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Intrinsic atomic characterizations of function spaces on domains

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1 Introduction

The two scales of function spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ on the euclidean n-space $\mathbb{R}^n$ with $s \in \mathbb{R}$, $0 < p \leqq \infty$ ($p < \infty$ for the F-scale), $0 < q \leqq \infty$, cover many well-known classical spaces:

(1) the Hölder–Zygmund spaces $\mathscr{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ with $s > 0$;

(2) the classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$ with $s > 0$, $1 < p < \infty$, $1 \leqq q \leqq \infty$;

(3) the fractional Sobolev spaces $H^s_p(\mathbb{R}^n) = F^s_{p2}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $1 < p < \infty$, where $W^s_p(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ with $s \in \mathbb{N}_0$, $1 < p < \infty$, are the classical Sobolev spaces

(4) and the (inhomogeneous) Hardy spaces $h_p(\mathbb{R}^n) = F^0_{p2}(\mathbb{R}^n)$ with $0 < p < \infty$.

Corresponding spaces on domains Ω in $\mathbb{R}^n$ have been studied in great detail, especially, of course, the spaces mentioned in (1)–(3). There are two possibilities:

(i) spaces on Ω are defined by restriction of corresponding spaces on $\mathbb{R}^n$;

(ii) intrinsic definition of the corresponding spaces on Ω.

Both possibilities have their advantages and disadvantages, and the related problems are quite clear.

As for (i), defining the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ by restriction, leads more or less immediately to the problem of an intrinsic characterization of these spaces. The state of the art has been described in [34], Ch. 5. If Ω is a bounded C^∞-domain in $\mathbb{R}^n$ and $s > n(\frac{1}{p}-1)_+$ then one has some intrinsic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ via oscillations, derivatives and differences of functions. There is not much known in this generality if Ω is non-smooth or if $s < n(\frac{1}{p}-1)_+$. It might be considered as the main aim of this paper to seal the gap by providing intrinsic atomic characterizations for all these spaces under very mild and natural restrictions on the (non-smooth) domain Ω. A second problem connected with the approach (i) is the question whether there exists a (linear bounded) extension operator from the spaces on Ω into the corresponding spaces on $\mathbb{R}^n$. If Ω is a bounded C^∞-domain in $\mathbb{R}^n$ then one has a satisfactory affirmative answer for all these spaces, see [34], 5.1.3. If Ω is non-smooth the situation is more delicate, but we shall not discuss this problem in this paper.

The second above-mentioned possibility (ii), defining directly spaces on Ω, is apparently restricted to some of these spaces. A lot has been done to study Sobolev spaces $W_p^k(\Omega)$ with $k \in \mathbb{N}$ and $1 < p < \infty$ (occasionally including $p = 1$ and $p = \infty$) in non-smooth domains, especially whether there exists a linear extension operator, see [23] and [24] and the references given there. Recall Whitney's intrinsic description of Hölder spaces (Lipschitz spaces) on closed sets in $\mathbb{R}^n$, and the construction of a linear and bounded extension operator to corresponding spaces on $\mathbb{R}^n$, see e.g. [28], Ch. 6. Extending this procedure Jonsson and Wallin studied more general spaces, especially Besov spaces, but also Hardy spaces on closed sets in $\mathbb{R}^n$, see [19], [36]. If there exist extension operators in these cases, then one has on admissible domains not only atomic characterizations but also characterizations via derivatives or differences of functions, approximation procedures etc.

The plan of the paper is the following. Sect. 2 deals with spaces on $\mathbb{R}^n$: definitions and atomic characterizations, where the latter is a modification of what has been done by Frazier and Jawerth, see [13] and [14], adapted to our needs. Sect. 3 deals with spaces on domains. First we describe the type of domains we have in mind and compare our definition with other proposals in the relevant literature. In particular if Ω is a bounded (ε, δ)-domain in $\mathbb{R}^n$ according to [16] which coincides with the interior of its closure, then it belongs to the class $IR(n)$ introduced in Definition 3.2(i), see also the more detailed discussion in 3.2. In 3.5 and 3.6 we prove our main result, the intrinsic atomic characterizations. The arguments are quite simple but we obtain a rather satisfactory and final answer. We wish to emphasize that our paper should also be considered as a contribution to the delicate question of adequate definitions and formulations of what is meant by atoms and atomic representations and which (non-smooth) domains should naturally be treated in this context. In particular we introduce in 3.3 the interior and boundary atoms in domains. By Definition 3.3/2(ii) an interior atom is essentially the same as its counterpart on $\mathbb{R}^n$ according to Definition 2.2(ii), inclusively the cancellation condition (3.3/9), which reflects the desired behavior of the Fourier transform of these types of atoms near the origin. As for the boundary atoms in domains introduced in Definition 3.3/2(iii) this cancellation condition is no longer needed. This is the point where the type of the underlying domain comes in: one tries to extend these boundary atoms from $\overline{\Omega}$ to $\mathbb{R}^n$ such that the extended atoms satisfy the necessary cancellation conditions. Boundary atoms of similar type may also be found in [25] and [1], where atomic decompositions of Hardy spaces $h_p(\Omega)$ with $0 \leqq p \leqq 1$ in some types of domains were considered. In these two papers the related atoms extend the classical atoms for Hardy spaces on $\mathbb{R}^n$. Our approach is different. As far as $\mathbb{R}^n$ is concerned we rely essentially on what has been done in [12], [13] and [14].

Section 4 complements the obtained results. In 4.1 we add a remark about C^∞-domains. In 4.2 we give a rather final description of the entropy numbers of compact embeddings between the considered function spaces. Atomic characterizations on the one hand and the correct asymptotics of entropy numbers on the other hand pave the way to a spectral theory of non-smooth (integral, partial differential, pseudodifferential) operators in non-smooth domains. We shift this task to a later paper. But in 4.2 we wish to provide an understanding why atomic decompositions and entropy numbers can be used to study spectral properties of non-smooth and degenerate operators. Finally, in 4.3 we add some complements, in order to compare our results with what is known in literature. But we restrict ourselves mostly to references.

All unimportant positive constants are denoted by c, occasionally with additional subscripts within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l, whereas (j) means formula (j) in the same subsection. Similarly we refer to remarks, theorems etc.

2 Function spaces on R^n

2.1 *Definitions*

Let $\mathbb{R}^n$ be the euclidean n-space. The Schwartz space $S(\mathbb{R}^n)$ and its dual space $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have the usual meaning here. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leqq \infty$, is the usual quasi-Banach space with respect to Lebesgue measure, quasi-normed by $\| \cdot | L_p(\mathbb{R}^n)\|$. Let $\varphi \in S(\mathbb{R}^n)$ be such that

$$(1) \qquad \operatorname{supp} \varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \text{ if } |x| \leqq 1 \, ;$$

let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for each $j \in \mathbb{N}$ (natural numbers) and put $\varphi_0 = \varphi$. Then since $1 = \sum_{j=0}^{\infty} \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the φ_j form a dyadic resolution of unity. Let $\hat{f}$ and $\check{f}$ be the Fourier transform and its inverse, respectively, of $f \in S'(\mathbb{R}^n)$. Then for any $f \in S'(\mathbb{R}^n)$, $(\varphi_j \hat{f})\check{}$ is an entire analytic function on $\mathbb{R}^n$.

Definition. (i) *Let $s \in \mathbb{R}$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$(2) \qquad \|f \,|\, B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})\check{}\,|\, L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}}$$

(*with the usual modification if $q = \infty$*) *is finite.*

(ii) *Let* $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqq \infty$. *Then* $F^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in S'(\mathbb{R}^n)$ *such that*

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left|(\varphi_j \hat{f})\check{}(\,\cdot\,)\right|^q \right)^{\frac{1}{q}} \,\middle|\, L_p(\mathbb{R}^n) \right\| \tag{3}$$

(*with the usual modification if* $q = \infty$) *is finite.*

Remark. Systematic treatments of the theory of these spaces may be found in [33] and [34]. In particular, both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces which are independent of the function $\varphi \in S(\mathbb{R}^n)$ chosen according to (1), in the sense of equivalent quasi-norms. This justifies our omission of the subscript φ in (2) and (3) in what follows. If $p \geqq 1$ and $q \geqq 1$ both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are Banach spaces. These two scales cover many well-known classical spaces, mentioned briefly in the introduction (1/1)–(1/4). Recall that for $k \in \mathbb{N}$ and $1 < p < \infty$

$$\|f \mid W^k_p(\mathbb{R}^n)\| = \sum_{|\alpha| \leqq k} \|D^\alpha f \mid L_p(\mathbb{R}^n)\| \tag{4}$$

is an equivalent norm in the Sobolev space $W^k_p(\mathbb{R}^n) = F^k_{p2}(\mathbb{R}^n)$. Of peculiar interest for us will be the Hölder spaces $\mathscr{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ with $0 < s = [s] + \{s\}$, where $[s] \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $0 < \{s\} < 1$, with the equivalent norm

$$\|f \mid \mathscr{C}^s(\mathbb{R}^n)\| = \sum_{|\alpha| \leqq [s]} \|D^\alpha f \mid L_\infty(\mathbb{R}^n)\| + \sum_{|\alpha| = [s]} \sup \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}}, \tag{5}$$

where the supremum is taken over all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $x \neq y$.

2.2 *Atoms*

We adapt the atoms introduced by Frazier and Jawerth in [12], [13] and [14] to our later purposes. Let again $\mathbb{N}$ be the natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components. Let $b > 0$ be given, $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$. Then $Q_{\nu k}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes, centered at $x^{\nu,k} \in \mathbb{R}^n$ with

$$|x^{\nu,k} - 2^{-\nu}k| \leqq b\,2^{-\nu} \tag{1}$$

and with side-length $2^{-\nu}$. Let Q be a cube in $\mathbb{R}^n$ and $r > 0$, then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times the side-length of Q. We always tacitly assume in the sequel, that $d > 0$ is chosen in dependence on b such that for all choices $\nu \in \mathbb{N}_0$ and all choices of $x^{\nu,k}$ in (1)

$$\bigcup_{k \in \mathbb{Z}^n} dQ_{\nu k} = \mathbb{R}^n\,. \tag{2}$$

Recall that $\mathscr{C}^\sigma(\mathbb{R}^n)$ with $0 < \sigma \notin \mathbb{N}$ may be normed by (2.1/5). Let $\mathscr{C}^0(\mathbb{R}^n)$ be the usual space of all complex-valued bounded continuous functions on $\mathbb{R}^n$ equipped with the L_∞-norm.

Definition. (i) *Let $0 \leqq \sigma \notin \mathbb{N}$. Then $a(x)$ is called a 1-atom (or more precisely 1_σ-atom) if*

(3) $$\operatorname{supp} a \subset d\, Q_{0k} \quad \textit{for some } k \in \mathbb{Z}^n$$

and

(4) $$a \in \mathscr{C}^\sigma(\mathbb{R}^n) \quad \textit{with } \|a\,|\,\mathscr{C}^\sigma(\mathbb{R}^n)\| \leqq 1\,.$$

(ii) *Let $s \in \mathbb{R}$ and $0 < p \leqq \infty$ Let $0 \leqq \sigma \notin \mathbb{N}$ and $L+1 \in \mathbb{N}_0$. Then $a(x)$ is called an (s,p)-atom (or more precisely $(s,p)_{\sigma,L}$-atom) if*

(5) $$\operatorname{supp} a \subset dQ_{\nu k} \quad \textit{for some } \nu \in \mathbb{N} \textit{ and some } k \in \mathbb{Z}^n\,,$$

(6) $$a \in \mathscr{C}^\sigma(\mathbb{R}^n) \quad \textit{with } \|a(2^{-\nu}\cdot)\,|\,\mathscr{C}^\sigma(\mathbb{R}^n)\| \leqq 2^{-\nu(s-\frac{n}{p})}$$

and

(7) $$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0 \quad \textit{for } |\beta| \leqq L\,.$$

Remark 1 We begin with some technical explanations. The number d has the above meaning, see (2), and is assumed to be fixed throughout this paper. Recall that Q_{0k} is a cube with side-length 1. If a satisfies (5) and (6) then

(8) $$2^{\nu(s-\frac{n}{p})}a(2^{-\nu}x)$$

is a 1_σ-atom, inclusively (3) with $a(2^{-\nu}\cdot)$ instead of a. Recall $x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$ if $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and β is a multi-index. Then the moment conditions (7) are equivalent to

(9) $$(D^\beta \hat{a})(0) = 0 \quad \text{for } |\beta| \leqq L\,.$$

If $L = -1$, then (7) simply means that there are no moment conditions. The reason for the normalizing factors in (6) (and also in (4)) is that there exists a constant c such that for all these atoms

(10) $$\|a\,|\,B^s_{pq}(\mathbb{R}^n)\| \leqq c\,, \qquad \|a\,|\,F^s_{pq}(\mathbb{R}^n)\| \leqq c\,.$$

In other words, atoms are normalized smooth building blocks, satisfying some moment conditions.

Remark 2 The above definition is adapted to our later needs, where we carry over this definition from $\mathbb{R}^n$ to (bounded, non-smooth) domains Ω in $\mathbb{R}^n$. Then the Whitney extension plays a decisive role. This explains why we used $\mathscr{C}^\sigma$ with $0 \leqq \sigma \notin \mathbb{N}$. In case of $\mathbb{R}^n$ one would otherwise prefer C^K, the space of all $f \in \mathscr{C}^0$ with $D^\alpha f \in \mathscr{C}^0$ if $|\alpha| \leqq K$. Doing so the normalizing condition in (6) can be re-written as

$$(11) \qquad |D^\alpha a(x)| \leqq 2^{-\nu(s-\frac{n}{p})} 2^{\nu|\alpha|}, \quad |\alpha| \leqq K .$$

This is the usual way to introduce atoms, see the above mentioned papers by Frazier and Jawerth or [34], 1.9.2, 3.2.2.

2.3 *Atomic characterizations*

First we introduce two sequence spaces. Let

$$(1) \qquad \lambda = \{\lambda_{\nu k} : \lambda_{\nu k} \in \mathbb{C},\ \nu \in \mathbb{N}_0 \text{ and } k \in \mathbb{Z}^n\}$$

and let $\chi_{\nu k}^{(p)}$ be the p-normalized characteristic function with respect to the above cube $Q_{\nu k}$, that means

$$(2) \qquad \chi_{\nu k}^{(p)}(x) = 2^{\frac{\nu n}{p}} \text{ if } x \in Q_{\nu k} \quad \text{and} \quad \chi_{\nu k}^{(p)}(x) = 0 \quad \text{if } x \in \mathbb{R}^n \setminus Q_{\nu k} ,$$

where $0 < p \leqq \infty$.

Definition. *Let $0 < p \leqq \infty$ and $0 < q \leqq \infty$.*

(i) *Then b_{pq} is the collection of all sequences λ given by (1) such that*

$$(3) \qquad \|\lambda \,|\, b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{k \in \mathbb{Z}^n} |\lambda_{\nu k}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

(with the usual modifications if p and/or q is infinite) is finite.

(ii) *Then f_{pq} is the collection of all sequences λ given by (1) such that*

$$(4) \qquad \|\lambda \,|\, f_{pq}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |\lambda_{\nu k} \chi_{\nu k}^{(p)}(\cdot)|^q \right)^{\frac{1}{q}} \,\Big|\, L_p(\mathbb{R}^n) \right\|$$

(with the usual modifications if q is infinite) is finite.

Proposition 1 *Let $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then*

$$(5) \qquad b_{p,\min(p,q)} \subset f_{pq} \subset b_{p,\max(p,q)} .$$

Proof. By the triangle inequality, in dependence on whether $p \geqq q$ or $p \leqq q$, one has

$$(6) \qquad \|\lambda \,|\, b_{p,\max(p,q)}\| \leqq \|\lambda \,|\, f_{pq}\| \leqq \|\lambda \,|\, b_{p,\min(p,q)}\| ,$$

see [33], p. 47, for this type of argument. One has to use

$$(7) \qquad \|\chi_{\nu k}^{(p)} \,|\, L_p(\mathbb{R}^n)\| = 1 \quad \text{and} \quad \|\lambda \,|\, b_{pp}\| = \|\lambda \,|\, f_{pp}\| .$$

Remark 1 This proposition is the direct counterpart of the embedding

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \subset F^s_{pq}(\mathbb{R}^n) \subset B^s_{p,\max(p,q)}(\mathbb{R}^n) , \tag{8}$$

both in formulation and proof, see [33], p. 47.

If $c \in \mathbb{R}$, then we put $c_+ = \max(c, 0)$ and denote by $[c]$ the largest integer less than or equal to c. Furthermore, if $0 < p \leqq \infty$ and $0 < q \leqq \infty$ then we use the abbreviations

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{9}$$

If the atom $a(x)$ is supported by $dQ_{\nu k}$ in the sense of (2.2/3) or (2.2/5) then we write $a_{\nu k}(x)$, hence

$$\operatorname{supp} a_{\nu k} \subset dQ_{\nu k}; \; \nu \in \mathbb{N}_0 \quad \text{and} \quad k \in \mathbb{Z}^n . \tag{10}$$

Proposition 2 *Let $s \in \mathbb{R}$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Let $0 \leqq \sigma \notin \mathbb{N}$, $s < \sigma$ and $L + 1 \in \mathbb{N}_0$ with $L \geqq \max([\sigma_p - s], -1)$. Let either $\lambda \in b_{pq}$ or $\lambda \in f_{pq}$ in the sense of the above definition. Let $a_{\nu k}(x)$ with $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$ be 1_σ-atoms ($\nu = 0$) and $(s,p)_{\sigma,L}$-atoms ($\nu \in \mathbb{N}$) in the sense of Definition* 2.2 (i) *and* (ii), *respectively, with* (10). *Then*

$$\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{\nu k} a_{\nu k}(x) \tag{11}$$

converges in $S'(\mathbb{R}^n)$.

Remark 2 The proof of this proposition is a by-product of the proof of the following theorem, which, in turn, is essentially covered by what has been done by Frazier and Jawerth, see especially [13]. Our modification compared with the original formulation described in Remark 2.2/2 is immaterial in this context. So we do not go into detail.

Theorem. (i) *Let $s \in \mathbb{R}$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Let $0 \leqq \sigma \notin \mathbb{N}$, $s < \sigma$ and $L + 1 \in \mathbb{N}_0$ with $L \geqq \max([\sigma_p - s], -1)$.*

Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{\nu k} a_{\nu k}(x), \quad \textit{convergence in } S'(\mathbb{R}^n) , \tag{12}$$

where $a_{\nu k}(x)$ are 1_σ-atoms ($\nu = 0$) or $(s,p)_{\sigma,L}$-atoms ($\nu \in \mathbb{N}$) in the sense of Definition 2.2(i) *and* (ii), *respectively, with* (10), *and $\lambda \in b_{pq}$. Furthermore,*

$$\inf \| \lambda \,|\, b_{pq} \| , \tag{13}$$

where the infimum is taken over all admissible representations (12), *is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n)$.*

(ii) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leqq \infty$. Let $0 \leqq \sigma \notin \mathbb{N}$, $s < \sigma$ and $L + 1 \in \mathbb{N}_0$ with $L \geqq \max([\sigma_{pq} - s], -1)$.*

Then $f \in S'(\mathbb{R}^n)$ belongs to $F^s_{pq}(\mathbb{R}^n)$ if and only if it can be represented by (12), *where the atoms $a_{\nu k}(x)$ have the same meaning as in part* (i) (*now, maybe, with a different value of L*), *and $\lambda \in f_{pq}$. Furthermore,*

$$\inf \|\lambda \mid f_{pq}\|\,, \tag{14}$$

where the infimum is taken over all admissible representations (12), *is an equivalent quasi-norm in $F^s_{pq}(\mathbb{R}^n)$* .

Remark 3 As we said this theorem is at least in principle covered by the work of Frazier and Jawerth, see [12], [13] and [14]. Our formulation is different and, as we hope, more handsome, even on $\mathbb{R}^n$, and we switched from requirements like (2.2/11) to their counterparts (2.2/6). This latter modification prepares the atomic approach to spaces on domains. Starting from (2.2/11) one has to replace $0 \leqq \sigma \notin \mathbb{N}$, $s < \sigma$ in the above theorem by $K \geqq ([s]+1)_+$. In this sense the above theorem offers also a slight improvement compared with what is known so far. But this is not crucial for our further considerations and would not justify to present a long and complicated proof. In other words we take the above theorem for granted.

3 Function spaces on domains

3.1 Definitions

An open connected set in $\mathbb{R}^n$ is called a domain. As usual $D'(\Omega)$ stands for all complex distributions on the domain Ω in $\mathbb{R}^n$. The restriction of $g \in S'(\mathbb{R}^n)$ to Ω is denoted by $g|\Omega$ and is considered as an element of $D'(\Omega)$.

Definition 1 *Let Ω be a domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$ and $0 < q \leqq \infty$.*

(i) *Let $0 < p \leqq \infty$. Then $B^s_{pq}(\Omega)$ is the restriction of $B^s_{pq}(\mathbb{R}^n)$ to Ω, quasi-normed by*

$$\|f \mid B^s_{pq}(\Omega)\| = \inf \|g \mid B^s_{pq}(\mathbb{R}^n)\| \tag{1}$$

where the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$.

(ii) *Let $0 < p < \infty$. Then $F^s_{pq}(\Omega)$ is the restriction of $F^s_{pq}(\mathbb{R}^n)$ to Ω, quasi-normed by*

$$\|f \mid F^s_{pq}(\Omega)\| = \inf \|g \mid F^s_{pq}(\mathbb{R}^n)\| \tag{2}$$

where the infimum is taken over all $g \in F^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$.

Remark 1 By standard arguments all these spaces are quasi-Banach spaces. Furthermore, simply by definition, all embedding theorems between spaces known on $\mathbb{R}^n$ can be carried over to the corresponding spaces on Ω.

Admissible domains. The above definition applies to any domain in $\mathbb{R}^n$. But in this generality the introduction of spaces on Ω via restriction of corresponding spaces on $\mathbb{R}^n$ is not very reasonable. Let, for example, g be a continuous function on $\mathbb{R}^n$. Then, of course, $f = g|\Omega$ can be extended continuously to $\overline{\Omega}$.

For instance, coming that way, the teeth of the comb domain in Fig. 1 are completely ignored. The atomic decomposition given in Theorem 2.3 provides the feeling that this argument applies also to more general functions and distributions, and related spaces. In other words, it seems to be reasonable to restrict the considerations to those domains Ω which coincide with the interior of their closure, $\Omega = \mathrm{int}(\overline{\Omega})$. This excludes the comb domain in Fig. 1, but it does not exclude the bow-tie domain in Fig. 2, where we took out two triangles as indicated of a square. The above arguments apply to the centre of this domain and thus make clear that some care is necessary. Furthermore, we assume in the sequel that Ω is bounded. This is neither necessary nor natural, but convenient for us. It is quite clear that adding some uniformity conditions most of what we have to say can be carried over to appropriate unbounded domains. An exception is subsection 4.2, where the boundedness of the underlying domains is natural to study the behaviour of entropy numbers. We formalize the above considerations.

Definition 2 *Let $MR(n)$ (minimally regular) be the collection of all bounded domains Ω in $\mathbb{R}^n$ with*

(3) $$\Omega = \mathrm{int}(\overline{\Omega}),$$

that means, Ω coincides with the interior of its closure $\overline{\Omega}$.

Remark 2 The comb domain in Fig. 1 does not belong to $MR(2)$, whereas the bow-tie domain in Fig. 2 belongs to $MR(2)$.

A discussion. Let $1 < p < \infty$ and $k \in \mathbb{N}$. Then as usual

(4) $$W_p^k(\Omega) = \{f \in L_p(\Omega) : D^\alpha f \in L_p(\Omega), |\alpha| \leqq k\}$$

are the classical Sobolev spaces. Let $\Omega \in MR(2)$ be the bow-tie domain in Fig. 2. Then, looking at traces on lines, it follows that the restriction of $W_p^k(\mathbb{R}^2)$ to Ω on the one hand and $W_p^k(\Omega)$ on the other hand do not coincide (at least if $kp > 2$, where all elements of $W_p^k(\mathbb{R}^2)$ are continuous). So we have in that case

(5) $$W_p^k(\mathbb{R}^2) = F^k_{p2}(\mathbb{R}^2) \quad \text{and} \quad W_p^k(\Omega) \neq F^k_{p2}(\Omega),$$

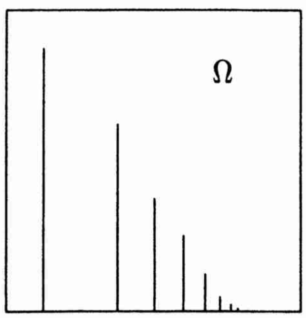

Fig. 1.

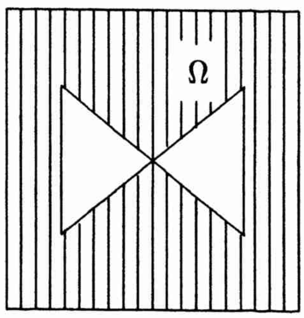

Fig. 2.

with $F^k_{p2}(\Omega) \subset W^k_p(\Omega)$, $k \in \mathbb{N}$, $1 < p < \infty$ and $kp > 2$. The equality in (5) is the well-known Paley–Littlewood characterization of Sobolev spaces, see, for example, [33], Theorem 2.5.6. In the second part of (5), the space $W^k_p(\Omega)$ is given by (4) and $F^k_{p2}(\Omega)$ by part (ii) of Definition 1. If $\Omega \notin MR(2)$ is the comb domain in Fig. 1, then we have also (5). How big the difference between $W^k_p(\Omega)$ and $F^k_{p2}(\Omega)$ is in that case can be imagined by the recent observation in [15] that in similar comb domains any given set of non-negative numbers may serve as the essential spectrum of the Neumann Laplacian on Ω. In 4.3 further discussions may be found, including references.

3.2 *Regular domains*

Our aim is to find intrinsic atomic characterizations of the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ introduced in Definition 3.1/1. We describe now those types of domains which, how we believe, are naturally connected with this task.

Definition. (i) *Let $IR(n)$ (interior regular) be the collection of all domains $\Omega \in MR(n)$ for which one finds a positive number c such that for any cube Q centered at $\partial\Omega$ with side-length less than or equal* 1

$$|\Omega \cap Q| \geqq c|Q| . \tag{1}$$

(ii) *Let $ER(n)$ (exterior regular) be the collection of all domains $\Omega \in MR(n)$ for which one finds a positive number c such that for any cube Q centered at $\partial\Omega$ with side-length l less than or equal* 1 *there exists a subcube Q^e with side-length cl and*

$$Q^e \subset Q \cap (\mathbb{R}^n \setminus \overline{\Omega}) . \tag{2}$$

(iii) *Let*

$$R(n) = IR(n) \cap ER(n) \tag{3}$$

be the collection of all domains $\Omega \in MR(n)$ which are both interior and exterior regular.

Remark. Of course, $\partial\Omega = \overline{\Omega} \setminus \Omega$ denotes the boundary of Ω. As we said, our restriction to bounded domains is convenient for us but not necessary, at least in this stage. Cubes may always be considered as having sides parallel to the axes.

A discussion. In analogy to $ER(n)$, let $\Omega \in MR(n)$ be a domain for which one finds a positive number c such that for any cube Q centered at $\partial\Omega$ with side-length l less than or equal 1 there exists a subcube Q^i with side-length cl and

$$Q^i \subset Q \cap \Omega . \tag{4}$$

Then we have $\Omega \in IR(n)$. In this sense the typical situation connected with the above definition is shown in Fig. 3. The domain in Fig. 4 with an inward

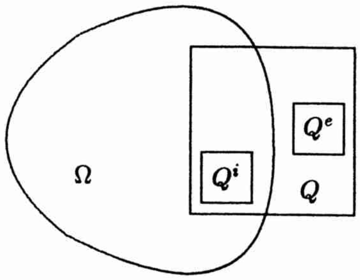

Fig. 3.

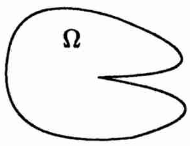

Fig. 4.

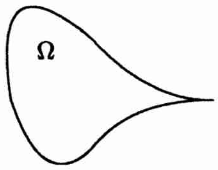

Fig. 5.

cusp belongs to $IR(2)$, but not to $ER(2)$; whereas the domain in Fig. 5 with an outward cusp belongs to $ER(2)$, but not to $IR(2)$. Attempts to specify (non-smooth) domains in connection with function spaces have a long history, which we will not discuss here. We only recall that domains $\Omega \in MR(n)$ satisfying the interior or exterior cone condition belong to $IR(n)$ or $ER(n)$ respectively. Minimally smooth domains $\Omega \in MR(n)$ which, by definition, have a Lipschitz boundary are regular in the sense of (3). We refer for definitions and details to [23], 1.1.9 and [28], p. 189. There are also conditions near to ours in the literature. In [13] there are domains D_s and NST (not so terrible), where the latter are near to the above class $IR(n)$. The condition in [25], Theorem 4, is similar as our condition (3). If $\Omega \in MR(n)$ is a so-called (ε, δ)-domain, see [16] and [23], 1.5.1, then it belongs to $IR(n)$. We refer especially to [23] and [24] for a detailed discussion of non-smooth (or bad) domains, mostly in connection with the extension property, which we comment on briefly at the end of this paper.

3.3 Atoms

We look for appropriate counterparts of the definitions and results in 2.2 and 2.3 now in suitable domains. We always assume $\Omega \in MR(n)$. First we need the counterpart of the spaces $\mathscr{C}^\sigma(\mathbb{R}^n)$.

Definition 1 *Let $\Omega \in MR(n)$ and let $0 < \sigma = [\sigma] + \{\sigma\}$ where $[\sigma] \in \mathbb{N}_0$ and $0 < \{\sigma\} < 1$. Then $\mathscr{C}^\sigma(\overline{\Omega})$ consists of all complex-valued continuous* *functions f on $\overline{\Omega}$ with the following two properties:*

(i) *f has classical derivatives $D^\alpha f$ in Ω for $|\alpha| \leqq [\sigma]$ and there exist continuous functions f_α on $\overline{\Omega}$ which coincide with $D^\alpha f$ on Ω,*

(ii)

$$(1)\quad \|f \mid \mathscr{C}^\sigma(\overline{\Omega})\| := \sup_{x\in\overline{\Omega}} \sum_{|\alpha|\leqq[\sigma]} |D^\alpha f(x)| + \sup \sum_{|\alpha|=[\sigma]} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{\sigma\}}} < \infty ,$$

where the second supremum is taken over all $x \in \overline{\Omega}$ and $y \in \overline{\Omega}$ with $x \neq y$.

Remark 1 Of course, $\mathscr{C}^\sigma(\overline{\Omega})$ is a Banach space. It coincides with the Lipschitz space $\mathrm{Lip}(\sigma, \overline{\Omega})$ in the sense of the usual "jet-definition" via $\{f_\alpha\}_{|\alpha|\leqq[\sigma]}$, $(f = f_0)$, see [19], pp. 22, 44/45 and [28], p. 173. In contrast to the more general situation treated there we benefit from the fact that $\overline{\Omega}$ is the closure of the bounded domain Ω. Let again $\mathscr{C}^0(\overline{\Omega})$ be the space of all complex-valued continuous functions on $\overline{\Omega}$. Then all these spaces $\mathscr{C}^\sigma(\overline{\Omega})$, now with $0 \leqq \sigma \notin \mathbb{N}$, can be extended to $\mathscr{C}^\sigma(\mathbb{R}^n)$ via Whitney's extension method, see [28], pp. 170–180. We will use this in the sequel.

To introduce atoms on domains $\Omega \in MR(n)$ we again rely on the cubes $Q_{\nu k}$ described at the beginning of 2.2. The numbers b and d have the same meaning as there. We may assume in addition that the centres $x^{\nu,k}$ of the cubes $Q_{\nu k}$ with $dQ_{\nu k} \cap \partial\Omega \neq \emptyset$ are located at $\partial\Omega$. In this sense we call $Q_{\nu k}$

$$(2)\qquad \text{an interior cube if } dQ_{\nu k} \subset \Omega, \quad \nu \in \mathbb{N}_0, k \in \mathbb{Z}^n ,$$

and

$$(3)\qquad \text{a boundary cube if } x^{\nu,k} \in \partial\Omega, \quad \nu \in \mathbb{N}_0, k \in \mathbb{Z}^n .$$

Other cubes are not of interest for us. Let, for brevity,

$$(4)\qquad \Omega^\nu = \{x \in \mathbb{R}^n : 2^{-\nu}x \in \Omega\}, \quad \nu \in \mathbb{N}_0 .$$

Definition 2 *Let $\Omega \in MR(n)$.*

(i) *Let $0 \leqq \sigma \notin \mathbb{N}$. Then $a(x)$ is called a 1-atom (or more precisely 1_σ-atom) in Ω if*

$$(5)\qquad \operatorname{supp} a \subset \overline{\Omega} \cap dQ_{0k}$$

for some interior or boundary cube Q_{0k} with $k \in \mathbb{Z}^n$ and

$$(6)\qquad a \in \mathscr{C}^\sigma(\overline{\Omega}) \quad \text{with } \|a \mid \mathscr{C}^\sigma(\overline{\Omega})\| \leqq 1 .$$

(ii) *Let $s \in \mathbb{R}$ and $0 < p \leqq \infty$. Let $0 \leqq \sigma \notin \mathbb{N}$ and $L + 1 \in \mathbb{N}_0$. Then $a(x)$ is called an (s, p)-atom (or more precisely $(s, p)_{\sigma,L}$ interior-atom) in Ω if*

$$(7)\qquad \operatorname{supp} a \subset dQ_{\nu k}$$

for some interior cube $Q_{\nu k}$ with $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}^n$,

(8) $$a \in \mathscr{C}^\sigma(\overline{\Omega}) \quad \text{with } \|a(2^{-\nu}\cdot)|\mathscr{C}^\sigma(\overline{\Omega^\nu})\| \leqq 2^{-\nu(s-\frac{n}{p})}$$

and

(9) $$\int_\Omega x^\beta a(x)dx = 0 \quad \text{for } |\beta| \leqq L\,.$$

(iii) *Let $s \in \mathbb{R}$ and $0 < p \leqq \infty$. Let $0 \leqq \sigma \notin \mathbb{N}$. Then $a(x)$ is called an (s,p)-atom (or more precisely $(s,p)_\sigma$-boundary atom) in Ω if*

(10) $$\operatorname{supp} a \subset \overline{\Omega} \cap dQ_{\nu k}$$

and (8) holds for some boundary cube $Q_{\nu k}$ with $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}^n$.

Remark 2 The above parts (i) and (ii) are the natural counterparts of Definition 2.2. As for part (iii) no moment conditions of type (9) are required. See Remark 2.2/1 for further explanations. In particular, 1-atoms and normalized boundary atoms in the sense of (2.2/8) are close to each other.

3.4 Atomic domains

First we introduce the counterparts of the sequence spaces b_{pq} and f_{pq} from Definition 2.3. Let $\Omega \in MR(n)$ and let $Q_{\nu k}$ be the dyadic cubes in the sense of 3.3, where we are only interested in interior and boundary cubes described in (3.3/2) and (3.3/3), respectively. In modification of (2.3/1) we put

(1) $$\lambda = \{\lambda_{\nu k} : \lambda_{\nu k} \in \mathbb{C},\ \nu \in \mathbb{N}_0,\ k \in \mathbb{Z}^n,\ Q_{\nu k} \text{ interior or boundary cube}\}\,.$$

Furthermore, $\sum_{k\in\mathbb{Z}^n}^{\nu,\Omega}$ means that for fixed $\nu \in \mathbb{N}_0$ the sum is taken over those $k \in \mathbb{Z}^n$ for which $Q_{\nu k}$ is an interior or boundary cube. Let again $\chi_{\nu k}^{(p)}(x)$ be the normalized characteristic function of $Q_{\nu k}$ in the sense of (2.3/2).

Definition 1 *Let $\Omega \in MR(n)$, and let $0 < p \leqq \infty$ and $0 < q \leqq \infty$.*

(i) *Then $b_{pq}(\Omega)$ is the collection of all sequences λ given by (1) such that*

(2) $$\|\lambda \,|\, b_{pq}(\Omega)\| = \left(\sum_{\nu=0}^{\infty}\left(\sum_{k\in\mathbb{Z}^n}^{\nu,\Omega}|\lambda_{\nu k}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

(with the usual modification if p and/or q is infinite) is finite.

(ii) *Then $f_{pq}(\Omega)$ is the collection of all sequences λ given by (1) such that*

(3) $$\|\lambda \,|\, f_{pq}(\Omega)\| = \left\|\left(\sum_{\nu=0}^{\infty}\sum_{k\in\mathbb{Z}^n}^{\nu,\Omega}|\lambda_{\nu k}\chi_{\nu k}^{(p)}(\cdot)|^q\right)^{\frac{1}{q}} \,\Big|\, L_p(\Omega)\right\|$$

(with the usual modification if q is infinite) is finite.

Remark 1 The desirable counterpart of Proposition 2.3/1 is not clear. It is connected with the question whether $L_p(\Omega)$ in (3) can be replaced by $L_p(\mathbb{R}^n)$. As we shall see in 3.5 this is the case if $\Omega \in IR(n)$. Then we have a full counterpart of (2.3/5). On the other hand let Ω be the cusp domain in Fig. 5 then one constructs easily a sequence of cubes $Q_{\nu k}$ converging to the edge of this domain and a corresponding sequence of numbers $\lambda = \{\lambda_{\nu k}\}$, such that λ belongs to $f_{pq}(\Omega)$ in (3), but λ does not belong neither to the corresponding space with $L_p(\mathbb{R}^n)$ instead of $L_p(\Omega)$, nor to any of the spaces $b_{uv}(\Omega)$ with $0 < u \leqq \infty$, $0 < v \leqq \infty$. In this sense the class $IR(n)$ is at least reasonable in this context.

Next we are interested in the counterpart of Proposition 2.3/2, which we now convert in a definition of domains having this property. As described in both parts of Theorem 2.3 we have in $\mathbb{R}^n$ natural restrictions for σ and L, if s, p, q are assumed to be given. This justifies to take over this knowledge to the situation considered now. Let again σ_p and σ_{pq} be given by (2.3/9).

Definition 2 *Let $s \in R$ and $0 < q \leqq \infty$.*

(i) *Let $0 < p \leqq \infty$,*

$$0 \leqq \sigma \notin \mathbb{N}, \ s < \sigma \quad \textit{and} \quad L+1 \in \mathbb{N}_0 \quad \textit{with } L \geqq \max([\sigma_p - s], -1)\,. \tag{4}$$

Then $Atom(B^s_{pq})^n$ (atomic B^s_{pq}-domain) denotes the collection of all domains $\Omega \in MR(n)$ such that for all such choices of σ and L

$$\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n}{}^{\nu,\Omega} \lambda_{\nu k} a_{\nu k}(x), \quad x \in \Omega, \quad \lambda \in b_{pq}(\Omega)\,, \tag{5}$$

converges in $D'(\Omega)$ to an element of $B^s_{pq}(\Omega)$, where $a_{\nu k}(x)$ are 1_σ-atoms ($\nu = 0$), $(s,p)_{\sigma,L}$-interior atoms ($\nu \in \mathbb{N}$), or $(s,p)_\sigma$-boundary atoms ($\nu \in \mathbb{N}$) in the sense of Definition 3.3/2 *with*

$$\operatorname{supp} a_{\nu k} \subset \overline{\Omega} \cap dQ_{\nu k}, \quad \nu \in \mathbb{N}_0\,. \tag{6}$$

(ii) *Let $0 < p < \infty$,*

$$0 \leqq \sigma \notin \mathbb{N}, \ s < \sigma \quad \textit{and} \quad L+1 \in \mathbb{N}_0 \quad \textit{with } L \geqq \max([\sigma_{pq} - s], -1)\,. \tag{7}$$

Then $Atom(F^s_{pq})^n$ (atomic F^s_{pq}-domain) denotes the collection of all domains $\Omega \in MR(n)$ such that for all such choices of σ and L

$$\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n}{}^{\nu,\Omega} \lambda_{\nu k} a_{\nu k}(x), \quad x \in \Omega, \quad \lambda \in f_{pq}(\Omega)\,, \tag{8}$$

converges in $D'(\Omega)$ to an element of $F^s_{pq}(\Omega)$, where $a_{\nu k}(x)$ are the same atoms as in part (i).

Remark 2 It should be noted that the convergence of (5) and (8) is required only in $D'(\Omega)$. But it comes out, that these series under the conditions in the theorems of the two following subsections converge also in $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, respectively, if $p < \infty$ and $q < \infty$. If $p = \infty$ and/or $q = \infty$ then the situation is different, but this is well-known even in $\mathbb{R}^n$ or on smooth domains (smooth functions are not dense in $\mathscr{C}^\sigma$, $\sigma > 0$).

Remark 3 The conditions (4) and (7) coincide with the corresponding conditions in Theorem 2.3. It is clear that (i) and (ii) shall be understood in that way that for any fixed couple (σ, L) satisfying these requirements the desired convergence takes place.

3.5 Atomic characterizations: large s

If one wishes to extend boundary atoms in the sense of Definition 3.3/2(iii) from Ω to $\mathbb{R}^n$, then the moment conditions (2.2/7) for these extended atoms cause some trouble. So we shift this task to the next subsection and deal with those cases where in the sense of Theorem 2.3 no moment conditions are necessarily required, that means where in Theorem 2.3, $L = -1$ is admissible. But otherwise we keep the moment conditions for the interior atoms in the sense of Definition 3.3/2(ii). The aim is to find the counterpart of Theorem 2.3. Let again σ_p and σ_{pq} be given by (2.3/9). Furthermore we use the notations introduced in the previous subsection.

Theorem. (i) *Let* $0 < p \leqq \infty$, $0 < q \leqq \infty$ *and* $s > \sigma_p$. *Then*

$$\text{Atom}(B^s_{pq})^n = MR(n)\,. \tag{1}$$

Let

$$s < \sigma \notin \mathbb{N} \quad \text{and} \quad L + 1 \in \mathbb{N}_0\,. \tag{2}$$

Then $f \in D'(\Omega)$ *belongs to* $B^s_{pq}(\Omega)$ *if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n}{}^{\nu,\Omega} \lambda_{\nu k} a_{\nu k}(x), \quad \text{convergence in } D'(\Omega)\,, \tag{3}$$

in the sense of Definition 3.4/2(i) *with* $\lambda \in b_{pq}(\Omega)$. *Furthermore,*

$$\inf \|\lambda | b_{pq}(\Omega)\|\,, \tag{4}$$

where the infimum is taken over all admissible representations (3), *is an equivalent quasi-norm in* $B^s_{pq}(\Omega)$.

(ii) *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $s > \sigma_{pq}$. *Then*

$$\text{Atom}(F^s_{pq})^n \supset IR(n)\,. \tag{5}$$

Let again

$$s < \sigma \notin \mathbb{N} \quad \text{and} \quad L + 1 \in \mathbb{N}_0\,. \tag{6}$$

Then $f \in D'(\Omega)$ *belongs to* $F^s_{pq}(\Omega)$ *if and only if it can be represented by* (3) *with* $\lambda \in f_{pq}(\Omega)$. *Furthermore,*

$$\inf \|\lambda | f_{pq}(\Omega)\|\,, \tag{7}$$

where the infimum is taken over all admissible representations (3), *is an equivalent quasi-norm in* $F^s_{pq}(\Omega)$.

Proof.

Step 1 We prove (i). Let $\Omega \in MR(n)$. First we have to show that the series (3.4/5) converges in $D'(\Omega)$. For this purpose we extend each atom $a_{\nu k}(x)$ individually from Ω to $\mathbb{R}^n$. Of course, only the boundary atoms in the sense of Definition 3.3/2 (iii) are of interest, where we may assume that this covers the 1-atoms in the sense of Definition 3.3/2(i) with $\nu = 0$ in (3.3/8) too. We rely on Whitney's extension method as described in [28], pp. 170–180, especially Theorem 4 on p. 177, and which can be applied to the spaces $\mathscr{C}^\sigma(\overline{\Omega})$, see Definition 3.3/1 and Remark 3.3/1. Let E_k be the linear extension operator, constructed in [28], p. 177, formula (18). By Theorem 4 on the same page, $E_{[\sigma]}$ generates a linear extension operator

$$(8) \qquad E_{[\sigma]}(\overline{\Omega}) : \mathscr{C}^\sigma(\overline{\Omega}) \to \mathscr{C}^\sigma(\mathbb{R}^n) \,,$$

with a bound being independend of $\overline{\Omega}$. Let D_c be the dilation operator on $\mathbb{R}^n$,

$$(9) \qquad D_c : f(x) \to f(cx), \quad c > 0 \,.$$

Then it follows from the explicit construction of $E_{[\sigma]}$ that

$$(10) \qquad E_{[\sigma]}(\overline{\Omega}) = D_{2^\nu} \circ E_{[\sigma]}(\overline{\Omega^\nu}) \circ D_{2^{-\nu}}, \quad \nu \in \mathbb{N}_0 \,,$$

where Ω^ν is given by (3.3/4). Let ψ be a C^∞ cut-off function with

$$(11) \qquad \operatorname{supp} \psi \subset 2dQ \quad \text{and} \quad \psi(x) = 1 \quad \text{if } x \in dQ \,,$$

where Q is the unit cube centered at the origin. Here d has the same meaning as in (3.3/10). We apply $E_{[\sigma]}(\overline{\Omega})$ to the $(s,p)_\sigma$-boundary atom $a_{\nu k}(x)$ and put

$$(12) \qquad b_{\nu k}(x) = \psi(2^\nu(x - x^{\nu,k}))(E_{[\sigma]}(\overline{\Omega})a_{\nu k})(x) \,,$$

where $x^{\nu,k}$ is the centre of $Q_{\nu k}$, see (3.3/3). By (3.3/8), (10), the existence of an independent bound for the norms of $E_{[\sigma]}(\overline{\Omega^\nu})$ of ν, and (2.2/6) it follows that $b_{\nu k}$ is an atom on $\mathbb{R}^n$ in the sense of Definition 2.2 with $L = -1$, besides a constant and after replacing d in (2.2/5) by, say, $2d$. Both, of course, is immaterial, and we neglect it simply. Then the counterpart of (3.4/5) is given by

$$(13) \qquad \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{\nu k} b_{\nu k}(x), \quad x \in \mathbb{R}^n, \quad \lambda \in b_{pq} \,,$$

with $\lambda_{\nu k} = b_{\nu k}(x) = 0$ if $Q_{\nu k}$ is an "exterior" cube. By Proposition 2.3/2 and Theorem 2.3(i), the series (13) converges in $S'(\mathbb{R}^n)$ and its limit belongs to $B^s_{pq}(\mathbb{R}^n)$. Hence, its restriction (3.4/5) to Ω converges in $D'(\Omega)$ and, by Definition 3.1/1, its limit belongs to $B^s_{pq}(\Omega)$. Thus the proof of (1) is complete. The rest of part (i) is now simple. By the above argument, any $f \in D'(\Omega)$, given by (3) with $\lambda \in b_{pq}(\Omega)$ belongs to $B^s_{pq}(\Omega)$. Conversely, by Definition 3.1/1 and Theorem 2.3(i) any $f \in B^s_{pq}(\Omega)$ can be represented in that way. Now it follows immediately from Definition 3.1/1 that (4) with (3) is an equivalent quasi-norm in $B^s_{pq}(\Omega)$.

Step 2 We prove (ii). Let $\Omega \in IR(n)$ in the sense of Definition 3.2(i). Let $Q_{\nu k}$ be a boundary cube with centre $x^{\nu,k} \in \partial\Omega$, see (3.3/3). Let temporarily $\chi_{\nu k,i}$ and $\chi_{\nu k}$ be the characteristic functions of $Q_{\nu k} \cap \Omega$ and $Q_{\nu k}$, respectively, where i refers to interior. Let $(Mg)(x)$ be the Hardy–Littlewood maximal function of a function $g(x)$. Then it follows easily from (3.2/1)

$$\chi_{\nu k}(x) \leqq c(M\chi_{\nu k,i})(x), \quad x \in \mathbb{R}^n, \quad \nu \in \mathbb{N}_0\,, \tag{14}$$

where c depends only on the constant in (3.2/1), which does not need to be the same as in (14). Let $0 < w < \min(1, p, q)$, let $\chi^{(p)}_{\nu k}$ be given by (2.3/2) and similarly $\chi^{(p)}_{\nu k,i}(x) = 2^{\nu n/p}\chi_{\nu k,i}(x)$. Then (14) yields

$$\chi^{(p)}_{\nu k}(x) \leqq c^{\frac{1}{w}}(M\chi^{(p)w}_{\nu k,i})(x)^{\frac{1}{w}} \tag{15}$$

with the same constant as in (14). We wish to show that $L_p(\Omega)$ on the right-hand side of (3.4/3) can be replaced by $L_p(\mathbb{R}^n)$ in the sense of equivalent quasi-norms. We have by (15)

$$\left\| \left(\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n}{}^{\nu,\Omega} |\lambda_{\nu k}\chi^{(p)}_{\nu k}(\,\cdot\,)|^q \right)^{\frac{1}{q}} \middle| L_p(\mathbb{R}^n) \right\| \tag{16}$$

$$\leqq c^{\frac{1}{w}} \left\| \left(\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |\lambda_{\nu k}(M\chi^{(p)w}_{\nu k,i})(\,\cdot\,)|^{\frac{q}{w}} \right)^{\frac{w}{q}} \middle| L_{\frac{p}{w}}(\mathbb{R}^n) \right\|^{\frac{1}{w}}$$

with the same constant c as in (14). Since $1 < \frac{q}{w} \leqq \infty$ and $1 < \frac{p}{w} < \infty$ we may apply the vector-valued Hardy–Littlewood maximal inequality in the sense of Fefferman and Stein, see [10], [29], pp. 50–56, [34], 2.2.2. Thus, the right-hand side of (16) can be estimated from above by

$$c' \left\| \left(\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |\lambda_{\nu k}\chi^{(p)}_{\nu k,i}(\,\cdot\,)|^q \right)^{\frac{1}{q}} \middle| L_p(\mathbb{R}^n) \right\| , \tag{17}$$

which, in turn, can be estimated from above by $c'\|\lambda|f_{pq}(\Omega)\|$ given by (3.4/3). Hence we obtain

$$\|\lambda|f_{pq}(\Omega)\| \sim \left\| \left(\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n}{}^{\nu,\Omega} |\lambda_{\nu k}\chi^{(p)}_{\nu k}(\,\cdot\,)|^q \right)^{\frac{1}{q}} \middle| L_p(\mathbb{R}^n) \right\| . \tag{18}$$

Now, after (18) has been established, one can follow the arguments of Step 1. This proves (5) and the other assertions of part (ii).

Remark 1 The most remarkable feature of the above theorem is the striking difference between part (i) and part (ii), especially between (1) and (5). It supports again the well-known fact that spaces of type B^s_{pq} are structurally simpler than spaces of type F^s_{pq}, see [33], 2.5.5, and the references given there.

Remark 2 The equivalence (18) is the crucial desired property. For its proof we used (14), which, in turn, was based on (3.2/1), with (3.2/4) as a preference case. Although (3.2/4) is quite natural in order to have $\Omega \in IR(n)$, there exist domains $\Omega \in IR(n)$ for which (3.2/4) is not true. If one takes out of a square in $\mathbb{R}^2$ infinitely many smaller squares such that one obtains a carpet-like domain, then it might happen, that (3.2/4) is violated, but not (3.2/1). Other candidates may be recruited from domains with irregular, say, fractal boundaries, or from appropriately constructed domains of type "rooms and passages". As for domains of these types we refer to [23], [24], [4], [15] and [8]. But as far as we know, the above question has not yet been studied in detail.

3.6 Atomic characterizations: general s

Whitney's extension method proved to be a convenient and effective vehicle to extend atoms from Ω (better $\overline{\Omega}$) to $\mathbb{R}^n$ as long as no moment conditions of type (2.2/7) are required. If moment conditions are necessary, then we have to amend what has been done so far in 3.5 by a special method creating moment conditions on $\mathbb{R}^n \backslash \overline{\Omega}$. For this purpose we need in addition to the previous assumptions $\Omega \in ER(n)$ in the sense of Definition 3.2(ii). We use the same notations and formulations as in the Theorems 2.3 and 3.5 without further explanations.

Theorem. (i) *Let* $0 < p \leqq \infty$, $0 < q \leqq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\text{Atom}(B^s_{pq})^n \supset ER(n)\,. \tag{1}$$

Let

$$0 \leqq \sigma \notin \mathbb{N}, s < \sigma \quad \text{and} \quad L + 1 \in \mathbb{N}_0 \quad \text{with } L \geqq \max([\sigma_p - s], -1)\,. \tag{2}$$

Then $f \in D'(\Omega)$ *belongs to* $B^s_{pq}(\Omega)$ *if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} {}^{\nu,\Omega}\lambda_{\nu k} a_{\nu k}(x), \quad \text{convergence in } D'(\Omega)\,, \tag{3}$$

in the sense of Definition 3.4/2(i) *with* $\lambda \in b_{pq}(\Omega)$. *Furthermore,*

$$\inf \|\lambda | b_{pq}(\Omega)\|\,, \tag{4}$$

where the infimum is taken over all admissible representations (3), *is an equivalent quasi-norm in* $B^s_{pq}(\Omega)$.

(ii) *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$Atom(F^s_{pq})^n \supset R(n)\,. \tag{5}$$

Let

$$0 \leqq \sigma \notin \mathbb{N}, s < \sigma \quad \text{and} \quad L + 1 \in \mathbb{N}_0 \quad \text{with } L \geqq \max([\sigma_{pq} - s], -1)\,. \tag{6}$$

Then $f \in D'(\Omega)$ *belongs to* $F^s_{pq}(\Omega)$ *if and only if it can be represented by* (3) *with* $\lambda \in f_{pq}(\Omega)$. *Furthermore,*

$$\inf \|\lambda | f_{pq}(\Omega)\| , \tag{7}$$

where the infimum is taken over all admissible representations (3), *is an equivalent quasi-norm in* $F^s_{pq}(\Omega)$.

Proof.

Step 1 The main idea of the proof of Theorem 3.5 culminated in the atomic expansion (3.5/13) in $\mathbb{R}^n$, by extending boundary atoms $a_{\nu k}$ in Ω to corresponding $\mathbb{R}^n$-atoms $b_{\nu k}$. In contrast to the situation in 3.5 we are now forced to ensure some moment conditions for the extended atoms $b_{\nu k}$. For that purpose we begin in this step with a preparation. Let $\varphi(x)$ be a C^∞-function on the real line $\mathbb{R}$ with a support near the origin and with

$$\int_{\mathbb{R}} \varphi(t)dt = 1 . \tag{8}$$

Let $L \in \mathbb{N}$ and let

$$\varphi_k(t) = \sum_{m=0}^{L} c_{km} \frac{d^m\varphi(t)}{dt^m}, \quad c_{km} \in \mathbb{C}, \quad k = 0, \ldots, L . \tag{9}$$

Since $\{\int_{\mathbb{R}} t^l \frac{d^m\varphi(t)}{dt^m} dt\}^{0,\ldots,L}_{l,m}$ is a triangular matrix with non-vanishing entries on the diagonal, the coefficients c_{km} can be uniquely calculated in such a way that

$$\int_{\mathbb{R}} t^l \varphi_k(t)\,dt = \delta_{l,k}, \quad k, l = 0, \ldots, L \tag{10}$$

($\delta_{l,k}$ Kronecker symbol).

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a multi-index and let

$$\varphi_\gamma(x) = \varphi_{\gamma_1}(x_1) \cdots \varphi_{\gamma_n}(x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n . \tag{11}$$

Then we have

$$\int_{\mathbb{R}^n} x^\beta \varphi_\gamma(x)dx = \delta_{\beta,\gamma}, \quad 0 \leqq |\beta| \leqq L, \quad 0 \leqq |\gamma| \leqq L . \tag{12}$$

Step 2 We prove (i). Let $\Omega \in ER(n)$. We follow Step 1 of the proof of Theorem 3.5. Let, as there, $a_{\nu k}(x)$ be a boundary atom, again 1-atoms are included (what is convenient but not really necessary, since this case is covered by 3.5). We extend as there $a_{\nu k}$ by (3.5/12) to an $\mathbb{R}^n$-atom $b_{\nu k}$ (besides immaterial constants which are again neglected) in the sense of Definition 2.2 without moment conditions of type (2.2/7). Let $Q_{\nu k}$ be the underlying cube and let $Q^e_{\nu k}$ be the related subcube in the sense of (3.2/2). Assume without restriction of generality that $Q^e_{\nu k}$ is centered at the origin and assume that the supports of the functions $\varphi_\gamma(2^\nu x)$, where $\varphi_\gamma(x)$ is given by (11), are contained in $Q^e_{\nu k}$. Let

$$d_\gamma = \int x^\gamma b_{\nu k}(2^{-\nu}x)dx, \quad |\gamma| \leqq L, \tag{13}$$

and let $\tilde{b}_{\nu k}(x)$ be given by

$$\tilde{b}_{\nu k}(2^{-\nu}x) = b_{\nu k}(2^{-\nu}x) - \sum_{|\gamma|\leqq L} d_\gamma \varphi_\gamma(x), \quad x \in \mathbb{R}^n . \tag{14}$$

We claim that $\tilde{b}_{\nu k}(x)$, besides immaterial constants, is the $\mathbb{R}^n$-atom we are looking for. It is an extension of $a_{\nu k}(x)$, satisfies (2.2/5) (maybe with $2d$ instead of d), and can be estimated as in (2.2/6), since $|d_\gamma| \leqq c2^{-\nu(s-\frac{n}{p})}$. Finally by (12) and (13) $\tilde{b}_{\nu k}$ has the required moment conditions of type (2.2/7). Now we are in the same position as in (3.5/13) with $\tilde{b}_{\nu k}(x)$ instead of $b_{\nu k}(x)$. By the same arguments as after (3.5/13) we obtain (1) and the other assertions of part (i).

Step 3 The proof of part (ii) is now almost clear. First we follow Step 2 of the proof of Theorem 3.5 where we need $\Omega \in IR(n)$. Afterwards we use the construction of the previous step where we need $\Omega \in ER(n)$. This proves (5), see (3.2/3). The rest is now the same as above.

Remark. If $s > \sigma_p$ in the B-case and $s > \sigma_{pq}$ in the F-case then we need no moment conditions for the extended atoms and we have the better assertions of Theorem 3.5 compared with the above theorem. To ensure the moment conditions we needed $\Omega \in ER(n)$. One might look for other conditions, but it is hard to see how to ensure the necessary moment conditions for the extended atoms without any additional assumption on the exterior of Ω as it was the case in Theorem 3.5. On the other hand it is not difficult to see that moment conditions are unavoidable in general. Let $0 < p \leqq \infty$ and $s < 0$. Then without moment conditions one cannot expect that (2.3/12) converges in $S'(\mathbb{R}^n)$ or that (3.4/5) converges in $D'(\Omega)$. Let, for example, $\Omega = Q$ be a cube in $\mathbb{R}^n$ with side-length 1, subdivided in the canonical way in $2^{\nu n}$ subcubes with side-length $2^{-\nu}$. Since $\sigma = 0$ is admissible in that case we choose continuous functions $a_{\nu k}(x)$ in (3.4/5) with (3.4/6) and

$$|a_{\nu k}(x)| \leqq 2^{-\nu(s-\frac{n}{p})}, \quad \nu \in \mathbb{N}_0 \tag{15}$$

in the sense of (3.3/8). To materialize $\lambda \in b_{p\infty}(\Omega)$ in (3.4/5) we may choose $\lambda_{\nu k} = 2^{-\nu n/p}$. Then we have

$$|\lambda_{\nu k} a_{\nu k}(x)| \leqq 2^{-\nu s}, \quad s < 0, \quad \nu \in \mathbb{N}_0 . \tag{16}$$

Now it is clear, that we may find, say, non-negative continuous functions $a_{\nu k}(x)$ such that (3.4/5) diverges in any sense.

On the other hand this consideration sheds new light on the role played by the moment conditions of type (3.3/9). We refer also to [35] where we studied problems of this type in greater detail.

4 Complements

4.1 *Atoms in C^∞-domains*

Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ and let Δ^m be the m-th power of the Laplacian, hence $(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2})^m$, $m \in \mathbb{N}$. Let again $\sigma_p = n(\frac{1}{p} - 1)_+$, $0 < q \leqq \infty$, and

$$(1) \qquad s > \begin{cases} -m - 1 + \frac{1}{p} & \text{if } 1 \leqq p \leqq \infty\,, \\ \sigma_p - m & \text{if } 0 < p < 1\,. \end{cases}$$

Then Δ^m maps isomorphically

$$(2) \qquad \{f \in B^{s+2m}_{pq}(\Omega) : D^\alpha f | \partial\Omega = 0 \quad \text{if } |\alpha| \leqq m-1\} \quad \text{onto } B^s_{pq}(\Omega)$$

and (with $p < \infty$)

$$(3) \qquad \{f \in F^{s+2m}_{pq}(\Omega) : D^\alpha f | \partial\Omega = 0 \quad \text{if } |\alpha| \leqq m-1\} \quad \text{onto } F^s_{pq}(\Omega)\,.$$

By the trace theorem, see [33], 3.3.3, p.200, the left-hand spaces in (2) and (3) make sense. In the case of the Sobolev spaces $F^s_{p2}(\Omega) = W^s_p(\Omega)$, $1 < p < \infty$, $0 \leqq s \in \mathbb{N}_0$, the above assertion is classical and goes back to Agmon, Douglis, Nirenberg. Also its extension to the fractional Sobolev spaces $H^s_p(\Omega) = F^s_{p2}(\Omega)$, $1 < p < \infty$, $0 \leqq s < \infty$, the classical Besov spaces $B^s_{pq}(\Omega)$, $1 < p < \infty$, $1 \leqq q \leqq \infty$, $0 < s < \infty$, and the Hölder–Zygmund spaces $\mathscr{C}^s(\Omega) = B^s_{\infty\infty}(\Omega)$, $0 < s < \infty$, is well-known, see [32], Ch. 5, and the references given there. The inclusion of spaces with $s \leqq 0$ goes back to Lions and Magenes, see [22]. Extensions to $p \leqq 1$ had been considered by the first-named author of this paper, see [33] and the references given there. Finally the above result in its full generality is due to J. Franke, see [11]. By (1) any space $B^s_{pq}(\Omega)$ and any space $F^s_{pq}(\Omega)$ may function as a target space. Let $f \in B^s_{pq}(\Omega)$ or $f \in F^s_{pq}(\Omega)$ and $f = \Delta^m g$ in the above sense, then any atomic decomposition of g produces an atomic decomposition of f since $\Delta^m a_{\nu k}$ is an (s,p)-atom in the sense of Definition 3.3/2 and (3.4/6) if $a_{\nu k}$ is a sufficiently smooth $(s+2m,p)$-atom. It can easily be seen that this includes the necessary moment conditions if $s < \sigma_p$ or $s < \sigma_{pq}$ in the sense of the Theorems 2.3 and 3.6. The theorem below must be understood in this sense. We use the same notations as in 3.5 and 3.6.

Theorem. *Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$.*

(i) *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$, $m \in \mathbb{N}$ and let $s \in \mathbb{R}$ be restricted by (1). Then $f \in D'(\Omega)$ belongs to $B^s_{pq}(\Omega)$ if and only if it can be represented as*

$$(4) \qquad f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} {}^{\nu,\Omega}\lambda_{\nu k} \Delta^m a_{\nu k}, \quad \textit{convergence in } D'(\Omega)\,,$$

with $\lambda \in b_{pq}(\Omega)$, where $a_{\nu k}(x)$ are 1_σ-atoms ($\nu = 0$), $(s+2m,p)_{\sigma,L}$-interior atoms ($\nu \in \mathbb{N}$), or $(s+2m,p)_\sigma$-boundary atoms ($\nu \in \mathbb{N}$) with (3.4/6),

$$(5) \qquad s + 2m < \sigma \notin \mathbb{N} \quad \textit{and} \quad L + 1 \in \mathbb{N}_0\,.$$

Furthermore,

$$\inf \|\lambda | b_{pq}(\Omega)\| , \tag{6}$$

where the infimum is taken over all admissible representations (4), *is an equivalent quasi-norm in* $B^s_{pq}(\Omega)$.

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $m \in \mathbb{N}$ *and let* $s \in \mathbb{R}$ *be restricted by* (1).

Then $f \in D'(\Omega)$ *belongs to* $F^s_{pq}(\Omega)$ *if and only if it can be represented by* (4) *with* $\lambda \in f_{pq}(\Omega)$, *where the atoms* $a_{\nu k}(x)$ *have the same meaning as in part* (i) *now with*

$$s + 2m < \sigma \notin \mathbb{N} \quad \text{and} \quad L + 1 \in \mathbb{N}_0, L \geqq \max([\sigma_{pq} - s - 2m], -1) . \tag{7}$$

Furthermore,

$$\inf \|\lambda | f_{pq}(\Omega)\| , \tag{8}$$

where the infimum is taken over all admissible representations (4), *is an equivalent quasi-norm in* $F^s_{pq}(\Omega)$.

Proof. We prove part (ii). The proof of part (i) is the same. Let f be given by (4) with $\lambda \in f_{pq}(\Omega)$. By the assumptions (7) for the atoms $a_{\nu k}(x)$ it follows that

$$g = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} {}^{\nu,\Omega} \lambda_{\nu k} a_{\nu k}(x) \tag{9}$$

belongs to $F^{s+2m}_{pq}(\Omega)$. Since in any case Δ^m is a bounded map from $F^{s+2m}_{pq}(\Omega)$ into $F^s_{pq}(\Omega)$ it follows

$$\|f | F^s_{pq}(\Omega)\| \leqq c_1 \|g | F^{s+2m}_{pq}(\Omega)\| \leqq c_2 \|\lambda | f_{pq}(\Omega)\| , \tag{10}$$

where the latter is a consequence of Theorem 3.6. On the other hand by the same theorem and by what has been said in front of the above theorem both inequalities in (10) can be replaced by equivalences if g and its representation (9) are appropriately chosen. This proves the desired assertion for any $f \in F^s_{pq}(\Omega)$ which can be represented in this way. But by the isomorphism property of Δ^m this applies to any $f \in F^s_{pq}(\Omega)$.

Remark 1 As for the smoothness assumptions we have

$$\Delta^m a_{\nu k} \in \mathscr{C}^{\sigma - 2m}(\overline{\Omega}) , \tag{11}$$

and $\sigma - 2m$ may be any number larger than s. Hence, $\Delta^m a_{\nu k}$ may be a "non-smooth atom". But, of course, this does not mean, that any such "non-smooth atom", equipped with the necessary support and moment conditions, is an admissible atom in the above sense.

Remark 2 The Theorems 3.5 and 3.6 pave the way, so we hope, to interesting applications to integral and (pseudo-) differential operators in non-smooth domains. Whether the above theorem can be used in that way is not so clear. But it may serve as a hint how to introduce atoms on more general structures such as boundaries of (smooth and non-smooth) domains, (Riemannian) manifolds etc. First one assumes that s in the corresponding spaces B^s_{pq} and F^s_{pq} is large. Then one has no trouble with "moment conditions" (what ever this means). Afterwards one can try to introduce atoms with the help of distinguished operators, for example Laplace–Beltrami operators etc.

4.2 Entropy numbers

The aim of this subsection is to support our opinion that the Theorems 3.5 and 3.6 can be used in spectral theory of integral and (pseudo-) differential operators in smooth and non-smooth domains. But these applications must be shifted to a later occasion.

First we define what is meant by entropy numbers. Let B_1 and B_2 be two quasi-Banach spaces with B_1 continuously and compactly embedded in B_2. In other words, given any $\varepsilon > 0$ there are finitely many balls in B_2 of radius ε which cover the unit ball $U_1 = \{u \in B_1 : \|u|B_1\| \leqq 1\}$ in B_1. Let $k \in \mathbb{N}$ and let $\mathrm{id} : B_1 \to B_2$ be the natural embedding. The kth entropy number e_k of id is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in B_2 of radius ε which cover U_1 (considered as a subset of B_2). We took over this definition from [5], p. 140, where one finds also the necessary references concerning entropy numbers and their history which we will not repeat here.

Theorem. *Let $\Omega \in MR(n)$. Let*

(1) $-\infty < s_2 < s_1 < \infty,\ 0 < p_1 \leqq p_2 \leqq \infty,\ 0 < q_1 \leqq \infty,\ 0 < q_2 \leqq \infty$

and let

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}\,. \tag{2}$$

Then the natural embedding

$$\mathrm{id} : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{3}$$

is compact. Let e_k be the corresponding entropy numbers, then there exist two positive numbers c_1 and c_2 such that

$$c_1 k^{-\frac{s_1-s_2}{n}} \leqq e_k \leqq c_2 k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N} \tag{4}$$

Proof. Let K_1 and K_2 be two open balls with $\overline{K_1} \subset \Omega$ and $\overline{\Omega} \subset K_2$. We may assume that the unit ball $\{f : \|f|B^{s_1}_{p_1q_1}(\Omega)\| \leqq 1\}$ of $B^{s_1}_{p_1q_1}(\Omega)$ is the restriction of a subset of $\{g : \|g|B^{s_1}_{p_1q_1}(K_2)\| \leqq c\}$ for some $c > 1$. By [5] and [6] there exist 2^{k-1} elements $g_l \in B^{s_2}_{p_2q_2}(K_2)$ with

$$\min_l \|g - g_l|B^{s_2}_{p_2q_2}(K_2)\| \leqq c k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}\,. \tag{5}$$

By restriction one has a similar assertion with Ω instead of K_2. This proves the right-hand side of (4). Recall that (4) is known for smooth domains, in particular for K_1, see [5] and [6]. Then the left-hand side of (4) follows from the above arguments with K_1 and Ω instead of Ω and K_2, respectively.

Remark 1 We have

$$B^s_{p,\min(p,q)}(\Omega) \subset F^s_{pq}(\Omega) \subset B^s_{p,\max(p,q)}(\Omega) \,. \tag{6}$$

This inclusion with $\mathbb{R}^n$ instead of Ω may be found in [33], 2.3.2, p. 47. But we introduced spaces on Ω via restriction procedures, see Definition 3.1/1, and such a procedure preserves embeddings of this type. Since q_1 and q_2 in (3) are independent of each other, (6) proves that B in (3) and (4) can be replaced by F on one or both sides (with $p_1 < \infty$ and /or $p_2 < \infty$).

Remark 2 We carried over (4) from smooth domains to arbitrary domains $\Omega \in MR(n)$ by the restriction argument without knowing whether there exists a linear and bounded extension operator from $B^s_{pq}(\Omega)$ into $B^s_{pq}(K_2)$. But we add a warning. The same argument cannot be applied, for example, to approximation numbers. This shows again the superiority of the entropy numbers compared with other geometric quantities measuring compactness.

Application to spectral theory. Let A be a compact linear operator acting in a complex quasi-Banach space B and let $\{e_k\}$ be the sequence of the entropy numbers of $\{Ab : b \in B, \|b|B\| \leqq 1\}$, the image of the unit ball. Let, on the other hand, $\{\mu_k\}$ be the sequence of the eigenvalues of A, each repeated according to algebraic multiplicity and ordered by

$$|\mu_1| \geqq |\mu_2| \geqq \cdots \,. \tag{7}$$

Then, by Carl's inequality,

$$|\mu_k| \leqq \sqrt{2}e_k, \quad k \in \mathbb{N} \,. \tag{8}$$

See [7] for details, references to original papers and to related books. In [7] we used this observation to study spectral properties of operators of type

$$A = a_2(x)Da_1(x) \tag{9}$$

in function spaces, preferably of type $H^s_p(\Omega) = F^s_{p2}(\Omega)$ in smooth domains. Here $a_1(x)$ and $a_2(x)$ are two singular functions, for example belonging to some spaces $L_r(\Omega)$, $1 \leqq r \leqq \infty$, whereas D stands, for example, for an (elliptic) pseudodifferential operator. The basic idea to handle spectral assertions for the operator A is the following. First one proves mapping properties of the pseudodifferential operators between function spaces of type, say, $B^s_{pq}(\Omega)$ or $F^s_{pq}(\Omega)$. Then one incorporates the singular functions $a_1(x)$ and $a_2(x)$ via Hölder inequalities in function spaces as studied in [27], which can be extended immediately to the non-smooth domains under consideration here. Finally a combination of (8) and (4) gives sharp (in order) results for the distribution of the eigenvalues $\{\mu_k\}$. Mapping properties for (exotic) pseudodifferential operators in function spaces on $\mathbb{R}^n$ can be obtained by using atoms, see [13], [34], Ch. 6., and [30], [31]. But now, using Theorems 3.5 and 3.6, one can study integral operators and pseudodifferential operators in non-smooth domains in the same way. On the basis of the so-obtained mapping properties between function spaces and the indicated technique one has a good chance to derive sharp assertions for the distribution of eigenvalues of operators of type (9). We hope to return to this subject in a later paper.

4.3 Complements

We comment briefly on a few key words connected with our approach. But instead of descriptions we restrict ourselves to references.

Domains. The necessary discussions and references about non-smooth domains have been given in 3.2 and Remark 3.5/2. In 3.3 we also commented on the related "jet- definition" of $\mathscr{C}^\sigma(\overline{\Omega})$.

Intrinsic definitions. Spaces of Sobolev and Besov type have been defined intrinsically, generalizing the "jet-definition" of $\mathscr{C}^\sigma(\overline{\Omega})$ on more general sets than we did. The necessary references, especially to the extensive work of Jonsson and Wallin, have been given in the Introduction.

Extensions. Closely connected with the intrinsically defined spaces is the problem whether there exists a linear and bounded extension operator from these spaces on $\mathbb{R}^n$. As far as spaces B^s_{pq} and F^s_{pq} on $\mathbb{R}^n_+$ or on smooth domains are concerned we refer to [32], [33] and [34] and the references given there. There are several papers discussing sufficient (and necessary) conditions for the existence of linear and bounded extension operators. As for Sobolev spaces in Lipschitz domains (minimally smooth) we refer to [28], p. 180–192. The extension problem for Sobolev spaces in the more general (ε, δ)-domains has been studied in [16]. But in this context consult [23], p. 29, for further references to earlier papers. The spaces C^s_p introduced in [2] coincide with $F^s_{p\infty}$ if $s > \sigma_p$, see [34], p. 248. The corresponding extension problem for minimally smooth domains has been treated in [2], and for (ε, δ)-domains in [26]. In [21] Kaljabin proved that Stein's extension method works also for spaces F^s_{pq} with $s > 0$, $1 < p < \infty$, $1 < q < \infty$, in minimally smooth domains. Extension operators for Besov spaces in (ε, δ)-domains have been constructed in [3]. Finally we refer to [9] which deals with anisotropic Sobolev spaces in (ε, δ)-domains. Further discussions may be found in [23], pp. 70/71.

Atoms. There are very few papers dealing with atoms on domains. As far as Hardy spaces are concerned we refer to [19], p. 86, [25] and [1]. As for Besov spaces we mention the recent work by Jonsson and Wallin, [20], [17], [18].

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