mathscr{D}_p = \{u \in H^{2,p}(D) \cap C_0(\overline{D}\backslash M); Wu \in C_0(\overline{D}\backslash M), Lu = 0\}.$$

Since we have $L^{p_1}(D) \subset L^{p_2}(D)$ for $p_1 > p_2$, it follows that

$$\mathscr{D}_{p_1} \subset \mathscr{D}_{p_2} \quad \text{if } p_1 > p_2.$$

Conversely, let v be an arbitrary element of $\mathscr{D}_{p_2}$:

$$v \in H^{2,p_2}(D) \cap C_0(\overline{D}\backslash M), \qquad Wv \in C_0(\overline{D}\backslash M), \qquad Lv = 0.$$

Then, since we have $v,\ Wv \in C_0(\overline{D}\backslash M) \subset L^{p_1}(D)$, it follows from an application of Theorem 3.1 with $p = p_1$ that there exists a unique function $u \in H^{2,p_1}(D)$ such that

$$\begin{cases} (W-\lambda)u = (W-\lambda)v & \text{in } D, \\ Lu = 0 & \text{on } \partial D, \end{cases}$$

if we choose λ sufficiently large. Hence we have $u - v \in H^{2,p_2}(D)$ and

$$\begin{cases} (W-\lambda)(u-v) = 0 & \text{in } D, \\ L(u-v) = 0 & \text{on } \partial D. \end{cases}$$

Therefore, by applying again Theorem 3.1 with $p = p_2$, we obtain that $u - v = 0$, so that $v = u \in H^{2,p_1}(D)$. This proves that $v \in \mathscr{D}_{p_1}$.

2) We shall make use of a λ-dependent localization argument in order to adjust the term $\|(W-\lambda)u\|_p$ in inequality (4.1) to obtain inequality (4.5), just as in [T2].

2-a) If x_0' is a point of ∂D and if χ is a C^∞ coordinate transformation such that χ maps $B(x_0', \eta_0) \cap D$ into $B(0, \delta) \cap \mathbf{R}_+^N$ and flattens a part of the boundary ∂D into the plane $x_N = 0$, then we let

$$G_0 = B(x_0', \eta_0) \cap D\,, \qquad G' = B(x_0', \eta) \cap D, 0 < \eta < \eta_0\,,$$
$$G'' = B(x_0', \eta/2) \cap D, 0 < \eta < \eta_0\,.$$

Here and in the following $B(x, \eta)$ denotes the ball of radius η about x.

Similarly, if x_0 is a point of D and if χ is a C^∞ coordinate transformation such that χ maps $B(x_0, \eta_0)$ into $B(0, \delta)$, then we let

$$G_0 = B(x_0, \eta_0)\,, \qquad G' = B(x_0, \eta), 0 < \eta < \eta_0\,,$$
$$G'' = B(x_0, \eta/2), 0 < \eta < \eta_0\,.$$

2-b) We take a function $\Phi \in C_0^\infty(\mathbf{R})$ such that Φ equals 1 near the origin, and define

$$\varphi(x) = \Phi(|x'|^2)\Phi(x_N), \quad x = (x', x_N)\,.$$

Here one may assume that the function φ is chosen so that

$$\begin{cases} \operatorname{supp} \varphi \subset B(0,1)\,, \\ \varphi(x) = 1 \quad \text{on } B(0, 1/2)\,. \end{cases}$$

We introduce a localizing function

$$\varphi_0(x, \eta) := \varphi\left(\frac{x - x_0}{\eta}\right) = \Phi\left(\frac{|x' - x_0'|^2}{\eta^2}\right)\Phi\left(\frac{x_N - t}{\eta}\right), \quad x_0 = (x_0', t)\,.$$

We remark that

$$\begin{cases} \operatorname{supp} \varphi_0 \subset B(x_0, \eta)\,, \\ \varphi_0(x, \eta) = 1 \quad \text{on } B(x_0, \eta/2)\,. \end{cases}$$

Then it is easy to verify the following (see [T2, Claim 7.5]):

Claim 4.3 *If $u \in B(\mathfrak{W})$, then we have $\varphi_0 u \in \mathscr{D}(W_p)$.*

3) Now let u be an arbitrary element of $D(\mathfrak{W})$. Then, by Claim 4.3, we can apply inequality (4.1) to the function $\varphi_0 u$ to obtain that

$$\begin{aligned} |\lambda|^{1/2}\|u\|_{C^1(\overline{G''})} + |\lambda|\,\|u\|_{C(\overline{G''})} &\leqq |\lambda|^{1/2}\|\varphi_0 u\|_{C^1(\overline{G'})} + |\lambda|\,\|\varphi_0 u\|_{C(\overline{G'})} \\ &= |\lambda|^{1/2}\|\varphi_0 u\|_{C^1(\overline{D})} + |\lambda|\,\|\varphi_0 u\|_{C(\overline{D})} \\ &\leqq C|\lambda|^{N/2p}\|(W - \lambda)(\varphi_0 u)\|_{L^p(D)}\,. \end{aligned} \tag{4.6}$$

3-a) We estimate the last term $\|(W - \lambda)(\varphi_0 u)\|_{L^p(D)}$ in terms of the supremum norm of $C(\overline{D})$.

First we write the term $(W - \lambda)(\varphi_0 u)$ in the following form:

$$(W - \lambda)(\varphi_0 u) = \varphi_0((W - \lambda)u) + [A, \varphi_0]u + [S, \varphi_0]u\,,$$

where $[A, \varphi_0]$ and $[S, \varphi_0]$ are the commutators of A and φ_0 and of S and φ_0, respectively:

$$[A, \varphi_0]u = A(\varphi_0 u) - \varphi_0 A u ,$$

$$[S, \varphi_0]u = S(\varphi_0 u) - \varphi_0 S u .$$

Now we need the following elementary inequality:

Claim 4.4 *We have, for all $v \in C^j(\overline{G'})$ $(j = 0, 1, 2)$,*

$$\|v\|_{H^{j,p}(G')} \leqq |G'|^{1/p} \|v\|_{C^j(\overline{G'})} ,$$

where $|G'|$ is the measure of G'.

Since we have, for some constant $c > 0$,

$$|G'| \leqq |B(x_0, \eta)| \leqq c\eta^N ,$$

it follows from an application of Claim 4.4 that

$$\begin{aligned} \|\varphi_0(W - \lambda)u\|_{L^p(D)} = \|\varphi_0(W - \lambda)u\|_{L^p(G')} &\leqq c^{1/p}\eta^{N/p} \|(W - \lambda)u\|_{C(\overline{G'})} \\ &\leqq c^{1/p}\eta^{N/p} \|(W - \lambda)u\|_{C(\overline{D})} . \end{aligned} \tag{4.7}$$

On the other hand, we can estimate the commutators $[A, \varphi_0]u$ and $[S, \varphi_0]u$ as follows:

Claim 4.5 *We have, as $\eta \downarrow 0$,*

$$\|[A, \varphi_0]u\|_{L^p(D)} \leqq C(\eta^{-1+N/p} \|u\|_{C^1(\overline{D})} + \eta^{-2+N/p} \|u\|_{C(\overline{D})}) , \tag{4.8}$$

$$\|[S, \varphi_0]u\|_{L^p(D)} \leqq C(\eta^{-1+N/p} \|u\|_{C^1(\overline{D})} + \eta^{-2+N/p} \|u\|_{C(\overline{D})}) . \tag{4.9}$$

Proof. Estimate (4.8) is proved in [T2, inequality (7.9)].

In order to prove estimate (4.9), we remark that

$$\begin{aligned} &S(\varphi_0 u)(x) \\ &= \int_{\mathbf{R}^N \setminus \{0\}} (\varphi_0(x+z)u(x+z) - \varphi_0(x)u(x) - z \cdot \nabla(\varphi_0 u)(x)) s(x,z) m(dz) \\ &= \varphi_0(x) \int_{\mathbf{R}^N \setminus \{0\}} (u(x+z) - u(x) - z \cdot \nabla u(x)) s(x,z) m(dz) \\ &\quad + \left(\int_{\mathbf{R}^N \setminus \{0\}} (u(x+z) - u(x)) z s(x,z) m(dz) \right) \cdot \nabla \varphi_0(x) \\ &\quad + \int_{\mathbf{R}^N \setminus \{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla \varphi_0(x)) u(x+z) s(x,z) m(dz) \\ &= \varphi_0(x) S u(x) + \left(\int_{\mathbf{R}^N \setminus \{0\}} (u(x+z) - u(x)) z s(x,z) m(dz) \right) \cdot \nabla \varphi_0(x) \\ &\quad + \int_{\mathbf{R}^N \setminus \{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla \varphi_0(x)) u(x+z) s(x,z) m(dz) . \end{aligned}$$

Hence we can write the commutator $[S, \varphi_0]u$ in the following form:

$$[S, \varphi_0]u(x) = \left(\int_{\mathbf{R}^N \setminus \{0\}} (u(x+z) - u(x)) z s(x,z) m(dz) \right) \cdot \nabla \varphi_0(x)$$

$$+ \int_{\mathbf{R}^N \setminus \{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla \varphi_0(x)) u(x+z) s(x,z) m(dz)$$

$$:= S_0^{(1)} u(x) + S_0^{(2)} u(x) .$$

First, just as in Lemma 1.6, we can estimate the term $S_0^{(1)} u$ as follows:

$$\begin{aligned} \|S_0^{(1)} u\|_{L^p(D)} &= \|S_0^{(1)} u\|_{L^p(G')} \\ &\leqq 2(\sigma(\eta)\|u\|_{C^1(\overline{D})} + \delta(\eta)\|u\|_{C(\overline{D})}) \|\nabla \varphi_0\|_{L^p(G')} \\ &\leqq 2\left(\sigma(\eta)\|u\|_{C^1(\overline{D})} + \left(\frac{C_1}{\eta} + C_2 \right) \|u\|_{C(\overline{D})} \right) \|\nabla \varphi_0\|_{L^p(G')} . \end{aligned}$$

But it follows from an application of Claim 4.4 that

$$\|\nabla \varphi_0\|_{L^p(G')} \leqq C \eta^{N/p} \|\nabla \varphi_0\|_{C(\overline{G'})} \leqq C' \eta^{-1+N/p} ,$$

$$\|\nabla^2 \varphi_0\|_{L^p(G')} \leqq C \eta^{N/p} \|\nabla^2 \varphi_0\|_{C(\overline{G'})} \leqq C' \eta^{-2+N/p} ,$$

since we have, as $\eta \downarrow 0$,

$$|\nabla \varphi_0| = O(\eta^{-1}), \qquad |\nabla^2 \varphi_0| = O(\eta^{-2}) .$$

Therefore we obtain that

$$\|S_0^{(1)} u\|_{L^p(D)} \leqq C(\eta^{-1+N/p} \|u\|_{C^1(\overline{D})} + \eta^{-2+N/p} \|u\|_{C(\overline{D})}) . \tag{4.10}$$

Similarly, arguing as in the proof of Lemma 3.2, we can estimate the term $S_0^{(2)} u$ as follows:

$$\begin{aligned} \|S_0^{(2)} u\|_{L^p(D)} &\leqq C\|u\|_{C(\overline{D})} \|\nabla^2 \varphi_0\|_{L^p(G')} \\ &\leqq C\|u\|_{C(\overline{D})} \eta^{N/p} \|\nabla^2 \varphi_0\|_{C(\overline{G'})} \\ &\leqq C \eta^{-2+N/p} \|u\|_{C(\overline{D})} . \end{aligned} \tag{4.11}$$

Thus, the desired estimate (4.9) follows by combining estimates (4.10) and (4.11). □

Therefore, combining estimates (4.6), (4.7), (4.8) and (4.9), we obtain that

$$
\begin{aligned}
&|\lambda|^{1/2}\|u\|_{C^1(\overline{G''})} + |\lambda|\|u\|_{C(\overline{G''})} \\
&\quad \leqq C|\lambda|^{N/2p}\|(W-\lambda)(\varphi_0 u)\|_{L^p(D)} \\
&\quad = C|\lambda|^{N/2p}\|\varphi_0((W-\lambda)u) + [A,\varphi_0]u + [S,\varphi_0]u\|_{L^p(D)} \\
&\quad \leqq C|\lambda|^{N/2p}(\eta^{N/p}\|(W-\lambda)u\|_{C(\overline{G'})} + \eta^{-1+N/p}\|u\|_{C^1(\overline{G'})} + \eta^{-2+N/p}\|u\|_{C(\overline{G'})}) \\
&\quad \leqq C|\lambda|^{N/2p}(\eta^{N/p}\|(W-\lambda)u\|_{C(\overline{D})} + \eta^{-1+N/p}\|u\|_{C^1(\overline{D})} + \eta^{-2+N/p}\|u\|_{C(\overline{D})}) .
\end{aligned}
\tag{4.12}
$$

3-b) We remark that the closure $\overline{D} = D \cup \partial D$ can be covered by a finite number of sets of the forms

$$
\begin{cases} B(x_0, \eta/2), & x_0 \in D , \\ B(x_0', \eta/2) \cap \overline{D}, & x_0' \in \partial D . \end{cases}
$$

Therefore, taking the supremum of inequality (4.12) over $x \in \overline{D}$, we find that

$$
\begin{aligned}
&|\lambda|^{1/2}\|u\|_{C^1(\overline{D})} + |\lambda|\|u\|_{C(\overline{D})} \\
&\quad \leqq C|\lambda|^{N/2p}\eta^{N/p}(\|(W-\lambda)u\|_{C(\overline{D})} + \eta^{-1}\|u\|_{C^1(\overline{D})} + \eta^{-2}\|u\|_{C(\overline{D})}) .
\end{aligned}
\tag{4.13}
$$

4) We now choose the localization parameter η. We let

$$
\eta = \frac{\eta_0}{|\lambda|^{1/2}} K ,
$$

where K is a positive constant (to be chosen later) satisfying

$$
0 < \eta = \frac{\eta_0}{|\lambda|^{1/2}} K < \eta_0 ,
$$

that is,

$$
0 < K < |\lambda|^{1/2} .
$$

Then we obtain from inequality (4.13) that

$$
\begin{aligned}
&|\lambda|^{1/2}\|u\|_{C^1(\overline{D})} + |\lambda|\|u\|_{C(\overline{D})} \\
&\quad \leqq C\eta_0^{N/p}K^{N/p}\|(W-\lambda)u\|_{C(\overline{D})} + (C\eta_0^{N/p-1}K^{-1+N/p})|\lambda|^{1/2}\|u\|_{C^1(\overline{D})} \\
&\qquad + (C\eta_0^{N/p-2}K^{-2+N/p})|\lambda|\|u\|_{C(\overline{D})} .
\end{aligned}
\tag{4.14}
$$

But, since the exponents $-1+N/p$ and $-2+N/p$ are negative, we can choose the constant K so large that

$$C\eta_0^{N/p-1}K^{-1+N/p} < 1,$$

and

$$C\eta_0^{N/p-2}K^{-2+N/p} < 1.$$

Then, the desired inequality (4.5) follows from inequality (4.14).

The proof of Lemma 4.2 is complete. □

III) The next lemma, together with Lemma 4.2, proves that the resolvent set of $\mathfrak{W}$ contains the set $\Sigma(\varepsilon) = \{\lambda = r^2 e^{i\vartheta}; r \geqq r(\varepsilon), \ -\pi + \varepsilon \leqq \vartheta \leqq \pi - \varepsilon\}$:

Lemma 4.6 *If $\lambda \in \Sigma(\varepsilon)$, then, for any $f \in C_0(\overline{D}\backslash M)$, there exists a unique function $u \in D(\mathfrak{W})$ such that $(\mathfrak{W} - \lambda I)u = f$.*

Proof. Since we have, for all $1 < p < \infty$,

$$f \in C_0(\overline{D}\backslash M) \subset L^p(D),$$

it follows from an application of Theorem 3 that if $\lambda \in \Sigma_p(\varepsilon)$, there exists a unique function $u \in H^{2,p}(D)$ such that

$$(W - \lambda)u = f \quad \text{in } D, \tag{4.15}$$

and

$$Lu = \mu\frac{\partial u}{\partial \mathbf{n}} + \gamma u = 0 \quad \text{on } \partial D. \tag{4.16}$$

But, by Sobolev's imbedding theorem, it follows that

$$u \in H^{2,p}(D) \subset C^{2-N/p}(\overline{D}) \subset C^1(\overline{D}) \quad \text{if } N < p < \infty.$$

Hence we have, by formula (4.16) and condition (H),

$$u = 0 \quad \text{on } M = \{x' \in \partial D; \mu(x') = 0\},$$

so that

$$u \in C_0(\overline{D}\backslash M).$$

Further, in view of equation (4.15), we find that

$$Wu = f + \lambda u \in C_0(\overline{D}\backslash M).$$

Summing up, we have proved that

$$\begin{cases} u \in D(\mathfrak{W}), \\ (\mathfrak{W} - \lambda I)u = f. \end{cases}$$

Now the proof of Theorem 4 is complete. □

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