Werk

Titel: 4 Local and global asymptotic Riemann-Roch Theorems for symmetric products of loc...

Jahr: 1996

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0222|log13

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$$= \frac{1}{N}\left(\sum_{0\leqq i\leqq n}(-1)^i\frac{1}{\# G}\sum_{g\in G}\operatorname{trace}(g^* : H^i(Y,\mathscr{F}_Y)) - \frac{1}{\# G}\cdot\chi(Y,\mathscr{F}_Y)\right)$$

$$= \frac{1}{N}\cdot\frac{1}{\# G}\cdot\sum_{g\neq\mathrm{id}}L(g:\mathscr{F}_Y) = \frac{1}{\# G}\cdot\sum_{g\in G-\{\mathrm{id}\}}\frac{\operatorname{trace}(\rho(g))}{\det(1_n - g)}. \qquad \square$$

Remark.

a) By this Atiyah–Bott type formula for $\mu_{X,x}$, it is obvious that $\mu_{X,x} : R(G) \to \mathbb{C}$ is a homomorphism of groups, but it is not obvious that $\mu_{X,x}$ takes values in $\mathbb{Q}$.

b) Theorem 3.17 together with Theorem 3.5 are a generalization of results of Atiyah–Segal–Singer ([A-Se 68], Theorem 3.5) and Reid ([Re 87], pp. 405–407).

(3.18) In (3.10) and (3.12) we introduced local Chern classes and local Chern numbers of locally free sheaves with respect to a given resolution of an isolated quotient singularity. We now want to calculate such numbers for two special locally free sheaves.

Proposition. *Let $\sigma : (\tilde{X},E) \to (X,x)$ be a good resolution of an isolated quotient singularity of dimension n. Then*

a) $c_n(x,\Omega^1_{\tilde{X}}) = (-1)^n \cdot (e(E) - \frac{1}{\# G_x})$.

b) $c_n(x,\Omega^1_{\tilde{X}}(\log E)) = (-1)^{n+1}\cdot\frac{1}{\# G_x}$.

Proof. Let us choose projective models X of (X,x) respectively $\tilde{X}$ of $(\tilde{X},E)$, such that $\tilde{X}$ is smooth and $\sigma : \tilde{X} - E \to X - \{x\}$ is biholomorphic.

a) Set $\tilde{\mathscr{F}} := \Omega^1_{\tilde{X}}$, $\mathscr{F} := (\sigma_*\tilde{\mathscr{F}})^{\vee\vee}$, hence $\mathscr{F} = \Omega^1_X$. By Proposition 3.14, $c_n(x,\tilde{\mathscr{F}}) = c_n(\tilde{\mathscr{F}}) - c_n(\mathscr{F})$. Hence, using Theorem 2.14, $c_n(x,\tilde{\mathscr{F}}) = (-1)^n\cdot e(\tilde{X}) - (-1)^n\cdot e_{\mathrm{orb}}(X) = (-1)^n\cdot((e(\tilde{X}-E)+e(E)) - (e(X)-(1-\frac{1}{\# G_x})))$ $= (-1)^n\cdot(e(E)-\frac{1}{\# G_x})$.

b) Set $\tilde{\mathscr{F}} := \Omega^1_{\tilde{X}}(\log E)$, $\mathscr{F} := (\sigma_*\tilde{\mathscr{F}})^{\vee\vee}$, hence $\mathscr{F} = \Omega^1_X$. Then $c_n(x,\tilde{\mathscr{F}}) = c_n(\tilde{\mathscr{F}}) - c_n(\mathscr{F}) = (-1)^n\cdot e(\tilde{X}-E) - (-1)^n\cdot e_{\mathrm{orb}}(X) = (-1)^n\cdot(e(\tilde{X}-E) - (e(X) - (1-\frac{1}{\# G_x}))) = (-1)^n\cdot(-\frac{1}{\# G_x})$, where we used Proposition 3.14, Theorem 2.14, and [B-H-H 87], p. 236. $\square$

4 Local and global asymptotic Riemann–Roch Theorems for symmetric products of locally free sheaves

We discuss asymptotic RR Theorems, i.e. formulas for the asymptotic behaviour of the Euler characteristic of the symmetric products $S^k\mathscr{F}$, where k goes to infinity; the main results are Theorem 4.1 (local version) and Theorem 4.7 (global version). Theorem 4.1 is related to a conjecture of J. Wahl, cf. section (4.8). In [Wa 93], local Chern numbers for locally free rank two sheaves on resolutions of normal surface singularities are introduced; using Theorem 4.1, these are compared to our local Chern numbers in Proposition 4.10.

(4.1) Definition. *For every $n \in \mathbb{N}$, let $Q_n \in \mathbb{Z}[Z_1,\dots,Z_n]$ be the following universal homogeneous polynomial of degree n, where $\deg Z_i := i$ for all i. Q_n is the degree n-part of the formal power series $(1+Z_1+\dots+Z_n)^{-1} = \sum_{l\geqq 0}(-1)^l \cdot (Z_1+\dots+Z_n)^l \in \mathbb{Z}[[Z_1,\dots,Z_n]]$.*

Theorem. *Let $\sigma : (\tilde{X},E) \to (X,x)$ be a good resolution of an isolated quotient singularity of dimension n. Let $\tilde{\mathcal{F}}$ be a locally free sheaf of* rank r *on $\tilde{X}$ and assume that $\mathcal{F} := (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$ is a V-free sheaf on X. Then*

$$\lim_{k\to\infty} \frac{\chi(x,S^k\tilde{\mathcal{F}})}{k^{n+r-1}} = \frac{-1}{(n+r-1)!} \cdot Q_n(c_1(x,\tilde{\mathcal{F}}),\dots,c_n(x,\tilde{\mathcal{F}}))\,,$$

where $S^k\tilde{\mathcal{F}}$ denotes the k-th symmetric product of $\tilde{\mathcal{F}}$.

Recall that (X,x) is allowed to be smooth (i.e. isomorphic to $(\mathbb{C}^n,0)$), and that "good resolution" means that E is a normal crossing divisor and that $\sigma(E)=\{x\}$. See (3.9) for the Definition of the local holomorphic Euler characteristic $\chi(x,S^k\tilde{\mathcal{F}})$, see (3.10) and (3.12) for the Definition of the local Chern number $Q_n(c_1(x,\tilde{\mathcal{F}}),\dots,c_n(x,\tilde{\mathcal{F}})) = Q_n(c(x,\tilde{\mathcal{F}}))$. Notice that $(\sigma_*(S^k(\tilde{\mathcal{F}})))^{\vee\vee} \cong \hat{S}^k\mathcal{F} := (S^k\mathcal{F})^{\vee\vee}$ is V-free for all $k \in \mathbb{N}$ because $\mathcal{F}$ is V-free.

Remark. $(1+\sum_{i\geqq 1} Z_i)^{-1} = 1 - Z_1 + Z_1^2 - Z_2 - Z_1^3 + 2Z_1Z_2 - Z_3 + \dots$, and hence $Q_1 = -Z_1$, $Q_2 = Z_1^2 - Z_2$, $Q_3 = -Z_1^3 + 2Z_1Z_2 - Z_3, \dots$. (The Q_i's are the polynomials which give the Segre classes when applied to the Chern classes, cf. [Fu 84], p. 50.)

(4.2) Lemma. *Let R be a commutative ring, let $a \in R$, $b_i \in R$ (for all $i \in \mathbb{N}$) be given elements, let $b_0 := 0$. Set $P_0 := 1$, $B_0 := 0$, $A_0 := 1$. Define $P_{n+1} := -\sum_{1\leqq i\leqq n+1} P_{n+1-i} \cdot b_i$ for all $n \geqq 0$, $B_{n+1} := a \cdot B_n + P_n$ for all $n \geqq 0$, $A_{n+1} := a \cdot A_n + b_{n+1}$ for all $n \geqq 0$. Then for all $n \geqq 1$ the following equality holds in R:*

$$A_n \cdot B_n - a^{2n-1} + a^{n-1} \cdot P_n = \sum_{0\leqq i\leqq n-2} \left(a^i \cdot \sum_{0\leqq j\leqq i} P_{n-1-i+j} \cdot b_{n-j} \right) .$$

Proof. For $n=1$, this is trivial. Assume that the equality is valid for n. We have to show that it is then valid for $n+1$. Now

$$\begin{aligned}
&A_{n+1}B_{n+1} - a^{2n+1} + a^nP_{n+1} \\
&= a^2(A_nB_n - a^{2n-1}) + aA_nP_n + ab_{n+1}B_n + b_{n+1}P_n + a^nP_{n+1} \\
&= a^2\left(\sum_{0\leqq i\leqq n-2}\left(a^i \sum_{0\leqq j\leqq i} P_{n-1-i+j}b_{n-j}\right) - a^{n-1}P_n\right) + a\sum_{0\leqq i\leqq n} a^ib_{n-i}P_n \\
&\quad + ab_{n+1}\sum_{0\leqq i\leqq n-1} a^iP_{n-1-i} + b_{n+1}P_n + a^nP_{n+1} \\
&=: \sum_{0\leqq i\leqq n+1} a^i \cdot \kappa_i\,.
\end{aligned}$$

We used that $B_{n+1} = \sum_{0\leqq l\leqq n} a^l \cdot P_{n-l}$ and that $A_n = \sum_{0\leqq l\leqq n} a^l \cdot b_{n-l}$ for all $n \geqq 1$. Then it is easy to see that $\kappa_i = \sum_{0\leqq j\leqq i} P_{n-i+j} \cdot b_{n+1-j}$ for all $i = 0,\ldots,n-1$ and that $\kappa_n = \kappa_{n+1} = 0$; and this proves the claim. □

(4.3) Lemma. *Let X be a compact manifold of dimension n, let $\mathscr{F}$ be a locally free sheaf of* rank r *on X. Let $Y := \mathbb{P}(\mathscr{F}) \xrightarrow{p} X$ be the projectivization of $\mathscr{F}$. Let $\zeta \in H^2_{dR}(Y,\mathbb{C})$ be the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$. Then $\int_Y \zeta^{r-1} \wedge p^*\eta = \int_X \eta$ for all $\eta \in \Gamma(X,\mathscr{E}^{2n})$.*

Proof. Let $X = \bigcup_i U_i$ be a covering of X by open subsets such that $\mathscr{F}|_{U_i}$ is trivial for all i. Let $\sum_i \phi_i \equiv 1$ be a partition of unity subordinate to $\{U_i\}$. Then $\int_X \eta = \sum_i \int_{U_i} \phi_i\eta$ and $\int_Y \zeta^{r-1} p^*\eta = \sum_i \int_{V_i} \zeta^{r-1} p^*(\phi_i\eta)$, where $V_i := p^{-1}(U_i)$. Now $V_i \cong U_i \times \mathbb{P}_{r-1}$ for all i and $\int_{p^{-1}(x)} \zeta^{r-1} = 1$ for all $x \in X$. Hence $\int_{V_i} \zeta^{r-1} p^*(\phi_i\eta) = \int_{U_i} (\phi_i\eta \cdot \int_{\mathbb{P}_{r-1}} \zeta^{r-1}) = \int_{U_i} \phi_i\eta$, and this proves the claim. □

(4.4) Proposition. *Let X be a manifold of dimension n, let $\mathscr{F}$ be a locally free sheaf of* rank r *on X. Then*

$$\lim_{k\to\infty} \frac{\chi(X,S^k\mathscr{F})}{k^{n+r-1}} = \frac{1}{(n+r-1)!} \cdot Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F}))\,.$$

Remark. In case $n = 2$, this was proved by Bogomolov by different methods, cf. [Bo 79], p. 535; the formula is then $\lim_{k\to\infty} \frac{\chi(X,S^k\mathscr{F})}{k^{r+1}} = \frac{1}{(r+1)!}(c_1(\mathscr{F})^2 - c_2(\mathscr{F}))$.

Proof. Set $Y := \mathbb{P}(\mathscr{F})$, $\mathscr{L} := \mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$, $m := n+r-1 = \dim Y$, $\zeta := c_1(\mathscr{L}) \in H^2_{dR}(Y,\mathbb{C})$. Then $\chi(X,S^k\mathscr{F}) = \chi(Y,\mathscr{L}^{\otimes k})$, cf. [B-P-V 84], Theorem I.5.1, and $\lim_{k\to\infty} \chi(Y,\mathscr{L}^{\otimes k}) = \frac{1}{m!}\zeta^m$, cf. [B-P-V 84], p. 21.

Hence we have to show that $\int_Y c_1(\mathscr{L})^m = \int_X Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F}))$. By Lemma 4.3, $\int_X Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F})) = \int_Y p^*Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F})) \cdot \zeta^{r-1}$, and thus it is sufficient to show that

$$Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F})) \cdot c_1(\mathscr{L})^{r-1} = c_1(\mathscr{L})^m \quad \text{in } H^{2m}_{dR}(Y,\mathbb{C})\,.$$

Let us assume, in the following, that $r \geqq n$; then we have $4n-2 \leqq 2m$. (If $r < n$, a similar proof works.) Set $a := c_1(\mathscr{L}), b_1 := c_1(\mathscr{F}),\ldots,b_n := c_n(\mathscr{F})$. Then it follows from Lemma 4.2 that $A_n \cdot B_n = (\sum_{0\leqq l\leqq n} c_1(\mathscr{L})^l \cdot c_{n-l}(\mathscr{F})) \cdot B_n = c_1(\mathscr{L})^{2n-1} - c_1(\mathscr{L})^{n-1} \cdot Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F}))$ in $H^{4n-2}_{dR}(Y,\mathbb{C})$ for a suitable element $B_n \in H^{2n-2}_{dR}(Y,\mathbb{C})$. (The right hand side of the formula of Lemma 4.2 vanishes in $H^{4n-2}_{dR}(Y,\mathbb{C})$ because $\deg(P_{n-1-i+j}(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F})) \cdot c_{n-j}(\mathscr{F})) = 2n-1-i > n$ for all $i \leqq n-2$.)

Now $A_n = 0$ in $H^{2n}_{dR}(Y,\mathbb{C})$, and hence $c_1(\mathscr{L})^{2n-1} = c_1(\mathscr{L})^{n-1} \cdot Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F}))$ in $H^{4n-2}_{dR}(Y,\mathbb{C})$. We multiply this equality by $c_1(\mathscr{L})^{m-2n+1}$, and obtain $c_1(\mathscr{L})^m = c_1(\mathscr{L})^{m-n} \cdot Q_n(c_1(\mathscr{F}),\ldots,c_n(\mathscr{F}))$. □

Corollary. *Let* ch_n *be the homogeneous polynomial of degree n which is the degree n-part of the Chern series* ch. *Then, in the situation of* *the Proposition*:

$$\lim_{k\to\infty} \frac{1}{k^{n+r-1}} \cdot \int_X \mathrm{ch}(S^k\mathscr{F}) \cdot \mathrm{td}(X) = \lim_{k\to\infty} \frac{1}{k^{n+r-1}} \cdot \mathrm{ch}_n(S^k\mathscr{F})$$
$$= \frac{1}{(n+r-1)!} \cdot Q_n(\mathscr{F}) \ (\textit{in } H^{2n}_{dR}(X, \mathbb{C}) \cong \mathbb{C}) .$$

Remark. This reverses the order of Bogomolov's proof of the Proposition in case $n = 2$: He first determines the asymptotic behaviour of $\mathrm{ch}(S^k\mathscr{F}) \cdot \mathrm{td}(X)$ (using the splitting principle for the calculation of the $c_i(S^k\mathscr{F})$'s in terms of the $c_j(\mathscr{F})$'s) and then deduces the proposition.

(4.5) Lemma. *Let (X,x) be an isolated quotient singularity. Then there is a rational constant $v(X,x)$ such that $|\mu_{X,x}(\mathscr{F})| \leqq \mathrm{rank}\,\mathscr{F} \cdot v(X,x)$ for all V-free sheaves $\mathscr{F}$ on (X,x).*

Proof. By Corollary 2.6, we know that there is a finite set $\{\mathscr{F}_1, \ldots, \mathscr{F}_t\}$ of V-free sheaves on (X,x) such that every $\mathscr{F}$ is isomorphic to a sum $\mathscr{F}_1^{\oplus l_1} \oplus \cdots \oplus \mathscr{F}_t^{\oplus l_t}$ for suitable $l_i \in \mathbb{N}_0$. Let us assume that $\frac{|\mu_{X,x}(\mathscr{F}_1)|}{\mathrm{rank}\,\mathscr{F}_1} \geqq \frac{|\mu_{X,x}(\mathscr{F}_i)|}{\mathrm{rank}\,\mathscr{F}_i}$ for all i. Then $\mu_{X,x}(\mathscr{F}) = \sum_{1\leqq i\leqq t} l_i \cdot \mu_{X,x}(\mathscr{F}_i)$, and hence

$$|\mu_{X,x}(\mathscr{F})| \leqq \sum l_i \cdot |\mu_{X,x}(\mathscr{F}_i)| = \sum l_i \cdot \mathrm{rank}\,\mathscr{F}_i \cdot \frac{|\mu_{X,x}(\mathscr{F}_i)|}{\mathrm{rank}\,\mathscr{F}_i}$$
$$\leqq \mathrm{rank}\,\mathscr{F} \cdot \frac{|\mu_{X,x}(\mathscr{F}_1)|}{\mathrm{rank}\,\mathscr{F}_1} =: \mathrm{rank}\,\mathscr{F} \cdot v(X,x). \qquad \square$$

Corollary. $\lim_{k\to\infty} \frac{1}{k^r} \cdot \mu_{X,x}(\hat{S}^k\mathscr{F}) = 0$ *for every V-free sheaf $\mathscr{F}$ on X, where $r := \mathrm{rank}\,\mathscr{F}$.*

Proof. $\lim_{k\to\infty} \frac{1}{k^r} \cdot \mathrm{rank}\,\hat{S}^k\mathscr{F} = 0$. $\square$

(4.6) *Proof of Theorem 4.1.* Let $Y \xrightarrow{\pi} Z = Y/G$ be a globalization of (X,x) as in Lemma 3.6, let $\mathscr{F}_Y$ and $\mathscr{F}_Z$ be the respective globalizations of $\mathscr{F}$. Let $\sigma : \tilde{Z} \to Z$ be the induced resolution of Z, $\mathscr{F}_{\tilde{Z}}$ the induced sheaf on $\tilde{Z}$. Set $m := n + r - 1$. Set $N := \#\,\mathrm{Sing}\,Z$. By Proposition 4.4, we know that $\lim_{k\to\infty} \frac{1}{k^m}\chi(Y, S^k\mathscr{F}_Y) = \frac{1}{m!}Q_n(\mathscr{F}_Y)$ and that $\lim_{k\to\infty} \frac{1}{k^m}\chi(\tilde{Z}, S^k\mathscr{F}_{\tilde{Z}}) = \frac{1}{m!}Q_n(\mathscr{F}_{\tilde{Z}})$. Now $\chi(Z, \hat{S}^k\mathscr{F}_Z) = \chi_{\mathrm{orb}}(Z, \hat{S}^k\mathscr{F}_Z) + N \cdot \mu_{X,x}(\hat{S}^k\mathscr{F})$, and hence $\lim_{k\to\infty} \frac{1}{k^m}\chi(Z, \hat{S}^k\mathscr{F}_Z) = \lim_{k\to\infty} \frac{1}{k^m}\chi_{\mathrm{orb}}(Z, \hat{S}^k\mathscr{F}_Z)$, by Lemma 4.5. Thus

$$\lim_{k\to\infty} \frac{1}{k^m}\chi(Z, \hat{S}^k\mathscr{F}_Z) = \lim_{k\to\infty} \frac{1}{k^m}\chi_{\mathrm{orb}}(Z, \hat{S}^k\mathscr{F}_Z) = \lim_{k\to\infty} \frac{1}{k^m}\frac{1}{\# G}\chi_{\mathrm{orb}}(Y, \hat{S}^k\mathscr{F}_Y)$$
$$= \frac{1}{\# G}\lim_{k\to\infty} \frac{1}{k^m}\chi(Y, \hat{S}^k\mathscr{F}_Y) = \frac{1}{\# G}\frac{1}{m!}Q_n(\mathscr{F}_Y) = \frac{1}{m!}Q_n(\mathscr{F}_Z).$$

Then

$$\lim_{k\to\infty} \frac{1}{k^m}\chi(x, S^k\tilde{\mathscr{F}}) = \lim_{k\to\infty} \frac{1}{k^m}\frac{1}{N}(\chi(Z, \hat{S}^k\mathscr{F}_Z) - \chi(\tilde{Z}, S^k\mathscr{F}_{\tilde{Z}}))$$
$$= \frac{1}{m!}\frac{1}{N}(Q_n(\mathscr{F}_Z) - Q_n(\mathscr{F}_{\tilde{Z}})) = \frac{-1}{m!}Q_n(c(x, \tilde{\mathscr{F}})),$$

where we used Lemma 3.9 and Proposition 3.14. $\square$

(4.7) We now generalize Proposition 4.4:

Theorem. *Let X be a compact Vi-manifold of dimension n. Let $\mathscr{F}$ be a locally V-free sheaf of* rank r *on X. Then*

$$\lim_{k\to\infty} \frac{\chi(X,S^k\mathscr{F})}{k^{n+r-1}} = \frac{1}{(n+r-1)!} \cdot Q_n(c_1(\mathscr{F}),\dots,c_n(\mathscr{F}))\,.$$

Proof. This follows from Theorem 4.1 and Proposition 4.4 by arguments which were already used in the proof of Theorem 3.5. □

(4.8) In sections (4.8) to (4.10) we want to compare some of our concepts and results to some concepts and results of J. Wahl, cf. [Wa 93]. There, Wahl considers the following situation: Let (X,x) be a normal surface singularity, let $\sigma:(\tilde{X},E)\to(X,x)$ be a good resolution (in the sense that E is a normal crossing divisor), let $\tilde{\mathscr{F}}$ be a locally free sheaf of rank 2 on $(\tilde{X},E)$. Then he defines local Chern numbers $\tilde{c}_1^2(x,\tilde{\mathscr{F}})$ (in $\mathbb{Q}$) and $\tilde{c}_2(x,\tilde{\mathscr{F}})$ (in $\mathbb{R}$), and he conjectures (and proves in some special cases) an asymptotic RR Formula for $\chi(x,S^k\tilde{\mathscr{F}}) = \dim\Gamma((\tilde{X},E)-E,S^k\tilde{\mathscr{F}})/\Gamma((\tilde{X},E),S^k\tilde{\mathscr{F}}) + h^0(R^1\sigma_*(S^k\tilde{\mathscr{F}}))$.

Definition ([Wa 93]).

a) *Let $\tilde{\mathscr{L}}$ be an invertible sheaf on $(\tilde{X},E)$. Then there is a unique divisor $D=\sum\alpha_iE_i$ ($\alpha_i\in\mathbb{Q}$ for all i) such that $E_j\cdot D=\deg_{E_j}\tilde{\mathscr{L}}$ for all j. Then set $\tilde{c}_1^2(x,\tilde{\mathscr{L}}):=D^2$. Similarly one defines a rational number $\tilde{c}_1(x,\tilde{\mathscr{L}})\cdot\tilde{c}_1(x,\tilde{\mathscr{M}})$ for invertible sheaves $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{M}}$ on $(\tilde{X},E)$.*

b) *Let $\tilde{\mathscr{F}}$ be a locally free sheaf on $(\tilde{X},E)$. Then $\tilde{c}_1^2(x,\tilde{\mathscr{F}}):=\tilde{c}_1^2(x,\det\tilde{\mathscr{F}})$.*

c) *Let $\tilde{\mathscr{F}}$ be a locally free sheaf of rank 2 on $(\tilde{X},E)$. Set*

$$\tilde{c}_2(x,\tilde{\mathscr{F}}) := \inf\left\{\frac{\tilde{c}_1(x,\tilde{\mathscr{L}})\cdot\tilde{c}_1(x,\tilde{\mathscr{M}})}{\deg f}\right\},$$

where one takes the infimum over all triples $((\tilde{Y},F)\xrightarrow{f}(\tilde{X},E),\tilde{\mathscr{L}},\tilde{\mathscr{M}})$ such that f is a generically finite holomorphic mapping and such that there is an exact sequence $0\to\tilde{\mathscr{L}}\to f^*\tilde{\mathscr{F}}\to\tilde{\mathscr{M}}\to 0$, where $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{M}}$ are invertible sheaves on $(\tilde{Y},F)$, with $F=f^{-1}(E)$. Here, $(\tilde{Y},F)$ means the germ of the smooth surface $\tilde{Y}$ along the compact curve $F\subset\tilde{Y}$.

Conjecture ([Wa 93]). *Let $\tilde{\mathscr{F}}$ be a locally free sheaf of rank 2 on $(\tilde{X},E)$. Then*

$$\lim_{k\to\infty}\frac{1}{k^3}\cdot\chi(x,S^k\tilde{\mathscr{F}}) = \frac{-1}{6}\cdot(\tilde{c}_1^2(x,\tilde{\mathscr{F}})-\tilde{c}_2(x,\tilde{\mathscr{F}}))\,.$$

Thus, the case $(n,r)=(2,2)$ of our Theorem 4.1 is exactly the Wahl-Conjecture for quotient singularities with respect to our local Chern numbers.

(4.9) Lemma. *Let (X,x) be a quotient surface germ, let $\sigma:(\tilde{X},E)\to(X,x)$ be a resolution, let $\tilde{\mathscr{F}}$ be a locally free sheaf on $(\tilde{X},E)$. Then $\tilde{c}_1^2(x,\tilde{\mathscr{F}}) = c_1^2(x,\tilde{\mathscr{F}})$, where $\tilde{c}_1^2$ is as in (4.8), c_1^2 as in (3.10) and (3.12).*

Proof. We may globalize: Z a projective Vi-manifold of dimension 2 such that $\operatorname{Sing} Z=\{z_0\}$ and $(Z,z_0)\cong(X,x)$, $\tilde{Z}\xrightarrow{\sigma}Z$ the induced resolution, $\mathscr{F}_{\tilde{Z}}$ a locally free sheaf on $\tilde{Z}$ such that $\mathcal{F}_{\tilde{Z}}|_{(\tilde{Z},E)} \cong \tilde{\mathcal{F}}$. Set $\mathcal{L}_{\tilde{Z}} := \det(\mathcal{F}_{\tilde{Z}})$, $\tilde{\mathcal{L}} := \det(\tilde{\mathcal{F}})$; hence $\tilde{\mathcal{L}} = \mathcal{L}_{\tilde{Z}}|_{(\tilde{Z},E)}$. There is a divisor $\tilde{D}$ on $\tilde{Z}$ such that $\mathcal{L}_{\tilde{Z}} \cong \mathcal{O}_{\tilde{Z}}(\tilde{D})$. Then $(\sigma_*\mathcal{L}_{\tilde{Z}})^{\vee\vee} \cong \mathcal{O}_Z(D)$, with $D := \sigma_*\tilde{D}$, a Weil divisor on Z. Let σ^*D be the numerical lifting of D, write $\sigma^*D = \tilde{D} - D^s$, hence D^s is a $\mathbb{Q}$-divisor with $\operatorname{supp} D^s \subseteq E$ and $D^s \cdot E_i = \tilde{D} \cdot E_i$ for all i. Thus $\tilde{c}_1^2(x,\tilde{\mathcal{F}}) = (D^s)^2$. On the other hand, $-c_1^2(x,\tilde{\mathcal{F}}) = c_1^2(\mathcal{F}_Z) - c_1^2(\mathcal{F}_{\tilde{Z}}) = c_1^2(\mathcal{L}_Z) - c_1^2(\mathcal{L}_{\tilde{Z}}) = D^2 - \tilde{D}^2 = -(D^s)^2$; here we used Proposition 3.14; and we used $\tilde{D}^2 = D^2 + (D^s)^2$, which follows from $\tilde{D}^2 = (\sigma^*D + D^s)^2 = (\sigma^*D)^2 + (D^s)^2 = D^2 + (D^s)^2$. □

(4.10) Proposition. *Let $\sigma : (\tilde{X},E) \to (X,x)$ be a resolution of a quotient surface singularity, let $\tilde{\mathcal{F}}$ be a locally free sheaf of rank 2 on $(\tilde{X},E)$. Then the following two conditions are equivalent:*

(i) $\tilde{c}_2(x,\tilde{\mathcal{F}}) = c_2(x,\tilde{\mathcal{F}})$

(ii) *the Wahl-Conjecture is true for the pair $((\tilde{X},E),\tilde{\mathcal{F}})$.*

Here, $\tilde{c}_2$ is as in (4.8), *and c_2 is as in* (3.10) *and* (3.12).

Proof. This is a direct consequence of Theorem 4.1 together with Lemma 4.9. □

Corollary. *If, in the situation of the Proposition, the local fundamental group of (X,x) is cyclic (in other words: $G_x \cong \mathbb{Z}_m$ for some $m \in \mathbb{N}$), then $\tilde{c}_2(x,\tilde{\mathcal{F}}) = c_2(x,\tilde{\mathcal{F}})$.*

Proof. For a cyclic group, every irreducible representation is of rank one, hence $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{M}$ for suitable divisorial sheaves $\mathcal{L}$ and $\mathcal{M}$ on (X,x), and thus $\tilde{\mathcal{F}}|_{(\tilde{X},E)-E} \cong \tilde{\mathcal{L}}|_{(\tilde{X},E)-E} \oplus \tilde{\mathcal{M}}\,|_{(\tilde{X},E)-E}$ for invertible sheaves $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{M}}$ on $(\tilde{X},E)$. But in this case, the Wahl-Conjecture is true, cf. [Wa 93], p. 83, and hence the claim follows. □

Remark. In the situation of the Proposition, let $\tilde{\mathcal{F}} = \Omega^1_{\tilde{X}}(\log E)$.

Then $\tilde{c}_2(x,\tilde{\mathcal{F}}) = -\frac{1}{\# G_x} = c_2(x,\tilde{\mathcal{F}})$, by Proposition 3.18 and by [Wa 93], p. 82.

(4.11) In view of the definition of $\tilde{c}_2(x,\tilde{\mathcal{F}})$, the following result may be of interest:

Lemma. *Let (X,x) be a quotient surface germ, let $\sigma : (\tilde{X},E) \to (X,x)$ be a resolution, let $\tilde{\mathcal{F}}$ be a locally free sheaf of rank 2 on $(\tilde{X},E)$, let $(Y,y) \xrightarrow{\pi} (X,x)$ be the smoothing covering, let $(\tilde{Y},\tilde{E})$ be the minimal resolution of the normalization of $(\tilde{X},E) \times_{(X,x)} (Y,y)$; let*

$$\begin{array}{ccc} (\tilde{Y},\tilde{E}) & \xrightarrow{\tau} & (Y,y) \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\pi} \\ (\tilde{X},E) & \xrightarrow{\sigma} & (X,x) \end{array}$$

be the resulting commutative diagram.