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Fejér means for multivariate Fourier series

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Dedicated to Professor D.J. Newman on the occasion of his 65th birthday

1 Introduction

Summability of multivariate Fourier series (briefly: F. S.) has been the object of intensive studies for this century, especially after the appearance of the classical paper of S. Bochner on, what is now called, the Bochner–Riesz means of F. S. in 1936 and again after the appearance of the papers of A. P. Calderón and A. Zygmund on singular integrals in the early 1950s, see A. Zygmund [13, Chap. XVII], E. M. Stein & G. Weiss [9, Chap. VII], and also V. L. Shapiro [8]. In connection with his investigations of multivariate orthogonal polynomial systems the second named author [11] ran into problems of summability of F. S. from a point of view which seems not to have been in the main stream of the theory developed so far. To be more precise, let us introduce the necessary notation.

Let $\mathbb{R}^d$ denote the Euclidean d-space, $d \geq 1$, and let $\mathbb{T}^d = \mathbb{R}^d (\mathrm{mod}\ 2\pi\mathbb{Z}^d)$ denote the d-dimensional torus. For a vector $x \in \mathbb{R}^d$, $|x|_p$ defines its ℓ_p-norm, $1 \leq p \leq \infty$.

Let $C(\mathbb{T}^d)$ denote the space of (complex-valued) continuous functions on $\mathbb{T}^d$. To the multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{Z}^d$ we associate the ℓ-p modulus to be the ℓ_p-norm of α considered as a point in $\mathbb{R}^d$, and if $\theta = (\theta_1, \ldots, \theta_d)$ denotes a point in $\mathbb{T}^d$, the product $\alpha \cdot \theta$ defines the inner product of α and θ. For an $f \in C(\mathbb{T}^d)$ its F. S. is defined by

$$f(\theta) \sim \sum_{\alpha} \hat{f}(\alpha) e^{i\alpha\cdot\theta}, \qquad \hat{f}(\alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta) e^{-i\alpha\cdot\theta} d\theta.$$

In a way summability of the n-th partial sum of the F. S. of f w. r. t. the ℓ-∞ modul of the index set is apparent; i. e., studying summability properties of

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$$S_{n,d}^{(\infty)}(f;\theta) = \sum_{|\alpha|_\infty \le n} \hat{f}(\alpha)e^{i\alpha\cdot\theta} = D_{n,d}^{(\infty)} * f(\theta), \qquad n \in \mathbb{N}_0,$$

see [13, Chap. XVII]. Bochner introduced the ℓ-2 partial sum

$$S_{R,d}^{(2)}(f;\theta) = \sum_{|\alpha|_2 \le R} \hat{f}(\alpha)e^{i\alpha\cdot\theta} = D_{n,d}^{(2)} * f(\theta), \qquad R > 0,$$

and the success of his means is based on their relation to the Laplace operator, see [9, Chap. VII]. But equally natural, if not more so, seems to us to study summability of the n-th partial sum of the F. S. of f w. r. t. the ℓ-1 modul of the index set; i. e., studying

$$S_{n,d}^{(1)}(f;\theta) = \sum_{|\alpha|_1 \le n} \hat{f}(\alpha)e^{i\alpha\cdot\theta} = D_{n,d}^{(1)} * f(\theta), \qquad n \in \mathbb{N}_0;$$

see Section 5 in [11] which shows the close relationship to the theory of series expansions w. r. t. orthogonal polynomials on $\mathbb{R}^d$.

In contrast to the first two methods, the last one seems to have been almost forgotten. The authors could just come up with a single reference by J. G. Herriot [6] in 1944. In [12] the second named author gave an explicit formula for the Dirichlet kernel $D_{n,d}^{(1)}$ as a divided difference of a univariate function, the arguments in the difference being the independent variables. This representation gives hope for the development of a rich theory of ℓ-1 summability.

Here, the authors want to restrict themselves to the study of just one topic. In 1904 L. Fejér proved that for a function in $C(\mathbb{T})$ the arithmetical means of the n-th partial sums of its F. S. converge uniformly to f, giving the first constructive proof of the Weierstraß theorem. Fundamental to his proof proved to be the fact that the arithmetical (or the Cesàro) means define a positive linear transformation on $C(\mathbb{T})$, leaving the const.-valued functions invariant. Here is an analogue to Fejér's theorem in multivariate ℓ-1 summability.

Theorem 1 *In ℓ-1 summability the Cesàro $(C, 2d-1)$ means of the F. S. of a function f in $C(\mathbb{T}^d)$ converge uniformly to f. In particular, the means define a positive linear polynomial approximate identity on $C(\mathbb{T}^d)$; the order of summability is best possible in the sense that the (C,δ) means are not positive for $0 < \delta < 2d-1$.*

Let us recall the definition of Cesàro summability. The sequence $\{s_n\}$ is summable by Cesàro's method of order δ: (C,δ), to s if

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k$$

converges to s as $n \to \infty$. If for each $n \in \mathbb{N}_0$ s_n is the n-th partial sum of the series $\sum_{k=0}^\infty c_k$, the Cesàro means can be rewritten as

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} c_k.$$

For the basic properties of Cesàro summability see [13, Chap. III].

The proof of the main part of the theorem reduces to proving that the $(C, 2d-1)$ means of the Dirichlet kernel $D_{n,d}^{(1)}$ define a positive linear transformations on $C(\mathbb{T}^d)$; in other words, we have to verify that

$$(1.1)\binom{n+2d-1}{2d-1}\sigma_{n,d}^{2d-1}(\theta) = \sum_{|\alpha|_1 \le n} \binom{n-|\alpha|_1+2d-1}{2d-1} e^{i\alpha\cdot\theta} \ge 0 \quad \text{on} \quad C(\mathbb{T}^d);$$

implicitly we defined $\sigma_{n,d}^{\delta}$ to be the (C, δ) means of the Dirichlet kernels.

For ℓ-2 summability the Riesz means of the F. S. of a function f in $C(\mathbb{T}^d)$:

$$S_{R;\lambda,\delta;d}^{(2)}(f;\theta) = \sum_{|\alpha|_2 \le R} \left(1 - \left[\frac{|\alpha|_2}{R}\right]^{\lambda}\right)^{\delta} \hat{f}(\alpha) e^{i\alpha\cdot\theta} \qquad \lambda, \delta > 0, \quad \text{and} \quad R > 0,$$

do not define a positive transformation on $C(\mathbb{T}^d)$ for $\lambda \ge 2$ ($\lambda = 2$ are the Bochner–Riesz means), while for $0 < \lambda < 2$ there exists a bound $\delta(\lambda) > 0$ ($\delta(1) = (d+1)/2$) such that the (R, λ, δ) means define positive transformations for $\delta \ge \delta(\lambda)$ on $C(\mathbb{T}^d)$, as was proved by B. I. Golubov [5] in 1981. We do not know about any results on the positivity of the Cesáro means in ℓ-p summability, $1 \le p \le \infty$, for F. S. in $C(\mathbb{T}^d)$ – by a personal communication we learned from Professor J. Korevaar that in the early 1950s he did study ℓ-1 summability of bivariate Fourier series and proved among others that the $(C, 3)$ means define a positive approximate identity; he did not publish his results. To us the theorem is surprising; even more so is its proof which depends heavily on results from the theory of special functions, in particular, on positive sums of Jacobi polynomials, which go back to R. Askey and G. Gasper in the 1970s, see the SIAM–Lectures [2] of Askey.

In the following section we will prove the theorem for bivariate F. S. Here, the proof can be reduced to better-known statements on trigonometric series. In Section 3 we will derive various representations of the ℓ-1 Dirichlet kernel, while Section 4 is reserved for the proof of the theorem. In the last section we will briefly discuss the closely related Abel means.

2 The bivariate case

Let us begin by giving an elementary proof of the theorem for bivariate F. S.; better, by reducing the proof by elementary arguments to known inequalities on Fourier series. These arguments, however, did not lead to an extension of the proof for multivariate F. S. in general.

In his paper, loc. cit., Herriot gives the following formula for $D_{n,2}^{(1)}$, which can be easily verified; he states

$$D_{n,2}^{(1)}(\theta_1,\theta_2) = \sum_{|\alpha_1|+|\alpha_2|\le n} e^{i(\alpha_1\theta_1+\alpha_2\theta_2)}$$
$$= -\frac{\cos\frac{1}{2}\theta_1\cos(n+\frac{1}{2})\theta_1 - \cos\frac{1}{2}\theta_2\cos(n+\frac{1}{2})\theta_2}{\sin\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1-\theta_2)}.$$

Setting

$$G_{n,2}(\phi) = -\cos\tfrac{1}{2}\phi\cos(n+\tfrac{1}{2})\phi, \qquad 0<\phi<\pi,$$

it is not too difficult to verify that the $(C,3)$ means of the derivative of $G_{n,2}(\phi)$ are positive on $0<\phi<\pi$. Indeed,

$$G_{n,2}'(\phi) = \tfrac{1}{2}\sin\tfrac{1}{2}\phi\cos(n+\tfrac{1}{2})\phi + (n+\tfrac{1}{2})\cos\tfrac{1}{2}\phi\sin(n+\tfrac{1}{2})\phi;$$

and formula (1.21) in [2, Lect. 1] explicitly states that for each $n\in\mathbb{N}_0$

$$(2.1)\qquad s_n(\phi) = \sum_{k=0}^{n}\binom{n-k+2}{2}(k+\tfrac{1}{2})\sin(k+\tfrac{1}{2})\phi > 0, \qquad 0<\phi<\pi.$$

Incidentally, this inequality goes back to Fejér too, cf. again [2, p. 4]. Furthermore,

$$c_n(\phi) = \sum_{k=0}^{n}\binom{n-k+2}{2}\cos(k+\tfrac{1}{2})\phi > 0, \qquad 0<\phi<\pi;$$

the function $c_n(\phi)$ has $-s_n(\phi)$ as its derivative on $0<\phi<\pi$, hence it is strictly decreasing on the interval and $c_n(\pi)=0$. Consequently, the $(C,3)$ means of

$$G_{n,2}(\theta_1) - G_{n,2}(\theta_2), \qquad 0<\theta_2<\theta_1<\pi,$$

are positive, and so are the $(C,3)$ means of the quotient

$$\frac{1}{\sin\frac{1}{2}(\theta_1+\theta_2)}\cdot\frac{G_{n,2}(\theta_1)-G_{n,2}(\theta_2)}{\sin\frac{1}{2}(\theta_1-\theta_2)}, \qquad 0<\theta_1,\theta_2<\pi.$$

In other words, for each $n\in\mathbb{N}_0$

$$\binom{n+3}{3}\sigma_{n,2}^3(\theta_1,\theta_2) = \sum_{|\alpha_1|+|\alpha_2|\le n}\binom{n-(|\alpha_1|+|\alpha_2|)+3}{3}e^{i(\alpha_1\theta_1+\alpha_2\theta_2)} \ge 0 \quad \text{on} \quad \mathbb{T}^2,$$

proving the first part of the theorem for $C(\mathbb{T}^2)$.

The fact that the result is best possible in the sense that the (C,δ) means do not suffice for $0<\delta<3$ follows from the fact that

$$s_{n,\epsilon}(\phi) = \sum_{k=0}^{n}\binom{n-k+2-\epsilon}{2-\epsilon}(k+\tfrac{1}{2})\sin(k+\tfrac{1}{2})\phi, \qquad 0<\phi<\pi,$$

changes sign in any interval $0<\phi<\phi_0$, $\phi_0>0$, for infinitely many n when $\epsilon>0$. This was proved by I. Fuchs, cf. [2, p. 90]. Following the arguments given above quickly verifies the claim.

3 Representations of the Dirichlet kernel

It is difficult, if not impossible, to come up with arguments why ℓ-1 summability of multivariate F. S. has received so little attention in the past. One plausible reason seems to us the lack of a closed form for the Dirichlet kernel with all its consequences. In [12] the second named author proved that for each $n \in \mathbb{N}_0$ the n-th ℓ-1 Dirichlet kernel can be written as a divided difference of a real-valued function.

Since the notions of a divided difference of a function and of a B-spline play a basic role in our paper, we want to give their definitions and state some of their properties. Let f be a real- or complex-valued function on $\mathbb{R}$, and let $n \in \mathbb{N}_0$. The n-th divided difference of f at the (pairwise distinct) knots, $x_0, x_1, \ldots, x_n$ in $\mathbb{R}$, is defined inductively as

$$[x_0]f = f(x_0) \quad \text{and} \quad [x_0, \ldots, x_n]f = \frac{[x_0, \ldots, x_{n-1}]f - [x_1, \ldots, x_n]f}{x_0 - x_n}.$$

It follows that the difference is a symmetric function of the knots; while, considered as a functional on the vector space of functions on $\mathbb{R}$, it is linear. Indeed, we have

$$[x_0, \ldots, x_n]f = \sum_{k=0}^{n} \frac{f(x_k)}{\prod\limits_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j)},$$

a representation which will be used explicitly in the proof given below. Moreover, depending on the smoothness of the function, knots may coalesce.

The B-spline of order n and with knots, $x_0 < \ldots < x_n$, is then defined by

$$\mathbb{R} \ni u \quad \longrightarrow \quad M_n(u|x_0, \ldots, x_n) = [x_0, \ldots, x_n]\left\{ \frac{(\cdot - u)_+^{n-1}}{(n-1)!} \right\}.$$

The spline function vanishes outside the interval (x_0, x_n), on the interval itself it is strictly positive, and

$$\int_{\mathbb{R}} M_n(u|x_0, \ldots, x_n)\, du = \frac{1}{n!}.$$

One of its fundamental properties can be stated as follows:

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be n-times continuously differentiable, then

$$[x_0, \ldots, x_n]f = \int_{\mathbb{R}} f^{(n)}(u)\, M_n(u|x_0, \ldots, x_n)\, du.$$

The relation can even be considered as a definition of the B-spline; for these and further properties see e. g. the survey [3] of C. de Boor.

We have

Lemma 1 *For each $n \in \mathbb{N}_0$,*

$$(3.1)\qquad D_{n,d}^{(1)}(\theta) = [\cos\theta_1, \dots, \cos\theta_d]\, G_{n,d}, \qquad \theta = (\theta_1, \theta_2, \dots, \theta_d) \text{ in } \mathbb{T}^d,$$

where

$$G_{n,d}(\cos\phi) = (-1)^{[\frac{d-1}{2}]} 2\cos\tfrac{1}{2}\phi(\sin\phi)^{d-2} \begin{cases} \cos(n+\frac{1}{2})\phi & \text{for } d \text{ even}, \\ \sin(n+\frac{1}{2})\phi & \text{for } d \text{ odd}. \end{cases}$$

Furthermore,

$$(3.2)\qquad D_{n,d}^{(1)}(\theta) = \int_{-1}^{1} G_{n,d}^{(d-1)}(t) M_{d-1}(t \,|\, \cos\theta_1, \dots, \cos\theta_d)\, dt,$$

where $M_{d-1}(\cdot\,|\cos\theta_1, \dots, \cos\theta_d)$ is the $(d-1)$st B-spline with knots at $-1 < \cos\theta_j < 1$, $1 \le j \le d$.

Representing the Dirichlet kernel as a divided difference of a univariate function and further as an integral reduces the study of ℓ-1 summability of F. S. to the investigation of summability properties of the univariate function $g_{n,d} := G_{n,d}^{(d-1)}$, the multidimensional variables being nicely hidden in the B-spline kernel; this will prove to be more than just useful. We will derive various representations for the function $g_{n,d}$; but to make the paper self-contained let us first sketch a proof of the lemma – since we are only dealing with ℓ-1 summability, we will in the following omit the superindex.
Sketch of proof. The kernel can be rewritten as

$$D_{n,d}(\theta) = 2^d \sum_{|\alpha|_1 \le n}{}' \cos\alpha_1\theta_1 \cdots \cos\alpha_d\theta_d,$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ and where $\sum'$ means that whenever an index-coefficient α_j is equal to zero the term containing $\cos\alpha_j\theta_j$ is to be halved. Let us write $D_{n,d} = 2^d \Sigma_{n,d}$. We need the following (not so) well-known trigonometric formulae:
for each $n \in \mathbb{N}_0$ and $0 \le \phi, \phi' \le \pi$,

$$\sum_{k=0}^{n}{}' \cos k\phi' \sin(n-k+\tfrac{1}{2})\phi = \sin\tfrac{1}{2}\phi \left\{ \frac{\cos\frac{1}{2}\phi\cos(n+\frac{1}{2})\phi - \cos\frac{1}{2}\phi'\cos(n+\frac{1}{2})\phi'}{\cos\phi - \cos\phi'} \right\},$$

$$\sum_{k=0}^{n}{}' \cos k\phi' \cos(n-k+\tfrac{1}{2})\phi = \cos\tfrac{1}{2}\phi \left\{ \frac{\sin\frac{1}{2}\phi'\sin(n+\frac{1}{2})\phi' - \sin\frac{1}{2}\phi\sin(n+\frac{1}{2})\phi}{\cos\phi - \cos\phi'} \right\}.$$

The proof of formula (3.1) is by induction which is based on the following relation

$$\Sigma_{n,d+1}(\theta) = \sum_{k=0}^{n}{}' \cos k\theta_{d+1}\, \Sigma_{n-k,d}(\theta'),$$

where $\theta = (\theta', \theta_{d+1})$, $\theta' \in [0,\pi]^d$, and $\theta_{d+1} \in [0,\pi]$, and on the well-known formula

$$\Sigma_{n,1}(\theta) = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} = \cos\tfrac{1}{2}\theta(\sin\theta)^{-1}\sin(n+\tfrac{1}{2})\theta.$$

Let us assume that the formula has been proved for integers up to d and let us assume that d is even. Then

$$\begin{aligned}
\Sigma_{n,d+1}(\theta) &= \frac{(-1)^{[\frac{d-1}{2}]}}{2^{d-1}} \sum_{l=0}^{d} \frac{\cos\frac{1}{2}\theta_l(\sin\theta_l)^{d-2}}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d}(\cos\theta_l - \cos\theta_j)} \sum_{k=0}^{n}{}' \cos k\theta_{d+1}\cos(n-k+\tfrac{1}{2})\theta_l \\
&= -\frac{(-1)^{[\frac{d-1}{2}]}}{2^{d}} \left\{ \sum_{l=0}^{d} \frac{\cos\frac{1}{2}\theta_l(\sin\theta_l)^{d-1}\sin(n+\frac{1}{2})\theta_l}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d+1}(\cos\theta_l - \cos\theta_j)} - \right. \\
&\qquad \left. - \sin\tfrac{1}{2}\theta_{d+1}\sin(n+\tfrac{1}{2})\theta_{d+1} \sum_{l=0}^{d} \frac{(1+\cos\theta_l)(\sin\theta_l)^{d-2}}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d+1}(\cos\theta_l - \cos\theta_j)} \right\}.
\end{aligned}$$

For d even, the function $(1+t)(1-t^2)^{\frac{d-2}{2}}$ is a polynomial of degree $d-1$. Hence, its d-th divided difference at the knots $t_j = \cos\theta_j$, $0 \le \theta_j \le \pi$ and $1 \le j \le d$, vanishes; i. e.,

$$\sum_{l=1}^{d} \frac{(1+\cos\theta_l)(\sin\theta_l)^{d-2}}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d+1}(\cos\theta_l - \cos\theta_j)} + \frac{(1+\cos\theta_{d+1})(\sin\theta_{d+1})^{d-2}}{\prod\limits_{\substack{j=1\\ j\neq d+1}}^{d+1}(\cos\theta_{d+1} - \cos\theta_j)} = 0;$$

incorporating the formula into the equation for $\Sigma_{n,d+1}$ gives

$$\begin{aligned}
2^{d+1}\,\Sigma_{n,d+1}(\theta) &= (-1)^{[\frac{d}{2}]} \sum_{l=0}^{d+1} \frac{2\cos\frac{1}{2}\theta_l(\sin\theta_l)^{d-1}\sin(n+\frac{1}{2})\theta_l}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d+1}(\cos\theta_l - \cos\theta_j)} \\
&= [\cos\theta_1, \ldots, \cos\theta_{d+1}]\, G_{n,d+1},
\end{aligned}$$

which completes the proof for d even. For d odd the proof is similar – incidentally, in [12] a detailed proof has been given for this case. □

We will derive explicit formulae for the function $g_{n,d}$, $n \in \mathbb{N}_0$; to do so, we set

$$G_{n,d} =: F_{n,d} + F_{n-1,d} \quad \text{and} \quad g_{n,d} =: f_{n,d} + f_{n-1,d},$$

where

$$F_{n,d}(t) = (-1)^{[\frac{d-1}{2}]}(1-t^2)^{[\frac{d-1}{2}]} \begin{cases} T_{n+1}(t) & \text{for } d \text{ even,} \\ U_n(t) & \text{for } d \text{ odd,} \end{cases}$$

and

$$f_{n,d} = F_{n,d}^{(d-1)},$$

T_n and U_n being the n-th Chebyshev polynomials of the first and second kind, respectively. We will make extensive use of the Gegenbauer polynomials $C_n^{(\lambda)}$ and, more generally, of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, respectively, as they are defined in the monograph [10, Chap. IV] of G. Szegö. – Actually, Szegö writes P_n^λ for the Gegenbauer polynomials instead of $C_n^{(\lambda)}$, but this will be the only inconsistency.

As a first formula we prove

Lemma 2 *For each $n \in \mathbb{N}_0$,*

$$f_{n,d} = (d-1)! \sum_{l=0}^{d-1} (-1)^l \binom{d-1}{l} C_{n-2l}^{(d)}. \tag{3.3}$$

Proof. For $d = 1$ and 2 the formula is true as can be easily checked. We prove the inductive step: $d \to d+2$. A quick calculation verifies that

$$F_{n,d+2}(t) = -(1-t^2)F_{n,d}, \qquad -1 \le t \le 1,$$

but then

$$\begin{aligned} &f_{n,d+2} \\ &= -\{(1-t^2)F_{n,d}\}^{(d+1)} \\ &= -\{(1-t^2)F_{n,d}^{(d+1)} - 2(d+1)tF_{n,d}^{(d)} - d(d+1)F_{n,d}^{(d-1)}\} \\ &= -\{(1-t^2)f_{n,d}'' - 2(d+1)tf_{n,d}' - d(d+1)f_{n,d}\} \\ &= (d-1)! \sum_{l=0}^{d-1} (-1)^l \binom{d-1}{l} \{-(1-t^2)C_{n-2l}^{(d)\,\prime\prime} + 2(d+1)tC_{n-2l}^{(d)\,\prime} + d(d+1)C_{n-2l}^{(d)}\}. \end{aligned}$$

Using the differential equation for Gegenbauer polynomials, formula (4.7.5) in G. Szegö [10, Chap. IV], as well as the formulae (4.7.14) and (4.7.28), which state

$$C_n^{(\lambda)\,\prime} = 2dC_{n-1}^{(\lambda-1)} \quad \text{and} \quad (n+2\lambda)C_n^{(\lambda)} = C_{n+1}^{(\lambda)\,\prime} - tC_n^{(\lambda)\,\prime},$$

respectively, we can rewrite the curly brackets inside the sum as

$$
\begin{aligned}
tC_{n-2l}^{(d)\,\prime} + [(n-2l)(n &- 2l + 2d) + d(d+1)]C_{n-2l}^{(d)} \\
&= 2dC_{n-2l}^{(d+1)} + (n-2l+d-1)(n-2l+d)C_{n-2l}^{(d)},
\end{aligned}
$$

and further by use of formula (4.7.29) in [10], which states

$$
(n+\lambda)C_n^{(\lambda)} = 2\lambda\{C_n^{(\lambda+1)} - C_{n-2}^{(\lambda+1)}\},
$$

as

$$
\begin{aligned}
d\{(n-2l+d+1)C_{n-2l}^{(d+1)} &- (n-2l+d-1)C_{n-2l-2}^{(d)}\} \\
&= d(d+1)\{C_{n-2l}^{(d+2)} - 2C_{n-2l-2}^{(d+2)} + C_{n-2l-4}^{(d+2)}\}.
\end{aligned}
$$

Inserting this form back into the sum completes the proof after some manipulations with binomial coefficients. □

There is a second representation of the function $f_{n,d}$ which may be of independent interest; its proof is similar to the one given above.

Lemma 3 *For each $n \in \mathbb{N}_0$,*

$$
f_{n,d} = (d-1)! \sum_{l=1}^{[\frac{d-1}{2}]} \binom{n+1}{d-2l+1} C_n^{(l)}. \tag{3.4}
$$

The two representations (3.3) and (3.4) of the function $f_{n,d}$ might suggest that it is the more interesting object compared to the function $g_{n,d}$ $(= f_{n,d} + f_{n-1,d})$. This impression changes if we considers the $(C, d-1)$ means of $f_{n,d}$ and $g_{n,d}$. Indeed,

Lemma 4 *For each $n \in \mathbb{N}_0$,*

$$
\sum_{k=0}^{n} \binom{n-k+d-2}{d-2} f_{n,d} = (d-1)! \sum_{l=0}^{d-1} \binom{d-1}{l} C_{n-l}^{(d)},
$$

and consequently,

$$
\sum_{k=0}^{n} \binom{n-k+d-2}{d-2} g_{n,d} = (d-1)! \sum_{l=0}^{d} \binom{d}{l} C_{n-l}^{(d)}. \tag{3.5}
$$

Proof. The formulae are proved if we can verify that for any $d \in \mathbb{N}_0$, any $n \in \mathbb{Z}$, and any biinfinite sequence $\{c_k\}$ we have

$$
\sum_{k=0}^{n} \binom{n-k+d}{d} \sum_{l=0}^{d+1} (-1)^l \binom{d+1}{l} c_{k-2l} = \sum_{l=0}^{d+1} \binom{d+1}{l} c_{n-l},
$$

where we incorporated formula (3.3) of Lemma 2 into the sum. For $d = 0$

$$\sum_{k=0}^{n}\{c_k - c_{k-2}\} = c_n + c_{n-1}.$$

To prove the inductive step: $d-1 \to d$, we first note that by use of the elementary combinatorial relation $\binom{d+1}{l} = \binom{d}{l} + \binom{d}{l-1}$, which we shall henceforth refer to as the Pascalian,

$$\sum_{l=0}^{d+1}(-1)^l \binom{d+1}{l} c_{n-2l} = \sum_{l=0}^{d}(-1)^l \binom{d}{l}\{c_{n-2l} - c_{n-2l-2}\}.$$

By induction

$$\sum_{k=0}^{n}\binom{n-k+d-1}{d-1}\sum_{l=0}^{d+1}(-1)^l\binom{d+1}{l}c_{k-2l} = \sum_{l=0}^{d}\binom{d}{l}\{c_{n-l} - c_{n-l-1}\},$$

and consequently,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n-k+d}{d}&\sum_{l=0}^{d+1}(-1)^l\binom{d+1}{l}c_{k-2l}\\ &= \sum_{k=0}^{n}\left\{\sum_{l=0}^{d}\binom{d}{l}\{c_{k-l} - c_{k-l-1}\}\right\} = \sum_{l=0}^{d}\binom{d}{l}\{c_{n-l} + c_{n-l-1}\}\\ &= \sum_{l=0}^{d+1}\binom{d+1}{l}c_{n-l}.\end{aligned}$$

□

The representation (3.5) somehow indicates that $d-1$ is a critical index for the Cesàro summability of multivariate F. S.

4 Proof of the theorem

To prove the positivity of the Cesàro $(C, 2d-1)$ means of the Dirichlet kernels $D_{n,d}$, let us introduce the following notation: We write $s_{n,d}^{\delta}$, $n \in \mathbb{N}$, for the (C, δ) sums of the function $g_{n,d}$ – writing sums instead of averages signifies that we suppress the normalizing factor $1/\binom{n+\delta}{n}$. And using the notation introduced in the first section we have

$$\binom{n+\delta}{\delta}\sigma_{n,d}^{\delta}(\theta) = \int_{-1}^{1} s_{n,d}^{\delta}(t)M_{d-1}(t\,|\cos\theta_1, \ldots, \cos\theta_d)\, dt.$$

We will prove that the sums $s_{n,d}^{2d-1}(t)$, $n \in \mathbb{N}_0$, are nonnegative on $[-1, 1]$. Since the B-spline kernel is nonnegative, this will prove that the Cesàro $(C, 2d-1)$ means of the Dirichlet kernels are nonnegative, proving the main part of the theorem.

The proof of the nonnegativity of $s_{n,d}^{2d-1}$ will be as follows: We will take the $(C, d-1)$ sums $s_{n,d}^{d-1}$ and will rewrite them as a linear combination of Jacobi polynomials. From this representation we will be able to conclude by use of an inequality of Gasper on Jacobi sums that the (C,d) sums of $\{s_{n,d}^{d-1}\}$, or the $(C, 2d-1)$ sums of $\{g_{n,d}\}$, are nonnegative on $[-1,1]$.

Let us begin by stating the central inequality of Gasper; the inequality was conjectured by R. Askey and proved by G. Gasper [4], see in addition [2, Lect. 8] where Askey discusses the inequality and some of its applications in greater generality and in great detail. Setting

$$Q_n^{(\alpha,\beta)} = \frac{P_n^{(\alpha,\beta)}}{P_n^{(\beta,\alpha)}(1)}, \qquad \alpha, \beta > -1,$$

we have

for each $n \in \mathbb{N}_0$ and $0 \le \alpha + \beta$, $\beta \ge -\frac{1}{2}$,

$$\sum_{k=0}^{n} \binom{n-k+\alpha+\beta}{n-k} \binom{k+\alpha+\beta}{k} Q_k^{(\alpha,\beta)}(t) \ge 0 \quad on \quad [-1,1] \tag{4.1}$$

– here we have to admit that the notation is not standard, but we will have to use this normalization so often that we decided to introduce an abbreviation. We need two further formulae on the reduction of the indices for Jacobi polynomials which can be easily derived from the formulae (22.7.18) and (22.7.19) of [1, Sec. 22],

for each $n \in \mathbb{N}_0$ and each $\alpha, \beta > 0$,

$$\begin{aligned} (n+\alpha+\beta)Q_n^{(\alpha,\beta)} &= \beta Q_n^{(\alpha,\beta-1)} + (n+\alpha)Q_n^{(\alpha-1,\beta)} \\ nQ_{n-1}^{(\alpha,\beta)} &= \beta Q_n^{(\alpha,\beta-1)} - (n+\beta)Q_n^{(\alpha-1,\beta)}. \end{aligned} \tag{4.2}$$

We start by replacing the Gegenbauer polynomials in the sum (3.5) by Jacobi polynomials to get

$$\sum_{l=0}^{d} \binom{d}{l} C_{n-l}^{(d)} = \sum_{l=0}^{d} \binom{d}{l} \binom{n-l+2d-1}{2d-1} Q_{n-l}^{(d-\frac{1}{2}, d-\frac{1}{2})}.$$

Next we will apply the Pascalian and the formulae (4.2) to reduce the indices of the polynomial $Q_{n-l}^{(\alpha,\beta)}$ as far as possible; indeed, we do the index-reduction d-times, to obtain

Lemma 5 *For each $n \in \mathbb{N}_0$,*

$$s_{n,d}^{d-1} = (d-1)! \sum_{l=0}^{[\frac{d}{2}]} \binom{n+d-1}{d-1} c_l^{(d)} Q_n^{(d-l-\frac{1}{2}, l-\frac{1}{2})}, \tag{4.3}$$

where

$$c_l^{(d)} = \binom{m}{l} \begin{cases} \dfrac{(\frac{1}{2}+l)_{m-l}(m+\frac{1}{2}-l)_l}{(m)_m}, & d = 2m, \\[2mm] \dfrac{(\frac{1}{2}+l)_{m-l}(m+\frac{3}{2}-l)_l}{(m+1)_m}, & d = 2m+1. \end{cases}$$

Here, $(a)_0 = 1$ and $(a)_l = a(a+1)\cdots(a+l-1)$, $l \in \mathbb{N}$, are the Pochhammer symbols.

Proof. For the proof we disregard the factor $(d-1)!$ in (4.3). Using (4.2) one verifies easily that

$$\begin{aligned} s_{n,d}^{d-1} &= \sum_{l=0}^{d-1} \binom{d-1}{l} \binom{n-l+2d-2}{2d-2} Q_{n-l}^{(d-\frac{1}{2},d-\frac{3}{2})} \\ &= \sum_{l=0}^{d-2} \binom{d-2}{l} \binom{n-l+2d-3}{2d-3} \left\{ \frac{d-\frac{3}{2}}{d-1} Q_{n-l}^{(d-\frac{1}{2},d-\frac{5}{2})} + \frac{\frac{1}{2}}{d-1} Q_{n-l}^{(d-\frac{3}{2},d-\frac{3}{2})} \right\}. \end{aligned}$$

After $2m$ and $2m+1$ reductions the sums read

$$\begin{aligned} s_{n,d}^{d-1} \quad = \quad & \sum_{l=0}^{d-2m} \binom{d-2m}{l} \binom{n-l+2d-2m-1}{2d-2m-1} \frac{1}{(d-m)_m} \cdot \\ \cdot \; & \sum_{j=0}^{m} \binom{m}{j} (d+\tfrac{1}{2}-2m+j)_{m-j}(m+\tfrac{1}{2}-j)_j \, Q_{n-l}^{(d-j-\frac{1}{2},d-2m+j-\frac{1}{2})} \end{aligned}$$

and

$$\begin{aligned} s_{n,d}^{d-1} \quad = \quad & \sum_{l=0}^{d-2m-1} \binom{d-2m-1}{l} \binom{n-l+2d-2m-2}{2d-2m-2} \frac{1}{(d-m)_m} \cdot \\ \cdot \; & \sum_{j=0}^{m} \binom{m}{j} (d+\tfrac{1}{2}-2m-1+j)_{m-j}(m+\tfrac{3}{2}-j)_j \, Q_{n-l}^{(d-j-\frac{1}{2},d-2m-1+j-\frac{1}{2})}, \end{aligned}$$

respectively. Let us verify the inductive step: $2m \to 2m+1$. Using the Pascalian again,

$$\begin{aligned} & s_{n,d}^{d-1} = \\ & \sum_{l=0}^{d-2m-1} \binom{d-2m-1}{l} \binom{n-l+2d-2m-2}{2d-2m-2} \frac{1}{(d-m)_m} \cdot \\ & \cdot \sum_{j=0}^{m} \binom{m}{j} (d+\tfrac{1}{2}-2m+j)_{m-j}(m+\tfrac{1}{2}-j)_j \cdot \\ & \cdot \left\{ \frac{n-l+2d-2m-1}{2d-2m-1} Q_{n-l}^{(d-j-\frac{1}{2},d-2m+j-\frac{1}{2})} \right. \\ & \left. + \frac{n-l}{2d-2m-1} Q_{n-l-1}^{(d-j-\frac{1}{2},d-2m+j-\frac{1}{2})} \right\}. \end{aligned}$$

We apply the reduction formulae (4.2) to the expression in the curly brackets and obtain

$$\frac{d-2m+j-\frac{1}{2}}{d-m-\frac{1}{2}} Q_{n-l}^{(d-j-\frac{1}{2},d-2m-1+j-\frac{1}{2})} + \frac{m-j}{d-m-\frac{1}{2}} Q_{n-l}^{(d-j-\frac{3}{2},d-2m+j-\frac{1}{2})}.$$

The inner sum $\sum_j$ can then be rewritten as

$$\sum_{j=0}^{m} \binom{m}{j} (d+\tfrac{1}{2}-2m+j)_{m-j}(m+\tfrac{1}{2}-j)_j \frac{d-2m+j-\frac{1}{2}}{d-m-\frac{1}{2}} Q_{n-l}^{(d-j-\frac{1}{2},d-2m-1+j-\frac{1}{2})}$$

$$+\sum_{j=1}^{m} \binom{m}{j-1} (d+\tfrac{1}{2}-2m-1+j)_{m-j+1}(m+\tfrac{3}{2}-j)_{j-1} \frac{m-j+1}{d-m-\frac{1}{2}} Q_{n-l}^{(d-j-\frac{1}{2},d-2m-1+j-\frac{1}{2})}.$$

Furthermore, for $j=0,1,\ldots,m$,

$$(d+\tfrac{1}{2}-2m-1+j)_{m-j} = \begin{cases} \dfrac{d-2m+j-\frac{1}{2}}{d-m-\frac{1}{2}}(d+\frac{1}{2}-2m+j)_{m-j}, \\ \dfrac{1}{d-m-\frac{1}{2}}(d+\frac{1}{2}-2m-1+j)_{m-j+1}, \end{cases}$$

and consequently,

$$s_{n,d}^{d-1} =$$

$$(d+\tfrac{1}{2}-(2m+1))_m Q_{n-l}^{(d-\frac{1}{2},d-(2m+1)-\frac{1}{2})} + \sum_{j=1}^{m} (d+\tfrac{1}{2}-(2m+1)+j)_{m-j} \cdot$$

$$\cdot \left\{ \binom{m}{j}(m+\tfrac{1}{2}-j)_j + \tbinom{m}{j-1}(m+\tfrac{3}{2}-j)_{j-1}(m-j+1) \right\} Q_{n-l}^{(d-j-\frac{1}{2},d-2m-1+j-\frac{1}{2})}.$$

Since

$$\begin{aligned} \binom{m}{j}(m+\tfrac{1}{2}-j)_j \quad + \quad & \binom{m}{j-1}(m-j+1)(m+\tfrac{3}{2}-j)_{j-1} = \\ = \quad & \binom{m}{j}\{(m+\tfrac{1}{2}-j)_j + j(m+\tfrac{3}{2}-j)_{j-1}\} \\ = \quad & \binom{m}{j}(m+\tfrac{3}{2}-j)_j, \end{aligned}$$

we proved the inductive step. We omit the verification that $2m+1 \to 2m+2$.

Setting $d=2m$ and $d=2m+1$, respectively, completes the proof of the lemma. □

Taking the (C, d) sums of the sequence $\{s_{n,d}^{d-1}\}$ and using the representation (4.3) of Lemma 5, we obtain for each $n \in \mathbb{N}_0$

$$s_{n,d}^{2d-1}(t) = (d-1)! \sum_{l=0}^{[\frac{d}{2}]} c_l^{(d)} \sum_{k=0}^{n} \binom{n-k+d-1}{d-1} \binom{k+d-1}{d-1} Q_k^{(d-l-\frac{1}{2}, l-\frac{1}{2})} \geq 0.$$

Clearly, for each index l, the inner sum Σ_k is nonnegative by Gasper's inequality (4.1) and the coefficients $c_l^{(d)}$ are positive and sum up to one – positivity follows from the definition and to prove that they sum up to one just set $n = 0$ in (4.3).

We thus proved that the $(C, 2d-1)$ Cesàro means of the sequence of functions $\{g_{n,d}\}$ are nonnegative on $[-1, 1]$, and consequently the $(C, 2d-1)$ sums of the Dirichlet kernels $D_{n,d}$ on $\mathbb{T}^d$ too. To show that the (C, δ) sums are not for $0 < \delta < 2d - 1$, let us study the $(C, d-1)$ sums of $\{g_{n,d}(t)\}$ at $t = -1$ which are, except for the factor $(d-1)!$, equal to

$$s_n^0 = (-1)^n \binom{n+d-1}{d-1}, \qquad n \in \mathbb{N}_0,$$

recall that $Q_n^{(\alpha,\beta)}(-1) = (-1)^n$. Setting

$$s_n^l = \sum_{k=0}^{n} s_k^{l-1}, \qquad l = 1, 2, \ldots, d \quad \text{and} \quad n \in \mathbb{N}_0,$$

we have by definition the recursion

$$s_0^l = s_0^{l-1} \quad \text{and} \quad s_n^l = s_{n-1}^l + s_n^{l-1}, \qquad l = 1, 2, \ldots, d \quad \text{and} \quad n \in \mathbb{N}.$$

We just proved that $s_n^d \geq 0$ for $n \in \mathbb{N}_0$; moreover, the representation

$$s_n^d = \sum_{k=0}^{n} (-1)^k \binom{n-k+d-1}{d-1} \binom{k+d-1}{d-1}, \qquad n \in \mathbb{N}_0, \tag{4.4}$$

implies that $s_n^d = (-1)^n s_n^d$, $n \in \mathbb{N}_0$. Hence, $s_{2m+1}^d = 0$, $m \in \mathbb{N}_0$. The terms s_n^d for even $n \in \mathbb{N}_0$ can be even evaluated explicitly; indeed, by formula #38 of [7, Sec. 4.2.5] we have $s_{2m}^d = \binom{m+d-1}{m}$, $m \in \mathbb{N}_0$. Using the recursion, we have

$$s_{2m}^{d-1} = s_{2m}^d \quad \text{and} \quad s_{2m+1}^{d-1} = -s_{2m}^d, \qquad m \in \mathbb{N}_0.$$

But then the (C, δ) sums of the sequence $\{s_n^{d-1}\}$, given by

$$\sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k^{d-1}, \qquad n \in \mathbb{N}_0,$$

are strictly negative for odd integers in $\mathbb{N}$ for $0 < \delta < 1$ – for odd integers we have an even number of terms of the sum, just add always two consecutive terms. So are the (C, δ) sums of $g_{n,d}$ for the argument t close to -1 for odd $n \in \mathbb{N}$ and for $0 < \delta < 2d - 1$. Since the B-spline $M(t | \cos\theta_1, \ldots, \cos\theta_d)$ is nonnegative and lives close to $t = -1$ for θ_j close to π, $j = 1, 2, \ldots, d$, this proves that the (C, δ) means of the Dirichlet kernels are not nonnegative for $0 < \delta < 2d - 1$, completing the proof of the theorem.

Remark 1 For the bivariate case

$$g_{n,2} = (n+1)C_n^{(1)} + nC_{n-1}^{(1)} = (C_n^{(2)} - C_{n-2}^{(2)}) + (C_{n-1}^{(2)} - C_{n-3}^{(2)})$$

and

$$\sum_{k=0}^{n} g_{k,2} = C_n^{(2)} + 2\,C_{n-1}^{(2)} + C_{n-2}^{(2)} = (n+1)\{\tfrac{1}{2}Q_n^{(\frac{3}{2},-\frac{1}{2})} + \tfrac{1}{2}Q_n^{(\frac{1}{2},\frac{1}{2})}\}$$

by using the reduction formulae (4.2) twice; Gasper's inequality (4.1) applied to the left-hand side of the equation then completes the proof.

Incidentally, our very first proof for $d = 2$ used an estimate by M. Schweitzer; Schweitzer proved that

$$\sum_{k=0}^{n} \binom{n-k+2}{2}(k+1)\sin(k+1)\phi \geq 0, \quad \text{on} \quad 0 \leq \phi \leq 2\pi/3,$$

where the interval $[0, 2\pi/3]$ is maximal, see formula (1.24) of [2, Lect. 1]. On the other hand, formula (1.19) of [2, Lect. 1] states that

$$(4.5) \qquad \sum_{k=0}^{n} \binom{n-k+3}{3}(k+1)\sin(k+1)\phi > 0, \quad \text{on} \quad 0 < \phi < \pi$$

– this formula again goes back to Fejér. With these inequalities in mind, it is somewhat surprising to see that

$$(4.6) \quad \sum_{k=0}^{n} \binom{n-k+2}{2}\{(k+1)\sin(k+1)\phi + k\sin k\phi\} > 0, \qquad 0 < \phi < \pi.$$

Indeed,

$$(k+1)\sin(k+1)\phi + k\sin k\phi = 2(k+\tfrac{1}{2})\cos\tfrac{1}{2}\phi\sin(k+\tfrac{1}{2})\phi + \sin\tfrac{1}{2}\phi\cos(k+\tfrac{1}{2})\phi.$$

The proof of (4.6) then follows from the inequalities (2.1) and (2.2) of Section 2. This equation did lead us to the proof of the theorem in general, but not directly and not in an obvious way; one further intermediate step was the following slightly modified equation

$$\begin{aligned}(k+1)\frac{\sin(k+1)\phi}{\sin\phi} + k\frac{\sin k\phi}{\sin\phi} &= (k+1)^2 Q_k^{(\frac{1}{2},\frac{1}{2})} + k^2 Q_{k-1}^{(\frac{1}{2},\frac{1}{2})} \\ &= (k+\tfrac{1}{2})Q_k^{(\frac{1}{2},-\frac{1}{2})} + (k+\tfrac{1}{2})Q_k^{(-\frac{1}{2},\frac{1}{2})}.\end{aligned}$$

The relations (4.5) and (4.6) have their counterparts for arbitrary dimensions ($d \in \mathbb{N}$) in the statements:

$$(4.5') \qquad \sum_{k=0}^{n} \binom{n-k+d}{d} C_k^{(d)}(t) \geq 0 \quad \text{on} \quad [-1, 1]$$

and

$$(4.6') \qquad \sum_{k=0}^{n} \binom{n-k+d-1}{d-1} \sum_{l=0}^{d} \binom{d}{l} C_{k-l}^{(d)}(t) \geq 0 \quad \text{on} \quad [-1,1],$$

respectively. The first inequality follows directly from Gasper's inequality (4.1), while the second one rephrases the content of this section.

5 The Abel means

Studying Cesàro means for multivariate Fourier series w. r. t. to ℓ-1 summability, it is natural to have a look at the Abel means too.

We set

$$p_{r,d}(\theta) = \sum_{\alpha} r^{|\alpha|_1} e^{i\alpha\theta}, \qquad 0 \leq r < 1,$$

and have on the one hand

$$p_{r,d}(\theta) = \prod_{k=1}^{d} p_{r,1}(\theta_k), \qquad \text{where} \quad p_{r,1}(\phi) = \frac{1-r^2}{1-2r\cos\phi + r^2}, \quad \phi \in \mathbb{T},$$

the proof being transparent. On the other hand,

$$\begin{aligned} p_{r,d}(\theta) &= (1-r)\sum_{k=0}^{\infty} r^k D_{k,d}(\theta) \\ &= (1-r)\int_{-1}^{1} \sum\nolimits_{k=0}^{\infty} r^k g_{k,d}(t) M(t\,|\cos\theta_1, \ldots, \cos\theta_d)\, dt. \end{aligned}$$

We use formula (3.3) to prove that

$$\begin{aligned} (1-r)\sum_{k=0}^{\infty} r^k f_{k,d} &= (d-1)!(1-r)\sum_{l=0}^{d-1}(-1)^l \binom{d-1}{l} \sum_{k=0}^{\infty} r^k C_{k-2l}^{(d)} \\ &= (d-1)!(1-r)\sum_{l=0}^{d-1}(-1)^l \binom{d-1}{l} r^{2l} \sum_{k=0}^{\infty} r^k C_k^{(d)}. \end{aligned}$$

The inner sum $\sum_k$ is nothing but the generating function for the Gegenbauer polynomials $C_k^{(d)}$, see formula (4.7.23) of [10, Chap. IV], while the sum $\sum_l$ equals $(1-r^2)^{d-1}$. Recalling that $g_{n,d} = f_{n,d} + f_{n-1,d}$, we get

$$(1-r)\sum_{k=0}^{\infty} r^k g_{k,d}(t) = (d-1)!\frac{(1-r^2)^d}{(1-2rt+r^2)^d}, \quad \text{on} \quad [-1,1],$$

and finally

$$(5.1) \qquad p_{r,d}(\theta) = (d-1)! \int_{-1}^{1} \frac{(1-r^2)^d}{(1-2rt+r^2)^d} M(t \,|\, \cos\theta_1, \dots, \cos\theta_d)\, dt.$$

Positivity of the kernel is evident, not so seems to us the representation above. In a forthcoming paper we will make extensive use of formula (5.1) to study ℓ-1 summability of F. S. on $\mathbb{T}^d$, in particular we will study the properties of the B-spline $M(t \,|\, \cos\theta_1, \dots, \cos\theta_d)$, $-1 < t < 1$, considered as a function on $\mathbb{T}^d$.

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