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Theta functions and cycles on some abelian fourfolds

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1 Introduction

1.1 Let X be an abelian variety over $\mathbb{C}$ and let K be an imaginary quadratic field with

$$K \subset \mathrm{End}(X) \otimes \mathbb{Q}\,.$$

The action t of K on T_0X, the tangent space at the origin of X, can be diagonalized thus, w.r.t. a suitable basis of T_0X:

$$t(k) = \mathrm{diag}(\sigma(k), \ldots, \sigma(k), \overline{\sigma}(k), \ldots, \overline{\sigma}(k)), \quad (k \in K)$$

with $\sigma : K \hookrightarrow \mathbb{C}$ an embedding of K. We say X is of type (p, q) if there are p entries $\sigma(k)$ and q entries $\overline{\sigma}(k)$.

In case X has type (p, p), Weil [W] constructed a two dimensional subspace $W \subset B^p(X) := H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. For general X, $\dim X > 2$, of type (p, p) one has in fact:

$$\dim NS(X)_{\mathbb{Q}} = \dim B^1(X) = 1, \qquad B^p(X) = D^p(X) \oplus W\,,$$

here $D^p(X) := \mathrm{Im}(B^1(X)^p \to B^p(X))$, $(\omega_1, \ldots, \omega_p) \mapsto \omega_1 \wedge \cdots \wedge \omega_p$. In particular such an X has Hodge classes which are not obtained as intersections (cup products) of divisors classes. In [vG2] we gave an elementary introduction to these abelian varieties.

1.2 For certain abelian varieties X of Weil-type with field $K = \mathbb{Q}(i)$, $\mathbb{Q}(\omega)$, $\omega^3 = -1$, C. Schoen [S] gave a construction for algebraic cycles on X whose classes span W.

In this paper we use theta functions to construct algebraic cycles on certain abelian 4-folds of Weil type. In fact we show that in the case under consideration the abelian varieties allow a rational map onto a smooth quadric $\overline{Q}$ in $\mathbb{P}^5$, the classes of the inverse images of the rulings of $\overline{Q}$ give classes which do not lie in D^2.

Theorem 3.7 *Let X be an abelian 4-fold of Weil-type* $(2,2)$, *with field* $\mathbb{Q}(i)$ *and* $\det H = 1$.

Then the space W of Weil–Hodge cycles is spanned by classes of algebraic cycles.

1.3 We refer to 2.1 and [vG2] for the explanation of '$\det H$'. It turns out, see 5.2, that the abelian varieties we consider are among those studied by Schoen [S], however, our method is different. We don't know how the cycles obtained by us and Schoen are related.

2 The abelian varieties

2.1 The abelian varieties of Weil-type we consider were introduced in [vG1], 10.6, 10.7. Let $\mathbb{S}_{2n}$ be the Siegel upper half space of $(2n) \times (2n)$ complex, symmetric matrices whose imaginary part defines a positive definite quadratic form on $\mathbb{R}^{2n}$ and let

$$\mathbb{H}_{2n} := \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ -\tau_{12} & \tau_1 \end{pmatrix} \in \mathbb{S}_{2n} : \tau_1 \in \mathbb{S}_n,\ {}^t\tau_{12} = -\tau_{12} \right\}$$

and

$$\Omega_\tau := \begin{pmatrix} I \\ \tau \end{pmatrix} .$$

We will write vectors as row vectors. The abelian varieties are then defined by: $X = X_\tau := \mathbb{C}^{2n}/(\mathbb{Z}^{4n}\Omega_\tau)$ with $\tau \in \mathbb{H}_{2n}$. The equality:

$$M\Omega_\tau = \Omega_\tau A, \quad \text{with } M := \begin{pmatrix} 0 & I & & \\ -I & 0 & & \\ & & 0 & I \\ & & -I & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

shows that X_τ has an automorphism ϕ of order four with:

$$\phi : X_\tau \to X_\tau, \qquad \phi_* = M \in \mathrm{End}(H_1(X_\tau, \mathbb{Z})), \qquad d\phi = A \in \mathrm{End}(T_0 X_\tau) .$$

Note that the eigenvalues of $d\phi$ are i and $-i$, both with multiplicity n, thus the X_τ are of Weil-type (n, n).

The principal polarization $E : \wedge^2 H_1(X, \mathbb{Z}) \to \mathbb{Z}$ on X_τ is given by an alternating $4n \times 4n$ matrix of the same shape as A, but with four $2n \times 2n$ blocks rather than $n \times n$ blocks. A simple computation verifies that ϕ_* preserves the polarization:

$$E(\phi_* x, \phi_* y) = E(x, y) \quad (\forall x, y \in H_1(X, \mathbb{Z})) .$$

The space $H_1(X, \mathbb{Q})$ has the structure of a $\mathbb{Q}(i)$-vector space via the action of ϕ_*. Define:

$$H : H_1(X, \mathbb{Q}) \times H_1(X, \mathbb{Q}) \to \mathbb{Q}(i); \qquad H(x, y) := E(x, \phi_* y) + iE(x, y) .$$

Then H is a Hermitian form ($H(x,y) = \overline{H(y,x)}$ and $H(x,\phi_* y) = iH(x,y)$) and its determinant (modulo norms of elements of $\mathbb{Q}^*$) is an isogeny invariant. An easy computation shows that

$$\det H = 1 \, .$$

2.2 We recall the definition of theta functions with characteristics $m, m' \in \mathbb{R}^g$ (cf. [I], p. 49):

$$\theta_{m,m'}(\tau,z) = \sum_{k\in\mathbb{Z}^g} \exp(\pi i[(k+m)\tau^t(k+m) + 2(k+m)^t(z+m')]) \, ,$$

here $\tau \in \mathbb{S}_g$, $z \in \mathbb{C}^g$. The zero locus of the function $\theta_{0,0}(\tau,z)$ (Riemann's theta function on X_τ) in $\mathbb{C}^g$ defines a symmetric divisor $\Theta_\tau \subset X_\tau$. The pair $(X_\tau, \mathcal{O}(\Theta_\tau))$ is a principally polarized abelian variety. Thus the Riemann theta function is the pull-back of a global section of $\mathcal{O}(\Theta_\tau)$ to $\mathbb{C}^g$. If $X_\tau \cong X_{\tau'}$ one has $\mathcal{O}(2\Theta_\tau) \cong \mathcal{O}(2\Theta_{\tau'})$, thus this line bundle is canonically defined (intrinsically: the unique totally symmetric line bundle algebraically equivalent to twice the principal polarization).

We define theta functions with half integral characteristics:

$$\theta\begin{bmatrix} m \\ m' \end{bmatrix}(\tau,z) := \theta_{m/2,m'/2}(\tau,z) \quad \text{for } m,m' \in \{0,1\}^g \, .$$

Let α the point of order two in X_τ defined by $(m + \tau m')/2 \in \mathbb{C}^g$ and let $(\Theta_\tau)_\alpha$ be the translate of Θ_τ by α. Then the function $\theta[{}^{m}_{m'}]$ is a basis of $H^0(X_\tau, \mathcal{O}((\Theta_\tau)_\alpha))$. The 2^g functions:

$$\theta\begin{bmatrix} m \\ 0 \end{bmatrix}(2\tau, 2z), \quad m \in \{0,1\}^g$$

are a basis of $H^0(X_\tau, \mathcal{O}(2\Theta_\tau))$.

From the definition of the theta function one easily proves the following lemma, which gives the action of ϕ^* on global sections of certain line bundles on X_τ and which gives a formula for the maps

$$(1-\phi)^* : H^0(X, \mathcal{O}(\Theta_\alpha)) \to H^0(X, \mathcal{O}(2\Theta)) \, ,$$

for the translates Θ_α of Θ with $\alpha \in \ker(1-\phi)$.

2.3 Lemma. *For* $\varepsilon_i, \varepsilon_i' \in \{0,1\}^n$, $z \in \mathbb{C}^{2n}$, $\tau \in \mathbb{H}_{2n}$ *and* $N \in \mathbb{Z}_{\geqq 0}$ *we have*:

1.
$$\theta\begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_1' & \varepsilon_2' \end{bmatrix}(N\tau, zA) = (-1)^{\varepsilon_1^{\prime t}\varepsilon_1'}\theta\begin{bmatrix} \varepsilon_2 & \varepsilon_1 \\ \varepsilon_2' & \varepsilon_1' \end{bmatrix}(N\tau, z) \, .$$

2.
$$\theta\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon' & \varepsilon' \end{bmatrix}(N\tau, z(I-A)) = \sum_{\rho\in\{0,1\}^n} (-1)^{(\rho+\varepsilon)^t\varepsilon'}\theta\begin{bmatrix} \rho & \rho+\varepsilon \\ 0 & 0 \end{bmatrix}(2N\tau, 2z) \, .$$

Proof. First we prove the second statement. For $\tau \in \mathbb{H}_{2n}$ (and ${}^tA$ the transposed of A):

$$A^2 = -I, \qquad {}^tA = -A, \qquad A\tau^tA = \tau, \qquad A\tau + \tau^tA = 0 \, .$$

Note that these relations remain valid upon replacing τ by $N\tau$, we will suppress N in the rest of the proof.

We compute $\theta_{(m_1,m_2),(m'_1,m'_2)}(\tau, z(I-A))$. Using the relations, we can write:

$$\tau = (I+A)(1/2)\tau^t(I+A), \qquad I = (1/2)(I+A)(I-A).$$

and we get:

$$(k+m)\tau^t(k+m) = (k+m)(I+A)(1/2)\tau^t(I+A)^t(k+m),$$

$$(k+m)^t(z(I-A)+m'I) = (k+m)(I+A)^t(z+(m'/2)(I+A)).$$

Writing $k=(k_1,k_2)$, $m=(m_1,m_2)$, $m'=(m'_1,m'_2)\in \mathbb{Z}^n\times\mathbb{Z}^n$ we get:

$$\begin{aligned}(k+m)(I+A) &= (k_1-k_2+m_1-m_2, k_1+k_2+m_1+m_2)\\ &= (2l_1+\rho+m_1-m_2, 2l_2+\rho+m_1+m_2),\end{aligned}$$

with $\rho\in\{0,1\}^n$ defined by $k_1\pm k_2\in\rho+2\mathbb{Z}^n$ and $l_1 := (k_1-k_2-\rho)/2$, $l_2 := (k_1+k_2-\rho)/2\in\mathbb{Z}^n$. Then:

$$(k+m)\tau^t(k+m) = (l_1+(\rho+m_1-m_2)/2, l_2+(\rho+m_1+m_2)/2)(2\tau)^t(l_1+\cdots)$$

and in the second term we get:

$$(z+(m'/2)(I+A)) = (2z+(m'_1-m'_2, m'_1+m'_2)/2),$$

thus by summation of the terms over $\rho\in\{0,1\}^n, l_1,l_2\in\mathbb{Z}^n$ we have:

$$\begin{aligned}&\theta_{(m_1,m_2),(m'_1,m'_2)}(\tau, z(I-A))\\ &\quad = \sum_{\rho\in\{0,1\}^n}\theta_{(m_1-m_2+\rho,\, m_1+m_2+\rho)/2,(m'_1-m'_2,m'_1+m'_2)/2}(2\tau, 2z).\end{aligned}$$

The isogeny formula follows from this, but since we want $(0\,0)$ rather than $(0\,\varepsilon')$ in the second row of the characteristic, we get the sign (cf. [I], θ.2, p.49).

For the first formula, we observe that

$$(k+m)^t(zA+m') = (k+m)^tA^t(z+m'A^{-1}),$$

$$(k+m)\tau^t(k+m) = (k+m)^tA\tau A^t(k+m),$$

and that

$$(k+m)^tA = (k_2+m_2, -k_1-m_1), \qquad m'A^{-1} = (k_2+m_2, -k_1-m_1).$$

Thus we have:

$$\theta_{(m_1,m_2),(m'_1,m'_2)}(\tau, zA) = \theta_{(m_2,-m_1),(m'_2,-m'_1)}(\tau, z).$$

Since we want characteristics in $\{0,1\}^{2n}$, we get the sign in the formula. $\square$

2.4 Corollary. *The automorphism ϕ of order two induces an automorphism ϕ^* on $H^0(X_\tau, \mathcal{O}(2\Theta_\tau))$ of order two (in fact, the corresponding theta functions are even). Let $H^0(X_\tau, \mathcal{O}(2\Theta_\tau))_\pm$ be the $\pm$ eigenspace of ϕ^*, then*

$$\dim H^0(X_\tau, \mathcal{O}(2\Theta_\tau))_+ = 2^{n-1}(2^n + 1)\,,$$

$$\dim H^0(X_\tau, \mathcal{O}(2\Theta_\tau))_- = 2^{n-1}(2^n - 1)\,.$$

Bases of $\dim H^0(X_\tau, \mathcal{O}(2\Theta_\tau))_\pm$ *are given by the non-zero*

$$\theta\begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ 0 & 0 \end{bmatrix}(2\tau, 2z) \pm \theta\begin{bmatrix} \varepsilon_2 & \varepsilon_1 \\ 0 & 0 \end{bmatrix}(2\tau, 2z)$$

with $\varepsilon_1, \varepsilon_2 \in \{0,1\}^n$.

2.5 *Remark.* The isogeny formula 2.3.2 implies that each of the functions $\theta[{\sigma \atop 0}](2\tau, 2z)$, ($\sigma \in \{0,1\}^{2n}$) can be expressed as a linear combination of the 2^{2n} functions $\theta[{\varepsilon \atop \varepsilon'}{\varepsilon \atop \varepsilon'}](\tau, z(I - A))$, ($\varepsilon, \varepsilon' \in \{0,1\}^n$). Thus these functions also provide a basis of $H^0(X_\tau, \mathcal{O}(2\Theta_\tau))$.

3 The cycles

3.1 In this section, $X = X_\tau$ with $\tau \in \mathbb{H}_{2n}$. Let $N = 2^{2n} - 1$. We study the geometry of the Kummer map:

$$K : X_\tau \to \mathbb{P}^N = \mathbb{P}H^0(X, \mathcal{O}(2\Theta))\,,$$

$$z \mapsto (\cdots : \theta[{\sigma \atop 0}](2\tau, 2z) : \cdots) \quad (\sigma \in \{0,1\}^{2n})\,.$$

In case X is indecomposable, the map K embeds $X/\pm 1$ in $\mathbb{P}^N$, if the ppav is a product of ppav's $X = X_1 \times \cdots \times X_k$ then $K(X) \cong K(X_1) \times \cdots \times K(X_k)$.

3.2 The morphism K is equivariant for the action of the automorphism of order 4, which acts on $\mathbb{P}^N$ by (see 2.3.1):

$$(\cdots : X_{\sigma_1\sigma_2} : \cdots) \mapsto (\cdots : X_{\sigma_2\sigma_1} : \cdots) \quad (\sigma_1, \sigma_2 \in \{0,1\}^n)\,.$$

The two eigenspaces of this involution we denote by (cf. 2.4)

$$\mathbb{P}_\pm := \mathbb{P}H^0(X_\tau, \mathcal{O}(2\Theta_\tau))_\pm$$

Let $\Pi_- : \mathbb{P}^N \to \mathbb{P}_-$ be the projection on to $\mathbb{P}_-$ from $\mathbb{P}_+$, and let:

$$K_- := \Pi_- \circ K : X \to \mathbb{P}_-\,,$$

be the composition of K with the projection Π_-.

3.3 Lemma. *For an indecomposable ppav the base locus B of the rational map $K_- : X \to \mathbb{P}_-$ consists of a set of 2^{2n} two-torsion points, in fact:*

$$B = \ker(\phi - 1) = \{x \in X : \phi(x) = x\} \cong (\mathbb{Z}/2\mathbb{Z})^{2n}\,.$$

Moreover, B is a maximal isotropic subgroup of $X[2]$ and $X/B \cong X$.

Proof. Since K is a morphism, $K(B)$ is base locus of Π_-, so $K(B) = K(X) \cap \mathbb{P}_+$. As K is equivariant for ϕ and $\mathbb{P}_+$ is an eigenspace, B is contained in the set of points $x \in X$ with $\phi(x) = \pm x$, that is $(\phi \pm 1)x = 0$. Note $\phi \circ (\phi + 1) = \phi - 1$, so $\ker(\phi + 1) = \ker(\phi - 1)$; since $(\phi - 1)^2 = -2\phi$ we get $B \subset \ker(\phi - 1) \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$.

To see that all points in $\ker(\phi - 1)$ are mapped to $\mathbb{P}_+$ (and not to $\mathbb{P}_-$), note first of all that $0 \in X$ is mapped to $\mathbb{P}_+$ since 2.3.1 (with $N = 2$ there) implies that $\theta[{}^{\sigma_1}_{0}\ {}^{\sigma_2}_{0}](2\tau, 0) = \theta[{}^{\sigma_2}_{0}\ {}^{\sigma_1}_{0}](2\tau, 0)$.

The map K is equivariant for translations by points of order two (they act on $\mathbb{P}^N$ as the Heisenberg group), thus each $a \in X[2]$ gives a projective transformation $U(a)$ satisfying

$$K(x + a) = U(a)K(x) \quad (a \in X[2])\,.$$

Let $a \in \ker(\phi - 1)$. Then translation by a and ϕ commute ($\phi(x + a) = \phi(x) + \phi(a) = \phi(x) + a$ for all $x \in X$). Therefore $U(a)$ maps the eigenspaces of ϕ into themselves (it cannot interchange them since the dimensions are different). Thus $K(a) = U(a)K(0)$ also lies in $\mathbb{P}_+$ and $B = \ker(\phi - 1)$.

That $X/B \cong X$ is trivial since $\phi - 1 \in \mathrm{End}(X)$. Note that $B = \ker(\phi - 1) = \mathrm{im}(\phi - 1)$, $\phi_*^2 = -1$ and $E(\phi_* x, \phi_* y) = E(x, y)$ give:

$$E((\phi_* - 1)x, (\phi_* - 1)y) = 2E(x, y) + E(\phi_*^2 x, \phi_* y) - E(\phi_* x, y) = 2E(x, y)\,.$$

The polarization E defines the Weil-pairing and this formula shows that the Weil-pairing is trivial on B. □

3.4 Note that the proof shows that $K(X) \cap \mathbb{P}_-$ is empty, so the projection of $K(X)$ to $\mathbb{P}_+$ is a morphism.

3.5 We recall the basic facts on the quadrics on $\mathbb{P}^N$. A basis for the vector space of these quadrics is provided by the

$$Q\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} := \sum_\sigma (-1)^{\sigma^t \varepsilon'} X_\sigma X_{\sigma+\varepsilon}, \quad (\sigma, \varepsilon, \varepsilon' \in (\mathbb{Z}/2\mathbb{Z})^{2n},\ \varepsilon^t \varepsilon' = 0)\,,$$

which are indexed by the $2^{2n-1}(2^{2n} + 1)$ even theta characteristics (each $Q_{[{}^{\varepsilon}_{\varepsilon'}]}$ is an eigenvector for the action of the Heisenberg group).

The pull-back of a quadric Q along K gives a section of $\mathcal{O}(4\Theta)$ and, since the $\theta[{}^{\sigma}_{0}](2\tau, 2z)$ are even functions in z, this section will also be given by an even theta function. A basis of $H^0(X_\tau, \mathcal{O}(4\Theta))_+$, the space of these even theta functions, is given by the $\theta[{}^{\varepsilon}_{\varepsilon'}](\tau, 2z)$ with $\varepsilon^t \varepsilon' = 0$. The following theta relation ([I], IV.1) expresses the pull-back of the $Q_{[{}^{\varepsilon}_{\varepsilon'}]}$'s in this basis:

$$\begin{aligned} K^* Q_{[{}^{\varepsilon}_{\varepsilon'}]} :=& \sum_\sigma (-1)^{\sigma^t \varepsilon'} \theta\begin{bmatrix}\sigma\\ 0\end{bmatrix}(2\tau, 2z)\theta\begin{bmatrix}\sigma+\varepsilon\\ 0\end{bmatrix}(2\tau, 2z)\\ =& \ \theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, 0)\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, 2z)\,.\end{aligned}$$

In particular, $K(X)$ lies in $Q_{[{\varepsilon \atop \varepsilon'}]}$ (we denote the quadric in $\mathbb{P}^N$ and its defining equation by the same symbol) iff the corresponding theta constant vanishes, i.e. $\theta[{\varepsilon \atop \varepsilon'}](\tau, 0) = 0$.

From Lemma 2.3.1 we have that on X_τ the $2^{n-1}(2^n-1)$ even theta constants $[{\varepsilon \atop \varepsilon'}\ {\varepsilon \atop \varepsilon'}]$ with $\varepsilon^t\varepsilon' \equiv 1 \mod 2$ vanish.

In the case $n = 2$ the 6 even theta constants with the following characteristics vanish:

$$\begin{bmatrix}1010\\1010\end{bmatrix}, \quad \begin{bmatrix}1010\\1111\end{bmatrix}, \quad \begin{bmatrix}1111\\1010\end{bmatrix}, \quad \begin{bmatrix}1111\\0101\end{bmatrix}, \quad \begin{bmatrix}0101\\0101\end{bmatrix}, \quad \begin{bmatrix}0101\\1111\end{bmatrix} .$$

Therefore $K(X)$ lies in 6 independent quadrics. We will show, by explicit computation, that there exists a (unique) quadric Q which is linear combination of these 6 quadrics and which is the cone over a smooth quadric in $\mathbb{P}^5$. This implies that the image of $K(X)$ in $\mathbb{P}^5$ is in this quadric.

3.6 Proposition. *Define a quadric Q in $\mathbb{P}^{15}$ by:*

$$Q := Q\begin{bmatrix}1010\\1010\end{bmatrix} - Q\begin{bmatrix}1010\\1111\end{bmatrix} - Q\begin{bmatrix}1111\\1010\end{bmatrix} + Q\begin{bmatrix}1111\\0101\end{bmatrix} - Q\begin{bmatrix}0101\\0101\end{bmatrix} + Q\begin{bmatrix}0101\\1111\end{bmatrix} .$$

Then for any $\tau \in \mathbb{H}_4$, we have $K(X_\tau) \subset Q$ and Q is the cone over a smooth quadric $\overline{Q}$ in $\mathbb{P}^5 = \mathbb{P}_-$, thus:

$$K_- : X - B \to \overline{Q} \subset \mathbb{P}^5 .$$

Proof. Using the definition of $Q[{\varepsilon \atop \varepsilon'}]$, it is clear that $Q[{1010 \atop 1010}]$ and $Q[{1010 \atop 1111}]$ involve the same 8 monomials $(X_\sigma X_{\sigma+(1010)} = X_{0000}X_{1010}, \dots)$ with coefficients ± 2. The signs differ iff $\varepsilon^t\sigma = (1010) \cdot {}^t\sigma = -1$, and thus:

$$Q\begin{bmatrix}1010\\1010\end{bmatrix} - Q\begin{bmatrix}1010\\1111\end{bmatrix} = 4(X_{0001}X_{1011} + X_{0100}X_{1110} - X_{0011}X_{1001} - X_{1100}X_{0110}) .$$

Similarly, one finds:

$$Q\begin{bmatrix}1111\\1010\end{bmatrix} - Q\begin{bmatrix}1111\\0101\end{bmatrix} = 4(X_{0001}X_{1110} + X_{0100}X_{1011} - X_{0010}X_{1101} - X_{1000}X_{0111}) ,$$

$$Q\begin{bmatrix}0101\\0101\end{bmatrix} - Q\begin{bmatrix}0101\\1111\end{bmatrix} = 4(X_{0010}X_{0111} + X_{1000}X_{1101} - X_{0011}X_{0110} - X_{1100}X_{1001}) .$$

As coordinates on $\mathbb{P}^9 = \mathbb{P}_+$ we take:

$$Z_{\sigma_1\sigma_2} := X_{\sigma_1\sigma_2} + X_{\sigma_2\sigma_1}$$

with the convention that in $Z_{\sigma_1\sigma_2}$ with $\sigma_1 = (a_1, b_1)$, $\sigma_2 = (a_2, b_2) \in \{0,1\}^2$ we have that $2a_1 + b_1 \leqq 2a_2 + b_2$, and on $\mathbb{P}^5$ we take:

$$Y_{01} := X_{0001} - X_{0100}, \qquad Y_{02} := X_{0010} - X_{1000}, \qquad Y_{03} := X_{0011} - X_{1100}\,,$$

$$Y_{12} := X_{0110} - X_{1001}, \qquad Y_{13} := X_{0111} - X_{1101}, \qquad Y_{23} := X_{1011} - X_{1110}\,,$$

it is convenient to agree that $Y_{ab} = -Y_{ba}$ if $a > b$. Using the identity

$$2(Z_{st}Z_{uv} + Z_{ts}Z_{vu}) = (Z_{st} - Z_{ts})(Z_{uv} - Z_{vu}) + (Z_{st} + Z_{ts})(Z_{uv} + Z_{vu})\,,$$

with $s, t, u, v \in (\mathbb{Z}/2\mathbb{Z})^2$, we get:

$$Q\begin{bmatrix}1010\\1010\end{bmatrix} - Q\begin{bmatrix}1010\\1111\end{bmatrix} = 2(Y_{01}Y_{23} + Y_{03}Y_{12} + Z_{01}Z_{23} - Z_{03}Z_{12})\,,$$

$$Q\begin{bmatrix}1111\\1010\end{bmatrix} - Q\begin{bmatrix}1111\\0101\end{bmatrix} = 2(-Y_{01}Y_{23} + Y_{02}Y_{13} + Z_{01}Z_{23} - Z_{02}Z_{13})\,,$$

$$Q\begin{bmatrix}0101\\0101\end{bmatrix} - Q\begin{bmatrix}0101\\1111\end{bmatrix} = 2(Y_{02}Y_{13} - Y_{03}Y_{12} + Z_{02}Z_{13} - Z_{03}Z_{12})\,.$$

the equation for Q is then:

$$Q = 4(Y_{01}Y_{23} - Y_{02}Y_{13} + Y_{03}Y_{12})\,,$$

so Q is the cone over a quadric $\overline{Q}$ in $\mathbb{P}^5$. Since $K(X) \subset Q$ we get $K_-(X) \subset \overline{Q}$. □

3.7 Theorem. *Let X be an abelian 4-fold of Weil type $(2,2)$, field $\mathbb{Q}(i)$ and* $\det H = 1$.

Then the space W of Weil–Hodge cycles is spanned by classes of algebraic subvarieties.

Proof. Since X is isogeneous to an X_τ (cf. [vG2]), it suffices to prove the result for these varieties. We show that for a general X_τ the rational map $K_- : X_\tau \to \overline{Q}$ is dominant and that the (strict) pull-back of a general linear subspace $\mathbb{P}^2 \subset \overline{Q}$ has a cohomology class which is not in $D^2(X)$. Using the action of $\mathbb{Q}(i)^*$ by pull-back on $B^2(X) = D^2(X) \oplus W, \cdot$ so $x \cdot (v, w) = ((x\bar{x})^2 v, x^4 w)$, with $W = \wedge^4_{\mathbb{Q}(i)} H^1(X, \mathbb{Q}) \cong K$, it then follows that W is spanned by classes of cycles. Since any two points in $\mathbb{H}_4$ can be connected by a holomorphic curve, the closure of such a cycle in the generic fiber specializes to a cycle with a class not in $D^2(X)$ for any X_τ for $\tau \in \mathbb{H}_4$.

To show that K_- is dominant and to compute the cycle class, we specialize to the case that $X \cong Y \times Y$ with Y a general abelian surface. In terms of the period matrix for X that means we take $\tau \in \mathbb{H}_4$ with $\tau_{12} = 0$. One has:

$$H^0(X, \mathcal{O}(2\Theta_X)) \cong H^0(Y, \mathcal{O}(2\Theta_Y)) \otimes H^0(Y, \mathcal{O}(2\Theta_Y))$$

(in fact, from the definition one has: $\theta[{}^{\sigma_1}_{0}{}^{\sigma_2}_{0}](2\tau, 2z) = \theta[{}^{\sigma_1}_{0}](2\tau_1, 2z_1)\theta[{}^{\sigma_2}_{0}]$ $(2\tau_1, 2z_2)$ with $z = (z_1, z_2) \in \mathbb{C}^2 \times \mathbb{C}^2$). The coordinate functions Y_{ij} on $\mathbb{P}^5$

(cf. proof of 3.6) pull back along K_- to a basis for $\wedge^2 H^0(Y, \mathcal{O}(2\Theta_Y)) \subset H^0(X, \mathcal{O}(2\Theta_X))$.

Therefore we get the following diagram:

$$\begin{array}{ccc} Y \times Y & \xrightarrow{K_-} & \overline{Q} \subset \mathbb{P}^5 \\ \downarrow{\scriptstyle K \times K} & & \uparrow\cong \;\; \uparrow\cong \\ \mathbb{P}(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^4) & \xrightarrow{P} & G(1,3) \subset \mathbb{P}(\wedge^2 \mathbb{C}^4) \end{array}$$

(note that the horizontal maps are not everywhere defined), here $G(1,3)$ is the Grassmanian of lines in $\mathbb{P}^3$ and P is the Plücker map which sends a pair of points to the line connecting them. Note that the base locus of P is exactly the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$. Since $K(Y) \cong Y/\langle \pm 1 \rangle$, the base locus of K_- is the union of the diagonal Δ of $Y \times Y$ and $\Delta^- := \{(y, -y) : y \in Y\}$. (The base locus of K_- is now larger (cf. 3.3) since $K : X \to \mathbb{P}^{15}$ factors over the group generated by $id_Y \times (-id_Y)$ and $(-id_Y) \times id_Y$.)

The equation for $\overline{Q}$ is just the quadratic Plücker relation, it is identically zero in $\mathbb{P}^3 \times \mathbb{P}^3$. We will from now on identify $\overline{Q} = G(1,3)$.

It is clear that K_- is dominant: to a general $l \in G(1,3)$ corresponds a line in $\mathbb{P}^3$ which intersects the Kummer variety $K(Y)$ in 4 points, over each pair of these points there lie 4 elements in $Y \times Y$ mapping to l under K_-. Therefore K_- is dominant for the general X_τ with $\tau \in \mathbb{H}_4$.

A basis for $H^4(\overline{Q}, \mathbb{Q})$ is given by the classes of a $\mathbb{P}^2_P \subset \overline{Q}$, parametrizing lines through a point $P \in \mathbb{P}(\mathbb{C}^4)$, and a $\mathbb{P}^2_V \in \overline{Q}$, parametrizing the lines in a plane $V \cong \mathbb{P}^2 \subset \mathbb{P}(\mathbb{C}^4)$. Another basis is given by $\mathbb{P}^2_V$ and H^2, with H the hyperplane section.

The (strict) pull back of $\mathbb{P}^2_V$ along P is $V \times V \subset \mathbb{P}^3 \times \mathbb{P}^3$ and this pulls back to $Z := (2\Theta_Y) \times (2\Theta_Y)$ in $Y \times Y$. The pull-back of H is $2\Theta_X = Y \times (2\Theta_Y) + (2\Theta_Y) \times Y$, twice the product polarization on $Y \times Y$. For general $\tau \in \mathbb{H}_4$, one has $D^2(X_\tau) = \langle \Theta_\tau^2 \rangle$, which specializes to $\langle \Theta_X^2 \rangle$. Therefore we must show that

$$[Z] \notin \langle \Theta_X^2, [\Delta], [\Delta^-] \rangle \subset H^4(Y \times Y, \mathbb{Q}) = \bigoplus_a H^a(Y, \mathbb{Q}) \otimes H^{4-a}(Y, \mathbb{Q})\,.$$

Note that $[Z] \in H^2 \otimes H^2$ (with $H^k := H^k(Y, \mathbb{Q})$). On the other hand,

$$\langle \Theta_X^2, [\Delta], [\Delta^-] \rangle \cap H^2 \otimes H^2 = \langle \Theta_X^2 - ([\Delta] + [\Delta^-]) \rangle\,,$$

in fact, $\Theta_X^2 = 2(\{P\} \times Y + \Theta_Y \times \Theta_Y + Y \times \{P\})$ has no (3,1), (1,3) Künneth components, and $[\Delta] + [\Delta^-] = 2(\{P\} \times Y + \delta_{2,2} + Y \times \{P\})$, with $[\Delta] = \sum \delta_{a,b}$ the Künneth decomposition of $[\Delta]$. Since $\delta_{2,2}$ has a non-trivial component in $H^{2,0}(Y) \otimes H^{0,2}(Y)$, whereas $[Z] = [\Theta_Y \times \Theta_Y] \in H^{1,1} \otimes H^{1,1}$ has a trivial component there, we conclude that $[Z] \notin \langle \Theta_X^2, [\Delta], [\Delta^-] \rangle$.

By specialization, this proves that $[Z] \notin \langle \Theta_{X_\tau}^2 \rangle$ for any X_τ. □

4 An isogeny

4.1 In this section we give another description of the map K_- from theorem 3.7, using an isogeny on X_τ. This new description relates the quadratic relation between certain theta functions to geometrical properties of X_τ, see 4.4.

4.2 A basis of $H^0(X, \mathcal{O}(4\Theta))$ is given by the $4^g = 2^{2g}$ functions

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau, 2z) \quad \varepsilon, \varepsilon' \in \{0,1\}^g .$$

Let $V \subset H^0(X, \mathcal{O}(4\Theta))$ be the subspace spanned by the six even theta functions with vanishing Nullwert:

$$V := \langle \ldots, \theta\begin{bmatrix} \sigma & \sigma \\ \sigma' & \sigma' \end{bmatrix}(\tau, 2z), \ldots \rangle_{(\sigma,\sigma' \in \{0,1\}^2, {}^t\sigma\sigma' = 1)} \subset H^0(X, \mathcal{O}(4\Theta)) .$$

4.3 Proposition. *The image of $X = X_\tau$ under the rational map*:

$$\Phi_V : X \to \mathbb{P}V, \qquad z \mapsto (\ldots : \theta[\begin{smallmatrix} \sigma & \sigma \\ \sigma' & \sigma' \end{smallmatrix}](\tau, 2z) : \ldots)$$

is a smooth quadric $\overline{Q} \subset \mathbb{P}^5$. The quadratic relation on the six theta functions is:

$$\left(\theta^2\begin{bmatrix}1010\\1010\end{bmatrix} - \theta^2\begin{bmatrix}1010\\1111\end{bmatrix} - \theta^2\begin{bmatrix}1111\\1010\end{bmatrix} + \theta^2\begin{bmatrix}1111\\0101\end{bmatrix} - \theta^2\begin{bmatrix}0101\\0101\end{bmatrix}\right.$$
$$\left. + \theta^2\begin{bmatrix}0101\\1111\end{bmatrix}\right)(\tau, 2z) = 0 .$$

There is a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{K_-} & \overline{Q} \subset \mathbb{P}^5 \\ {\scriptstyle 1-\phi}\uparrow & \nearrow {\scriptstyle \Phi_V} & \\ X & & \end{array} .$$

Proof. We consider the subspace $\Gamma(\mathcal{O}(2\Theta))_-$ where $\phi^* = -I$. Remark 2.5 shows that the functions $\theta[\begin{smallmatrix} \varepsilon & \varepsilon \\ \varepsilon' & \varepsilon' \end{smallmatrix}](\tau, z(I - A))$ give a basis of $\Gamma(\mathcal{O}(2\Theta))$. From formula 2.3.1 we have:

$$\theta\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon' & \varepsilon' \end{bmatrix}(\tau, zA) = (-1)^{\varepsilon'^t\varepsilon'}\theta\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon' & \varepsilon' \end{bmatrix}(\tau, z) ,$$

and thus a basis for $\Gamma(\mathcal{O}(2\Theta))_-$ is given by the $2^{n-1}(2^n - 1)$ functions $\theta[\begin{smallmatrix} \varepsilon & \varepsilon \\ \varepsilon' & \varepsilon' \end{smallmatrix}](\tau, z(I - A))$ with ${}^t\varepsilon\varepsilon' \equiv 1 \bmod 2$. Since $(I - A)^2 = I - 2A + A^2 = -2A$ we get:

$$V = (1 - \phi)^* \Gamma(2\Theta)_- .$$

In the case $n = 2$ we used the basis with the six Y_{ab}'s. One easily finds, with $\Theta[\sigma] := \theta[{\sigma \atop 0}](2\tau, 2z)$:

$$\theta\begin{bmatrix}1010\\1010\end{bmatrix}(\tau, z(I - A)) = -\Theta[0010] - \Theta[0111] + \Theta[1000] + \Theta[1101]$$
$$= -Y_{02} - Y_{13}\,,$$
$$\theta\begin{bmatrix}1010\\1111\end{bmatrix}(\tau, z(I - A)) = -\Theta[0010] + \Theta[0111] + \Theta[1000] - \Theta[1101]$$
$$= -Y_{02} + Y_{13}\,,$$

similarly one has:

$$\theta\begin{bmatrix}0101\\0101\end{bmatrix}(\tau, z(I - A)) = -Y_{01} - Y_{23} \quad \theta\begin{bmatrix}1111\\1010\end{bmatrix}(\tau, z(I - A)) = -Y_{03} + Y_{12}$$
$$\theta\begin{bmatrix}0101\\1111\end{bmatrix}(\tau, z(I - A)) = -Y_{01} + Y_{23} \quad \theta\begin{bmatrix}1111\\0101\end{bmatrix}(\tau, z(I - A)) = -Y_{03} - Y_{12}\,.$$

From these equations one can express the Y_{ab}'s in the $\theta[{\varepsilon \atop \varepsilon'}{\varepsilon \atop \varepsilon'}](\tau, z(I - A))$. The quadratic relation between the Y_{ab}'s then gives the relation in the proposition. □

4.4 Remark. The existence of a quadratic relation between the six even theta functions with a vanishing Nullwert can also be obtained from the following observations.

First of all, each vanishing Nullwert corresponds to a singular point on the theta divisor Θ of X, the singular point is a point of order two in X. Thus Θ has at least six singular points. Furthermore, for $x \in \mathrm{Sing}(\Theta)$ we have that

$$f_x := \theta_{0,0}(\tau, z - x)\theta_{0,0}(\tau, z + x) \in \Gamma_{00} := \{s \in H^0(X, \mathcal{O}(2\Theta)) : \mu_0(s) \geqq 4\}\,,$$

where $\mu_0(s)$ is the multiplicity at $0 \in X$ of the section s. The six functions $f_x \in \Gamma_{00}$ we obtain are the six theta functions $\theta[{\sigma \atop \sigma'}{\sigma \atop \sigma'}]^2(\tau, z)$ with $\sigma^t\sigma' = 1$.

Since the general X we consider is not a product of lower dimensional ppav's, one has (see [vGvdG]):

$$\dim \Gamma_{00} = 2^4 - 1 - 10 = 5\,.$$

Thus the six functions $\theta[{\sigma \atop \sigma'}{\sigma \atop \sigma'}]^2(\tau, z) \in \Gamma_{00}$ are linearly dependent and therefore one has also a linear relation between the six $\theta[{\sigma \atop \sigma'}{\sigma \atop \sigma'}]^2(\tau, 2z)$.

The results of Debarre [D] imply that $\mathrm{Sing}(\Theta_\tau)$, for general $\tau \in \mathbb{H}_4$ consists of exactly 6 points (that the general X_τ is not the jacobian of a hyperelliptic curve follows from the fact that the sum of any three vanishing theta characteristics of X_τ is even, rather than odd).

Geometrically, the argument above shows that $K(\mathrm{Sing}(\Theta))$, which in general consists of 6 points, spans at most a $\mathbb{P}^4 \subset \mathbb{P}^{15}$, in fact the map K may also be given by (cf. [vGvdG]):

$$K : X_\tau \longrightarrow \mathbb{P}H^0(X_\tau, \mathcal{O}(2\Theta_\tau))^*, \qquad x \mapsto \theta_{0,0}(\tau, z - x)\theta_{0,0}(\tau, z + x)\,.$$

4.5 Remark. We indicate an alternative derivation of the quadratic relation in the previous proposition, using the identities (with $\sigma_1, \sigma_2 \in \{0,1\}^2$):

$$\theta\begin{bmatrix}\sigma_1 & \sigma_2\\ 0 & 0\end{bmatrix}(2\tau, 0) = \theta\begin{bmatrix}\sigma_2 & \sigma_1\\ 0 & 0\end{bmatrix}(2\tau, 0) \qquad \tau \in \mathbb{H}_{2n}\,.$$

The multiplication formula for theta functions ([I], Chapter IV, Thm. 2, p. 139) shows that the expression for $\theta^2[\begin{smallmatrix}\varepsilon & \varepsilon\\ \varepsilon' & \varepsilon'\end{smallmatrix}](\tau, z)$ in terms of the standard basis of $\Gamma(\mathcal{O}(2\Theta_\tau))$ is given by:

$$\theta\begin{bmatrix}\varepsilon & \varepsilon\\ \varepsilon' & \varepsilon'\end{bmatrix}^2(\tau, z)$$
$$= \sum_{\sigma_1,\sigma_2 \in \{0,1\}^2} (-1)^{(\sigma_1,\sigma_2)^t\varepsilon'}\theta\begin{bmatrix}\sigma_1+\varepsilon & \sigma_2+\varepsilon\\ 0 & 0\end{bmatrix}(2\tau, 0)\theta\begin{bmatrix}\sigma_1 & \sigma_2\\ 0 & 0\end{bmatrix}(2\tau, 2z)\,.$$

It is now a straight-forward computation to verify the relation. One can also use the proof of 3.6 as follows. Define:

$$B_{Q_m}(X, Y) := \sum_{\sigma}(-1)^{{}^t\sigma\varepsilon'}X_\sigma Y_{\sigma+\varepsilon}, \quad \text{with } m = [\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}]$$

an even theta characteristic. Note that B_{Q_m} is just the bilinear form associated to the quadratic form Q_m in the proof of Proposition 3.6. Upon substituting $Y_\sigma := \theta[\begin{smallmatrix}\sigma\\ 0\end{smallmatrix}](2\tau, 0)$ one obtains linear forms in the X_σ, these 6 forms are linearly dependent exactly when there is a linear relation between the $\theta^2[\begin{smallmatrix}\varepsilon & \varepsilon\\ \varepsilon' & \varepsilon'\end{smallmatrix}](\tau, z)$.

In the proof of 3.6, we showed that a certain linear combination Q of the six Q_m's is a quadric in the coordinates Y_{st} of $\mathbb{P}^5$. Thus Q is a linear combination of terms $(X_{st} - X_{ts})(X_{uv} - X_{vu})$, and therefore its associated bilinear form B_Q is a linear combination of terms $(Y_{uv} - Y_{vu})X_{st}$. Since we have $\theta[\begin{smallmatrix}\sigma_1 & \sigma_2\\ 0 & 0\end{smallmatrix}] = \theta[\begin{smallmatrix}\sigma_2 & \sigma_1\\ 0 & 0\end{smallmatrix}]$, upon substituting $Y_\sigma := \theta[\begin{smallmatrix}\sigma\\ 0\end{smallmatrix}](2\tau, 0)$ in B_Q we get zero.

5 Prym varieties

5.1 Let C_{n+1} be a smooth curve of genus $n+1$. A subgroup $G \cong \mathbb{Z}/4\mathbb{Z} \subset \mathrm{Pic}^0(C_{n+1})$ defines a cyclic etale 4:1 covering of C_{n+1} and an intermediate etale 2:1 cover:

$$C_{n+1} \leftarrow C_{2n+1} \leftarrow C_{4n+1}\,.$$

The Prym variety P of the C_{4n+1}/C_{2n+1} is a principally polarized abelian variety of dimension $2n$. The covering automorphism of C_{4n+1} over C_{n+1} induces an action of $\mathbb{Z}[i]$ on P.

5.2 Schoen constructs cycles on P by taking the inverse image of $\mathbb{P}^n = |K| \subset C_{n+1}^{(2n)}$ in $C_{4n+1}^{(2n)}$ (with $C^{(i)} = \mathrm{Sym}^i C$), which is reducible, and mapping the components to P. He shows that suitable linear combinations of these cycles span the space of Weil-Hodge cycles $W \subset H^{2n}(P, \mathbb{Q})$.

In the case $n = 2$ we show below that the general $X_\tau, \tau \in \mathbb{H}_4$, is such a Prym variety.

5.3 Theorem. *Let P be as in* 5.1. *Then*:

1. *The abelian variety P is of Weil type (n,n), with field $\mathbb{Q}(i)$ and* $\det H = 1$.

2. *There exist bases of $H_1(P,\mathbb{Z})$ and $H_1(P,\mathbb{R})$ (which has the structure of a complex vector space via its identification with T_0P) such that the period matrix Ω satisfies*:

$$\Omega = \begin{pmatrix} I \\ \tau \end{pmatrix} : H_1(P,\mathbb{Z}) \cong \mathbb{Z}^{4n} \to H_1(X,\mathbb{R}) = T_0P \cong \mathbb{C}^{2n} \quad \textit{with } \tau \in \mathbb{H}_{2n}\,.$$

In particular, P is isomorphic to an X_τ with $\tau \in \mathbb{H}_{2n}$.

3. *In case $n = 2$, the general abelian variety of Weil type $(2,2)$ and field $\mathbb{Q}(i)$ with* $\det H = 1$ *is isogeneous to a Prym variety as in* 5.1.

Proof. We consider the action of the automorphism ϕ of order 4 on $H_1(P,\mathbb{Z})$. According to [F], p. 62, there is a symplectic basis $\hat{A}_i$, $\hat{B}_i$ $(0 \leqq i \leqq 4n)$ of $H_1(C_{4n+1},\mathbb{Z})$ with:

$$\phi^k(\hat{A}_i) = \hat{A}_{i+kn}\,, \qquad \phi^k(\hat{B}_i) = \hat{B}_{i+kn} \quad (1 \leqq i \leqq n)\,.$$

Since $H_1(P,\mathbb{Z}) = H_1(C_{4n+1},\mathbb{Z})^{\phi_*=-I}$ (the classes anti-invariant under ϕ_*), a basis of $H_1(P,\mathbb{Z})$ is given by

$$\alpha_i := \hat{A}_i - \hat{A}_{i+2n}, \qquad \beta_i := \hat{B}_i - \hat{B}_{i+2n} \quad (1 \leqq i \leqq 2n)\,.$$

An easy computation shows that the action of ϕ on $H_1(P,\mathbb{Z})$ is given by the matrix M, i.e. $\phi(\alpha_i) = \alpha_{i+n}$, $\phi(\beta_i) = \beta_{i+n}$ with $1 \leqq i \leqq n$. Taking the α_i as a $\mathbb{C}$-basis of $H_1(P,\mathbb{R})$, the period matrix must satisfy $M\Omega = \Omega A$, which implies that $\tau \in \mathbb{H}_{2n}$. This proves the second point.

Using this result, the first point is easy to verify (cf. 2.1).

For the last point, we consider the Prym map:

$$\mathscr{P} : \mathscr{M}_{3,\mathbb{Z}/4\mathbb{Z}} \to \mathscr{A}_4, \qquad (C,G) \mapsto P$$

where $\mathscr{M}_{3,\mathbb{Z}/4\mathbb{Z}}$ is the moduli space of genus 3 curves with a cyclic subgroup of order 4 in Pic^0 and $\mathscr{A}_4$ is the moduli space of principally polarized abelian varieties. We already proved that $\mathrm{Im}(\mathscr{P})$ lies in the 4-fold $\mathscr{H} := \mathrm{Im}(\mathbb{H}_4 \to \mathbb{S}_4 \to \mathscr{A}_4)$. To show that $\mathrm{Im}(\mathscr{P})$ is Zariski dense in $\mathscr{P}$ it suffices to show that the differential of $\mathscr{P}$ has rank 4 at some point of $\mathscr{M}_{3,\mathbb{Z}/4\mathbb{Z}}$, or, equivalently, that the codifferential of $\mathscr{P}$ is injective at some point of $\mathscr{H}$.

The remainder follows the arguments of Schoen, [S], p. 26–p. 30, we just show the (obvious) modifications. The cotangent space of $\mathscr{M}_{3,\mathbb{Z}/4\mathbb{Z}}$ at a smooth point $[(C,G)]$ is $H^0(C,\Omega_C^{1\otimes 2})$. Let $\pi : C_9 \to C$ be the 4:1 etale map, then, using the action of ϕ, we have:

$$\pi_*\Omega^1_{C_9} = \Omega^1_C \oplus (\Omega^1_C \otimes \alpha^{\otimes 2}) \oplus (\Omega^1_C \otimes \alpha) \oplus (\Omega^1_C \otimes \alpha^{\otimes 3}) \quad \text{with } G = \langle \alpha \rangle\,.$$

Then $T_P^*\mathscr{A}_4$ and $T_P^*\mathscr{H}$, the cotangent spaces of $\mathscr{A}_4$ and $\mathscr{H}$ at P are given by:

$$T_P^*\mathscr{A}_4 = \mathrm{Sym}^2(H^0(C,\Omega_C^1\otimes\alpha)\oplus H^0(C,\Omega_C^1\otimes\alpha^3)),$$
$$T_P^*\mathscr{H} = H^0(C,\Omega_C^1\otimes\alpha)\otimes H^0(C,\Omega_C^1\otimes\alpha^3)).$$

The codifferential of $\mathscr{P}:\mathscr{M}_{3,\mathbb{Z}/4\mathbb{Z}}\to\mathscr{H}$ is now just the multiplication map:

$$H^0(C,\Omega_C^1\otimes\alpha)\otimes H^0(C,\Omega_C^1\otimes\alpha^3))\to H^0(C,\Omega_C^{1\otimes 2}).$$

The proof that this map injective for general $[C]\in\mathscr{M}_3$ follows from the Base Point Free Pencil trick, as in [S], p. 30. □

5.4 Remark. Since $\dim\mathscr{M}_4 = 9 = 3^2$, one might hope that the Prym map: $\mathscr{P}:\mathscr{M}_{4,\mathbb{Z}/4\mathbb{Z}}\to\mathscr{A}_6$ would be generically finite on its image. If that is the case, then $\mathrm{Im}(\mathscr{P})$ is Zariski dense in $\mathscr{H}_6$ and Schoen's cycles would solve the Hodge conjecture for the general abelian sixfold of Weil-type with field $\mathbb{Q}(i)$ and $\det H = 1$.

5.5 Remark. The previous Proposition also 'explains' the 6 singular points of order two on the theta divisor of the abelian fourfolds we considered. Recall that a theta characteristic on a curve C is a line bundle L with $L^{\otimes 2}=\Omega_C^1$.

For a curve of genus $n+1$ and a point $\beta\in\mathrm{Pic}^0(C)$ of order two, there are $2^{n-1}(2^n-1)$ pairs of odd theta characteristics L, L' with $L'=L\otimes\beta$ (this is most easily checked using the classical notation for characteristics). Let $\pi':C'\to C$ be the etale 2:1 cover defined by β, then

$$\pi'^*L\cong\pi'^*L',\qquad H^0(C',\pi'^*L)=H^0(C,L)\oplus H^0(C,L'),\qquad (\pi'^*L)^{\otimes 2}=\Omega_C^1,$$

thus π'^*L is an even theta characteristic with at least two independent sections.

Let now $\alpha\in\mathrm{Pic}^0(C)$ with $\alpha^{\otimes 2}=\beta$. Then $\gamma:=\pi'^*\alpha$ has order two in $\mathrm{Pic}(C')$ and moreover, by Serre duality, $H^0(C,L\otimes\alpha)\cong H^0(C,L\otimes\alpha^{-1})^*=H^0(C,L'\otimes\alpha)^*$. Therefore, also $(\pi'^*L)\otimes\gamma=\pi'^*(L\otimes\alpha)$ is an even theta characteristic on C'.

Now it is well known (for example from the classical Schottky–Jung identities) that pairs $M, M\otimes\gamma$ of even theta characteristics on C' with $H^0(C',M)>0$ correspond to singular points, of order two and even multiplicity, on the theta divisor of the Prym variety of the covering $C''\to C'$ defined by γ. These singular points correspond precisely to the vanishing (even) theta nulls.

Starting from a genus $n+1$ curve, the $2n$-dimensional Prym varieties introduced in 5.1, thus have at least $2^{n-1}(2^n-1)$ vanishing (even) theta nulls, in agreement with the fact that X_τ, for $\tau\in\mathbb{H}_{2n}$ has that number of vanishing theta nulls (simply put $z=0$ in 2.3.1).

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