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Harmonic maps between flat surfaces with conical singularities

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Introduction

In this paper we study the local and global mapping behavior of energy minimizing maps from a surface Σ into a surface S carrying a cone metric of nonpositive curvature. By this we mean that S is flat except at a finite set $\mathscr{C}$ of singularities and that near each $q \in \mathscr{C}$ the surface looks like the cone over a curve of length $2\pi\alpha$. We require that $\alpha = \alpha(q) > 1$ so that each of the vertices contributes to negative curvature. In appropriate local parameters of the underlying smooth Riemann surfaces the energy functional takes the form of the variational integral

$$E(f) = \int_\Sigma \alpha^2 |w|^{2(\alpha-1)}(|w_z|^2 + |w_{\bar{z}}|^2)\, dx\, dy$$

where $f = w(z)$ is the local expression of the map.

The study of the present harmonic map problem was proposed by Eells and Lemaire as an example for harmonic maps into spaces with singularities in [3, 4, 11]. A general existence and regularity theory for such mappings was developed only recently by Gromov and Schoen in [7]; our analysis relies on the methods and results contained in the first part of that paper. Harmonic maps into a special threedimensional cone were considered by Lin [12].

In [6] Gerstenhaber and Rauch proposed constructing the Teichmüller mapping by a maximum-minimum approach involving harmonic maps. This has been investigated further by Reich and Strebel [14, 15] and Miyahara [13], who observed that the Teichmüller map is harmonic in a certain sense with respect to the cone metric induced by the Teichmüller quadratic differential on the target. The existence problem for these cone metrics was also studied by Leite [10].

This paper is organized as follows. In section 1 we prove existence of minimizers in a homotopy class using the Bochner formula in an approximation argument. In section 2 we classify the homogenous harmonic maps into three different types, one of them being the typical singularity of a Teichmüller map.

We then obtain some regularity results for local minimizers through blow up arguments based on [7]. These will be improved to a complete description of the mapping behavior of minimizers in a homotopy class of degree one in section 3. A global condition of this type enters naturally because the main problem is to analyse the preimage of the vertices under the map. We show that the minimizer is unique and that the preimage of a cone point is either an isolated point or a finite union of analytic arcs meeting at certain angles. In section 4 we prove that the second case occurs always except when the cone metric is the Teichmüller metric on the target. We finally prove that the Teichmüller map minimizes energy in its homotopy class with respect to that metric.

1 Existence in a homotopy class

The existence problem will be solved here through approximation by nondegenerate metrics of nonpositive curvature. The same method was used in [10], although Leite did not prove the minimizing property.

Let $c: S^1 \to S^2$ be a constant speed parametrization of a simply closed, smooth curve of length $2\pi\alpha$ with $\alpha \geq 1$, and let $C \subset \mathbb{R}^3$ be the *cone generated by* c with the origin as vertex. An intrinsic model for C is obtained from the *isometric embedding*

$$J: (\mathbb{C}, ds^2) \xrightarrow{\sim} C \subset \mathbb{R}^3,\ J(w) = |w|^\alpha c\left(\frac{w}{|w|}\right), \tag{1}$$

where

$$ds^2 = \alpha^2 |w|^{2(\alpha-1)} |dw|^2. \tag{2}$$

Any branch of w^α provides a local isometry of ds^2 with the standard euclidian plane. A geodesic segment joining $w_1, w_2 \in \mathbb{C}$ is the preimage of a line segment under w^α, if the w_i lie in a common sector with angle less than $\frac{\pi}{\alpha}$, or the union of the ray segments joining w_i to 0 otherwise. Applying formally the Gauss–Bonnet theorem yields $K\,dA = 2\pi(1-\alpha)\,\delta_0$, where δ_0 is the Dirac measure at $0 \in \mathbb{C}$. Now let S be a closed Riemann surface of genus $g \geq 2$.

Definition 1 *ds^2 is a cone metric on S if for any $q \in S$ there is an $\alpha = \alpha(q) \geq 1$ and a local parameter w such that $ds^2 = \alpha^2 |w|^{2(\alpha-1)} |dw|^2$ near q. The finite set of concave vertices is $\mathscr{C} = \{q \in S : \alpha(q) > 1\}$.*

Given another compact Riemann surface Σ with local uniformizer $z = x + iy$, we consider the class of maps $f \in C^0(\Sigma, S)$ with the following property: For any open $\Omega \subset \Sigma$ such that $f(\Omega)$ is contained in the domain of an isometric embedding as in (1), the map $F = J \circ f|_\Omega$ belongs to $W^{1,2}(\Omega, \mathbb{R}^3)$. We define the energy density of f on Ω by

$$e(f)(z) = \frac{1}{2}\left(|F_x|^2 + |F_y|^2\right) dx\, dy.$$

Since $dF(z) = 0$ for a. e. $z \in F^{-1}\{0\}$ we have alternatively

$$e(f) = \alpha^2 \, |w|^{2(\alpha-1)} \, (|w_z|^2 + |w_{\bar{z}}|^2) \, dx \, dy$$

and obtain a welldefined global energy

$$E(f) = \int_\Sigma e(f).$$

Assuming that Σ is also closed and that γ is a homotopy class of maps we finally let

$$\Gamma = \{f \in \gamma : E(f) < \infty\}. \tag{3}$$

Lemma 1 *For ds^2 as in definition 1 there is a family ds_ε^2, $\varepsilon > 0$, of nondegenerate metrics with nonpositive curvature such that $ds_\varepsilon^2 \geq ds_\tau^2$ for $\varepsilon \geq \tau > 0$ and $ds_\varepsilon^2 \to ds^2$ as $\varepsilon \to 0$. The convergence is in C^0 on S and locally in C^∞ on $S \setminus \mathscr{C}$.*

For example one can take $ds_\varepsilon^2 = \alpha^2 \, (\varepsilon^3 \eta(\frac{r}{\varepsilon}) + r^2)^{(\alpha-1)} \, |dw|^2$ where $r = |w|$ and $\eta \geq 0$ is a suitable cutoff function. By the theorem of Eells–Sampson [5] there is a smooth map $f_\varepsilon \in \gamma$ that minimizes energy with respect to ds_ε^2, and the application of Moser's iteration technique to the Bochner formula for $\Delta \|df_\varepsilon\|^2$ gives

$$\sup\{\|df_\varepsilon(p)\| : p \in \Sigma\} \leq K \, E_\varepsilon(f_\varepsilon)^{1/2}$$

with K independent of ε, see [17]; here we use a fixed metric on Σ. This implies for $p_1, p_2 \in \Sigma$

$$d(f_\varepsilon(p_1), f_\varepsilon(p_2)) \leq K \, E_\varepsilon(f_\varepsilon)^{1/2} \operatorname{dist}(p_1, p_2)$$

where $d(\cdot, \cdot)$ is the distance for ds^2. As $E_\tau(f_\tau) \leq E_\tau(f_\varepsilon) \leq E_\varepsilon(f_\varepsilon)$ for $\tau \leq \varepsilon$ we find a subsequence of $\{f_\varepsilon\}$ which converges uniformly with respect to d to a map $f \in \Gamma$. The estimates for higher derivatives in [5] imply that f is smooth and harmonic on the complement of

$$\mathscr{P} = f^{-1}(\mathscr{C}). \tag{4}$$

Note that the Lipschitz continuity of f means that the local expression $f = w(z)$ belongs to $C^{1/\alpha}$, cf. [10]. In order to prove the minimizing property we need

Lemma 2 *For $g \in \Gamma$ and $\eta > 0$ there exists $h \in C^\infty(\Sigma, S) \cap \Gamma$ such that $|E(h) - E(g)| < \eta$.*

Proof. Let $\Sigma_\delta = \{p \in \Sigma : d(g(p), \mathscr{C}) < \delta\}$ and let $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(s) = 0$ for $s \leq 1/2$, $\varphi(s) = 1$ for $s \geq 1$ and $\varphi'(s) \geq 0$. Define

$$\Phi_\delta : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi_\delta(X) = \varphi\left(\frac{R}{\delta}\right) X; \; R = |X|.$$

Then $\|d\Phi_\delta(X)\| \leq K$ with K independent of X, δ. On each component of Σ_δ we can replace $G = J \circ g$ by $G_\delta = \Phi_\delta \circ G$. It follows that $|E(g_\delta) - E(g)| \leq (K^2 + 1) \, E(g, \Sigma_\delta) \leq \eta/2$ for δ sufficiently small. By the Sobolev chain rule g_δ belongs to $W^{1,2}(\Sigma, S)$ with respect to a nondegenerate metric. As g_δ is also continuous the claim now follows from standard Sobolev space theory. ∎

Proposition 1 *There exists a minimizing map $f \in \Gamma$ which is Lipschitz continuous with respect to the metric ds^2.*

Proof. Let f be as above, $g \in \Gamma$ and h as in lemma 2. Then $E(f) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(f_\varepsilon) \leq \lim_{\varepsilon \to 0} E_\varepsilon(h) = E(h) \leq E(g) + \eta$. Thus f is minimizing and also $E_\varepsilon(f_\varepsilon)$ converges to $E(f)$ as ε tends to 0. ∎

Of course this result is only a special case of theorem 4.4 in [7].

2 Local regularity results

Let $D = \{z \in \mathbb{C}: |z| < 1\}$. Depending on what seems more convenient we use either $f: D \to (\mathbb{C}, ds^2)$ or $F = J \circ f: D \to C \subset \mathbb{R}^3$ to describe the *same* locally minimizing map. It is a standard fact that for a local minimizer the formula

$$\Phi(v, v) = |dF \cdot v|^2 - |dF \cdot iv|^2 - 2i\langle dF \cdot v, dF \cdot iv \rangle \tag{5}$$

defines a holomorphic quadratic differential, the *Hopf differential* of F. Alternatively one has

$$\Phi = 4\alpha^2 \, |f|^{2(\alpha-1)} f_z \bar{f}_z \, dz^2. \tag{6}$$

For a nonzero holomorphic quadratic differential Φ on a Riemann surface Σ there exists at each $p \in \Sigma$ a *natural parameter* z with $z(p) = 0$ such that around p

$$\Phi = \left(\frac{m+2}{2}\right)^2 z^m \, dz^2. \tag{7}$$

Here $m \in \mathbb{N}_0$ is the vanishing order of Φ at p and z is unique up to multiplication with an $m+2$-th root of 1. If $m = 0$ the lines $\{x = \text{const}\}$ are the integral curves of the distribution $\{v \in T\Sigma: \Phi(v, v) \leq 0\}$; they are called *vertical trajectories*. For a harmonic map these are the lines of minimal stretch since

$$|dF \cdot v|^2 = \frac{1}{2}(|dF|^2|v|^2 + \mathrm{Re}(\Phi(v, v))). \tag{8}$$

We shall need the following *reference map* $w_0: \mathbb{C} \to \mathbb{C}$ which depends on parameters $k \in [0, 1]$, $m \in \mathbb{N}_0$:

$$w_0(z) = \begin{cases} z & \text{for} \quad k = 0 \\ \left(\frac{1}{2}\left(k^{-\frac{1}{2}} z^{\frac{m+2}{2}} + k^{\frac{1}{2}} \bar{z}^{\frac{m+2}{2}}\right)\right)^{\frac{2}{m+2}} & \text{for} \quad 0 < k \leq 1. \end{cases} \tag{9}$$

Here we agree that w_0 maps the rays over the $m+2$-th roots of 1 into themselves; these rays form the image of w_0 if $k = 1$.

Now suppose $F: \mathbb{C} \to C$ is a nonconstant, locally minimizing map which is homogenous of degree $\tau \geq 1$, thus f is homogenous of degree τ/α as a map into $\mathbb{C}$. By passing to $F_b(z) = F(bz)$ with a suitable $b \in \mathbb{C}^*$ we may assume that the Hopf differential Φ is given by (7) unless $\Phi \equiv 0$; in the latter case we require instead that

$$\frac{1}{2\pi}\int_{\partial D} |F|^2\, ds = 1. \tag{10}$$

We can now classify the homogenous harmonic maps.

Lemma 3 *Let f be a homogenous degree τ harmonic map. After the above normalizations and after passing to $R \circ f$ with suitable $R \in O(2)$, one of the following alternatives hold:*

(a) $f(z) = w_0(z)^{\tau/\alpha}$ *where* $k = 0,\ \tau/\alpha \in \mathbb{N}$.

(b) $f(z) = w_0(z)^{\tau/\alpha}$ *where* $0 < k < 1,\ \tau/\alpha \in \mathbb{N}$ *and* $m = 2(\tau - 1) \in \mathbb{N}_0$.

(c) *For* $l \in \{1, \ldots, m+2\}$ *there exists* $\omega_l \in S^1$ *such that* $f(re^{i\theta}) = |w_0(re^{i\theta})|^{\tau/\alpha}\,\omega_l$ *for* $\left(l - \frac{1}{2}\right)\frac{2\pi}{m+2} \le \theta \le \left(l + \frac{1}{2}\right)\frac{2\pi}{m+2}$, *where* $k = 1,\ m = 2(\tau - 1) \in \mathbb{N}_0$ *and* $\angle(\omega_l, \omega_{l+1}) \ge \pi/\alpha$.

Proof. We have $\frac{\partial}{\partial\theta}|F|^2 = 2\langle F, F_\theta\rangle = \frac{2r}{\tau}\langle F_r, F_\theta\rangle = -\frac{1}{\tau}\,\mathrm{Im}(\Phi(z,z))$. It is easy to see that $\Phi \equiv 0$ implies that **(a)** holds; let us therefore assume (7). For $z = re^{i\theta}$ we get by integration along θ

$$|F|^2(z) = \frac{1}{2}\left(\mu\, r^{m+2} + \mathrm{Re}\left(z^{m+2}\right)\right) \tag{11}$$

with $\mu = 2|F|^2(1) - 1$. $|F|^2 \ge 0$ implies $\mu \ge 1$. F is a harmonic map into a cone, thus for $|F| > 0$

$$|dF|^2(z) = \frac{1}{2}\Delta|F|^2(z) = \left(\frac{m+2}{2}\right)^2 \mu r^m. \tag{12}$$

Now (8) yields

$$d\sigma^2 := f^* ds^2 = \frac{1}{2}\left(\frac{m+2}{2}\right)^2 (\mu r^m |dz|^2 + \mathrm{Re}(z^m dz^2)). \tag{13}$$

We can also compute the Jacobian of f:

$$Jf = \left(\frac{m+2}{2}\right)^2 \frac{\sqrt{\mu^2 - 1}}{2}\, r^m. \tag{14}$$

Case 1: $\mu > 1$

Let $k = \mu - \sqrt{\mu^2 - 1} \in (0, 1)$, i. e. $\mu = \frac{1}{2}(k^{-1} + k)$, and denote by $\zeta = \zeta(w)$ the inverse of w_0 as in (9). Locally we can write $\zeta = g_3 \circ g_2 \circ g_1$ where $g_1(w) = w^{\frac{m+2}{2}}$, $g_3(w_2) = w_2^{\frac{2}{m+2}}$ and $g_2(w_1) = \sqrt{\frac{2}{\mu+1}}\, u_1 + i\sqrt{\frac{2}{\mu-1}}\, v_1$ with $w_1 = u_1 + iv_1$. We compute

$$\begin{aligned}\zeta^* d\sigma^2 &= g_1^* g_2^* g_3^* d\sigma^2 \\ &= g_1^* g_2^* \left(\frac{1}{2}(\mu|dw_2|^2 + \mathrm{Re}(dw_2^2))\right) \\ &= g_1^* g_2^* \left(\frac{\mu+1}{2} du_2^2 + \frac{\mu-1}{2} dv_2^2\right) \\ &= g_1^*(du_1^2 + dv_1^2) \\ &= g_1^* |dw_1|^2 = \left(\frac{m+2}{2}\right)^2 |w|^m |dw|^2.\end{aligned}$$

Therefore $\pi = f \circ \zeta: (\mathbb{C}, (\frac{m+2}{2})^2 |w|^m |dw|^2) \to (\mathbb{C}, ds^2)$ is a local isometry. The restriction $\pi: \partial D \to \partial D$ is then an isometric covering, which implies that $f(z) = \pi(w_0(z)) = R\, w_0(z)^j$ for some $j \in \mathbb{N}$, $R \in O(2)$.

Case 2: $\mu = 1$

Let again w_0 be as in (9), but now with $k = 1$. By (13) f is constant along the vertical trajectories of Φ, while along the rays $\theta_l = l \cdot \frac{2\pi}{m+2}$ one has $|F|^2(re^{i\theta_l}) = r^{m+2}$. Setting $\omega_l = f(e^{i\theta_l})$ we arrive at type (**c**). Now if $m = 0$ then f is independent of y ($z = x + iy$) and a standard comparison argument implies that $f|_{\mathbb{R}}$ is a *minimizing* geodesic, i. e. $\angle(\omega_1, \omega_2) \geq \pi/\alpha$. For $m \geq 1$ we consider on $\{\zeta: \mathrm{Im}(\zeta) > 0\}$ the minimizing map $h(\zeta) = f(\zeta^{\frac{2}{m+2}})$ to conclude that $\angle(\omega_l, \omega_{l+1}) \geq \pi/\alpha$ holds in general. ∎

Now let $f: D \to (\mathbb{C}, ds^2)$ be a locally minimizing map with $f(0) = 0$. By theorem 2.3 of [7] f is locally Lipschitz continuous. Define

$$\begin{aligned}E(\sigma) &= \int_{\{|z|<\sigma\}} (|F_x|^2 + |F_y|^2)\, dx\, dy, \quad I(\sigma) = \int_{\{|z|=\sigma\}} |F(z)|^2\, ds \quad \text{and} \\ \mathrm{ord}(\sigma) &= \sigma E(\sigma)/I(\sigma).\end{aligned}$$

The following statements are proved in (2.5), theorem 2.3 and proposition 3.3 of [7], see also [12]:

Lemma 4 *Suppose f as above is not constant.*

(i) *$\mathrm{ord}(\sigma)$ is monotonically nondecreasing with $\tau = \lim_{\sigma\to 0} \mathrm{ord}(\sigma) \geq 1$.*
(ii) *The function $\sigma^{-2\tau-1} I(\sigma)$ is nondecreasing.*
(iii) *As $\sigma_i \to 0$ the sequence $F_{\sigma_i}(z) = \left(\frac{I(\sigma_i)}{\sigma_i}\right)^{-1/2} F(\sigma_i z)$ contains a subsequence which converges locally uniformly and locally in energy to a nonconstant homogenous degree τ harmonic map $F_*: \mathbb{C} \to C$.*

Any limit F_* as in lemma 4 is called a *tangent map* of f at 0.

Lemma 5 *Let f be a locally minimizing map as above.*

(i) *All tangent maps of f at 0 have the same stretch $k = k_f(0) \in [0, 1]$.*

(ii) *If* $k_f(0) < 1$ *then 0 is an isolated point of* $f^{-1}\{0\}$.

Proof. Let $m \in \mathbb{N}_0 \cup \{\infty\}$ be the vanishing order of the Hopf differential Φ at 0, and assume w. l. o. g. that Φ is an in (7) if $m < \infty$. If F_{σ_i} converges to F_* then $\Phi_{\sigma_i} = \frac{\sigma_i^{m+3}}{I(\sigma_i)} \Phi$ converges to some Φ_*. By (ii) of lemma 4 we must have $\tau \leq \frac{m+2}{2}$. Now if $\tau < \frac{m+2}{2}$ then $\Phi_* \equiv 0$ and all tangent maps are of type (**a**). For $\tau = \frac{m+2}{2}$ we put $\mu = \lim_{\sigma\to 0} \frac{I(\sigma)}{\pi\, \sigma^{m+3}}$ and obtain $\Phi_* = \frac{1}{\pi\mu}\left(\frac{m+2}{2}\right)^2 z^m\, dz^2$. On the other hand the definition of F_{σ_i} gives

$$\int_{\partial D} |F_*|^2\, ds = \lim_{i\to\infty} \int_{\partial D} |F_{\sigma_i}|^2\, ds = 1.$$

It now follows from (11) that the stretch parameter of F_* is given by $k = \mu - \sqrt{\mu^2 - 1}$ (and that $\mu \geq 1$). This proves the first claim, and the second follows immediately because if 0 is not an isolated point of $f^{-1}\{0\}$, then at least one and therefore all tangent maps are of type (**c**). ∎

Due to the simplicity of the metric ds^2 it is easy to describe the behavior of f at an *isolated point of* $f^{-1}\{0\}$. Suppose that for the local minimizer $f: D \to (\mathbb{C}, ds^2)$ we have $f^{-1}\{0\} = \{0\}$. By passing to $f(\bar{z})$ if necessary we can assume that the local winding number of f around 0 is equal to some $j \in \mathbb{N}_0$. Taking branches of $g(z) = f(z)^\alpha$ and of z^α one obtains the singlevalued map $z^{-j\alpha} g(z)$ on $D \setminus \{0\}$. As g is locally a pair of standard harmonic functions, we get functions $\psi_1(z) = z^{-j\alpha} g_z(z)$ and $\psi_2(z) = z^{j\alpha} \bar{g}_{\bar{z}}(z)$ which are holomorphic on $D \setminus \{0\}$ with at most a pole at the origin. Thus for some $a_1, a_2 \in \mathbb{C}^*$, $m_1, m_2 \in \mathbb{Z}$ one has expansions

$$\begin{cases} \psi_1(z) &= a_1 z^{m_1} + \ldots \quad , \\ \psi_2(z) &= a_2 z^{m_2} + \ldots \quad . \end{cases} \tag{15}$$

Let m be the order of the Hopf differential, τ the order of f and k the stretch of f at zero. Then one has the following possibilities:

$$\begin{aligned} k = 0 \quad &\Rightarrow \quad m_1 = -1,\ m_2 = m+1,\ \tau = j\alpha < \frac{m+2}{2} \qquad (16) \\ 0 < k < 1 \quad &\Rightarrow \quad m_1 = -1,\ m_2 = m+1,\ \tau = j\alpha = \frac{m+2}{2} \text{ and } \left|\frac{a_2}{a_1}\right| = k. \\ k = 1 \quad &\Rightarrow \quad m_1 + j\alpha = m_2 - j\alpha = \frac{m}{2} = \tau - 1 \text{ and } |a_2| = |a_1|. \end{aligned}$$

Obviously (15) can be translated into an expansion for f by integrating to get g and taking the $\frac{1}{\alpha}$-th root. In any case we have for $|z| = r \to 0$

$$k_f(z) = \left|\frac{g_{\bar{z}}}{g_z}\right|(z) = \begin{cases} \left|\frac{a_2}{a_1}\right| r^{2(\frac{m+2}{2}-\tau)}(1 + O(r)) & \text{if } k_f(0) = 0 \\ k_f(0) + O(r) & \text{if } k_f(0) > 0. \end{cases} \tag{17}$$

Unfortunately we do not have such a simple description of f at a nonisolated point of $f^{-1}\{0\}$. Note that the preimage of a point may be complicated even for harmonic maps into regular metrics while the zero set of the Jacobian is relatively simple [20]. All we can prove here is

Lemma 6 *Let f be as in lemma 4 with $\tau = 1$. Let $z = x + iy$ be the natural parameter for the Hopf differential near 0, i. e. $\Phi \equiv dz^2$. For $\varepsilon > 0$ there is a $\delta > 0$ such that if $z, z' \in Q_\delta = \{z: |x|, |y| \leq \delta\}$ and $f(z) = f(z') = 0$, then $|x - x'| \leq \varepsilon |y - y'|$.*

Proof. By passing to the natural parameter and rescaling we have $F \in \mathrm{Lip}(D, \mathbb{C})$ and $\Phi \equiv dz^2$. Assume by contradiction that there are sequences $z_i, z_i' \to 0$ such that $F(z_i) = F(z_i') = 0$ but $|x_i - x_i'| \geq \varepsilon |y_i - y_i'|$. Selecting a subsequence one has $\frac{1}{\sigma_i}(z_i' - z_i) \to z_0 \in \partial D$ where $\sigma_i = |z_i - z_i'|$, $z_0 = x_0 + iy_0$ and $|x_0| \geq \varepsilon |y_0|$. Now let $F_i: D_2 \to C$, $F_i(z) = \frac{1}{\sigma_i} F(z_i + \sigma_i z)$. From the proof of proposition 3.3 in [7] we infer that the F_i subconverge uniformly and in energy to a minimzing map $F_*: D_2 \to C$. The Hopf differential of F_* is again $\Phi_* \equiv dz^2$. Now for $r > 0$ and $\rho \in (0, 2)$ one gets (with obvious notation) $1 \leq \mathrm{ord}(F_*, \rho) = \lim_{i\to\infty} \mathrm{ord}(F_i, \rho) = \lim_{i\to\infty} \mathrm{ord}(F, z_i, \sigma_i \rho) \leq \lim_{i\to\infty} \mathrm{ord}(F, z_i, r) = \mathrm{ord}(F, r)$. Letting $r \to 0$ we conclude from lemma 3.2 in [7] that F_* is homogenous of degree one. But then the classification of tangent maps (lemma 3) gives

$$0 \neq F_*(z_0) = \lim_{i\to\infty} F_i \left(\frac{z_i' - z_i}{\sigma_i} \right) = \lim_{i\to\infty} \frac{1}{\sigma_i} F(z_i') = 0.$$

This contradiction proves the lemma. ∎

Lemma 5 and 6 hold with the same proof for harmonic maps into nonpositively curved surfaces which have singularities asymptotic to the considered type in an appropriate sense, but we shall not pursue this.

3 Minimizing maps of degree one

From now on we assume that Σ, S are closed surfaces of the same genus $g \geq 2$, and that γ is the homotopy class of an orientation preserving diffeomorphism. For the minimizer $f \in \Gamma$ constructed in section 1 we have the holomorphic Hopf differential $\Phi \in QD(\Sigma)$ and the stretch function $k_f: \Sigma \to [0, 1]$ from lemma 5.

Definition 2 *A vertical arc of Φ is a compact topological interval contained in a vertical trajectory with $\Phi \neq 0$ at interior points.*

The following result should be compared with the corresponding statement for regular target metrics of nonpositive curvature [18, 16].

Theorem 1 *Let f be the minimizer in a degree one homotopy class constructed in section 1. Then*

(i) *$f: \Sigma \setminus \mathscr{P} \to S \setminus \mathscr{C}$ is diffeomorphic ($\mathscr{P} = f^{-1}(\mathscr{C})$).*
(ii) *If the Hopf differential Φ vanishes, then f is biholomorphic.*
(iii) *If $\Phi \not\equiv 0$, then the preimage $\mathscr{P}_q$ of any point $q \in S$ has one of the following types:*
 (a) *$\mathscr{P}_q$ is a single point p with $k_f(p) = 0$.*
 (b) *$\mathscr{P}_q$ is a single point p with $0 < k_f(p) < 1$.*

(c) *$\mathscr{P}_q$ is a nonempty, simply connected union of finitely many vertical arcs and $k_f|_{\mathscr{P}_q} \equiv 1$.*

Proof. Let $\Omega = \{p \in \Sigma \setminus \mathscr{P} : \Phi(p) \not= 0\}$. The function

$$h = \log \frac{|w_{\bar{z}}|^2}{|w_z|^2} \tag{18}$$

is harmonic on Ω and is known to be an important tool in the study of harmonic diffeomorphisms, see [9]. By [16, 18] the approximating maps from section 1 are orientation preserving diffeomorphisms. Thus $\mathscr{P}_q$ is connected for any $q \in S$, $h \leq 0$ on Ω and

$$h = \log k_f^2 \text{ on } \Omega. \tag{19}$$

Lemma 5 shows that if $\mathscr{P}_q$ contains more than one point, then $k_f|_{\mathscr{P}_q} \equiv 1$. In particular $\Phi \equiv 0$ implies that f is biholomorphic, so from now on we assume $\Phi \not\equiv 0$. For $p \in \mathscr{P}_q$ with $\Phi(p) \not= 0$ we infer from lemma 6 the existence of a square Q_δ, defined in terms of the natural parameter z at p, such that either $\mathscr{P}_q \cap Q_\delta = \{p\}$ or $\mathscr{P}_q \cap Q_\delta = \{iy : 0 \leq y \leq \delta\}$ or $\mathscr{P}_q \cap Q_\delta = \{iy : -\delta \leq y \leq \delta\}$. Taking into account the trajectory structure of Φ [19] this shows that $\mathscr{P}_q$ is either a point or a connected union of finitely many vertical arcs. Since the map induced by f on the fundamental groups is injective, $\mathscr{P}_q$ cannot contain a loop that is nontrivial in Σ. But it cannot contain a contractible loop either, again by the trajectory structure of Φ. Therefore Ω is connected and the maximum principle yields

$$h < 0 \text{ on } \Omega. \tag{20}$$

Now $f : \Sigma \setminus \mathscr{P} \to S \setminus \mathscr{C}$ must be homeomorphic since otherwise there would be a continuum of points with the same image, which contradicts (20). The fact that the Jacobian of f is strictly positive on $\Sigma \setminus \mathscr{P}$ follows from [8]; alternatively one could use lemma 3. Finally, if $\mathscr{P}_q = \{p\}$ is an isolated point, then (17) and the maximum principle for h yield $k_f(p) < 1$. Thus all assertions of the theorem are proved. ∎

We now have enough information to settle the uniqueness question.

Theorem 2 *If γ is the homotopy class of a diffeomorphism, then there is a unique minimizer f in Γ.*

Proof. Let f_0 be the minimizer constructed above, and suppose that $f_1 \in \Gamma$ is a different minimizer. There is a geodesic homotopy

$$f : [0,1] \times \Sigma \to S, \quad f(t,p) = f_t(p)$$

connecting f_0 to f_1. Let $\mathscr{P}(t) = f_t^{-1}(\mathscr{C})$, $\Omega(t) = \Sigma \setminus \mathscr{P}(t)$ and $Y_t : \Omega(t) \to TS$, $Y_t(p) = \frac{\partial f}{\partial t}(t,p)$. Note that for $\Omega \subset\subset \Omega(t)$ there exists $\tau > 0$ such that $f|_{[t-\tau, t+\tau] \times \Omega}$ is smooth. By theorem 4.1 of [7] we have for any $\Omega \subset \Sigma$

$$\frac{d^2}{dt^2} E(f_t, \Omega) \geq \int_\Omega |\nabla(d(f_0, f_1))|^2 \, dx \, dy$$

in the distributional sense. This gives

$$d(f_0, f_1) \equiv d > 0. \tag{21}$$

On the other hand for $\Omega \subset\subset \Omega(t)$ one gets $\int_\Omega |\nabla Y_t|^2\, dx\, dy = \frac{d^2}{dt^2} E(f_t, \Omega) = 0$ where ∇ denotes covariant derivative along f_t. Thus one obtains a parallel vector field on $S \setminus \mathscr{C}$ by setting $Z(q) = Y_0(f_0^{-1}\{q\})$. Equation (21) implies $ds(Z) \equiv d > 0$.

Now let $q \in \mathscr{C}$; we have to exploit the negative curvature at the vertices. We want to show that the local vector field $Z_0(w) = \alpha|w|^{\alpha-1} Z(w)$ has a continuous extension into $w = 0$; here w is as in definition 1 so that $|dw|(Z_0) \equiv d$. First note that $\mathscr{P}(0) \subset \Omega(t)$ for small $t > 0$ because $d(f_0, f_t) = d \cdot t > 0$. f_t maps $\mathscr{P}_q = f_0^{-1}\{q\}$ into the distance circle of radius $d \cdot t$ around q and for $p \in \mathscr{P}_q$ $f(\cdot, p)$ is a geodesic ray emanating from q. Thus $Y_t(p)$ is proportional to the normal of the distance circle. Since Y_t is parallel it follows (using the structure of $\mathscr{P}_q$ from theorem 1) that the image $f_t(\mathscr{P}_q)$ is a single point $q(t)$; $q(t)$ is a geodesic ray with $d(q(t), q) = d \cdot t$. Now let $p_i \in \Omega(0)$ with $p_i \to p_0 \in \mathscr{P}_q$. Let $q_i = f_0(p_i)$, $\tilde{q}_i = f_1(p_i)$ so that $q_i \to q$ and $\tilde{q}_i \to \tilde{q} = f_1(p_0)$. Let $\gamma_i = f(\cdot, p_i)$. Then γ_i converges to the geodesic segment $q(\cdot)$. Thus if we parametrize $q(\cdot)$ by arclength with respect to $|dw|^2$ for small $t > 0$, then $Z_0(q_i) \to d \cdot q'(0)$. As this is true for *any* sequence $p_i \to \mathscr{P}_q$, we have obtained a continuous extension of Z_0 by $Z_0(0) = d \cdot q'(0)$. But Z is parallel and therefore Z_0 rotates around 0 by the angle $2\pi(1 - \alpha(q)) < 0$. This contradiction proves that a minimizer different from f_0 cannot exist. ∎

We are now interested in the following two questions: Do the line singularities as in **(c)** of theorem 1 really occur? On the other hand, if they occur, how complicated can the set $\mathscr{P}_q$ be?

The next result will allow us to show that the singular lines do occur except in a very special situation (see section 4). It also implies a bound for the number of vertical arcs in $\mathscr{P}_q$. For example, *if $\alpha(q) \le 3/2$ then $\mathscr{P}_q$ is either a point or one vertical arc disjoint from the zeroes of Φ.*

Definition 3 *For f as in theorem 1 and $q \in S$, we let*

$$\delta(q) = \alpha(q) - \frac{1}{2}\left(\left(\sum_{p \in \mathscr{P}_q} m_p\right) + 2\right),$$

where m_p is the vanishing order of the Hopf differential at p.

By the harmonicity of the function h defined in (18), we have on $\Omega = \{p \in \Sigma \setminus \mathscr{P} : \Phi(p) \neq 0\}$ the closed 1-form

$$\omega(v) = \frac{1}{8\pi}\, dh(iv). \tag{22}$$

For any $q \in S$ the form ω has a welldefined period $\mathrm{per}(\omega, q) \in \mathbb{R}$ around $\mathscr{P}_q = f^{-1}\{q\}$.

Theorem 3 *Let f be the minimizer in a homotopy class of degree one. For any $q \in S$ we have*

$$\delta(q) = \mathrm{per}(\omega, q) = \oint_{\mathscr{S}_q} \omega.$$

According to the different possibilities in theorem 1, $\delta(q)$ is negative in case (**a**), *zero in case* (**b**) *and positive in case* (**c**).

Proof. We first describe the asymptotic behavior of f at any point and then prove the statement.

Step 1: *Local expansions of f*

Let $\tau = \mathrm{ord}(f, p) \geq 1$, $k = k_f(p) \in [0, 1]$ and assume that z, w are local parameters such that $\Phi = \left(\frac{m+2}{2}\right)^2 z^m\, dz^2$ and $ds^2 = \alpha^2 |w|^{2(\alpha-1)}\, |dw|^2$ where $z = 0$ corresponds to p, $w = 0$ to $f(p)$. Depending on k we have to consider three cases, the first two following readily from the discussion of isolated singularities in section 2.

(**a**) If $k = 0$, then $\tau = \alpha$ and for $\mu = \min\{\frac{1}{\alpha}, m + 2 - 2\alpha\} > 0$ and appropriate $c \in \mathbb{C}^*$ one has ($r = |z|$):

$$w(z) = cz + O(r^{1+\mu}). \tag{23}$$

Furthermore

$$\frac{\partial h}{\partial r}(z) = 4\left(\frac{m+2}{2} - \alpha\right)\frac{1}{r} + O(1). \tag{24}$$

(**b**) If $0 < k < 1$, then $\tau = \alpha = \frac{m+2}{2}$ and (after a rotation of w if necessary)

$$w(z) = w_0(z) + O(r^{1+1/\alpha}) \tag{25}$$

with w_0 as in (9). Also the point p is a *removable singularity* of h and ω.

(**c**) If $k = 1$ then $f(p) = q \in \mathscr{C}$ by theorem 1 and p is not an isolated point of $\mathscr{S}_q$. We may assume that $w(z)$ is nonzero on a wedge

$$W = \left\{r\, e^{i\theta} : 0 < r \leq r_0,\ -\frac{1}{2}\frac{2\pi}{m+2} < \theta < \left(l - \frac{1}{2}\right)\frac{2\pi}{m+2}\right\}, \tag{26}$$

but that $w(z)$ is zero on the sides of W. Here $\tau = \frac{m+2}{2}$ and $l \in \{1, \ldots, m+2\}$. On W we use the coordinate

$$\zeta(z) = e^{-i\varphi} z^{\frac{\tau}{l}}, \quad z(\zeta) = e^{i\frac{l}{\tau}\varphi} z^{\frac{l}{\tau}}$$

where $\varphi = \frac{l-1}{2l}\pi$, $1^{\tau/l} = 1^{l/\tau} = 1$. Thus ζ lies in the right half disk $\{\zeta = \xi + i\eta : \xi > 0,\ |\zeta| \leq r_0^{\tau/l}\}$. Finally we take a branch $g(z)$ of $w(z)^\alpha$ and obtain for $g(\zeta) = g(z(\zeta))$ and suitable $a \in \mathbb{R}$ the expansion

$$e^{-il\varphi} g(\zeta) = \frac{1}{2}\left(\zeta^l + (-1)^{l-1}\bar{\zeta}^l\right) + \frac{a}{2}\left(\zeta^{l+1} + (-1)^l \bar{\zeta}^{l+1}\right) + \cdots \tag{27}$$

We claim that a is in fact strictly positive. Namely, the function $\log\left(|g_{\bar\zeta}|^2/|g_\zeta|^2\right)$ is harmonic and strictly negative on the right half disk. Therefore the Hopf boundary point lemma implies

$$0 < -\frac{\partial}{\partial\xi}\log\frac{|g_{\bar\zeta}|^2}{|g_\zeta|^2}(0) = 4\,\frac{l+1}{l}\,a. \tag{28}$$

If $m = 0$ and $l = 1$ one obtains

$$\frac{\partial h}{\partial x}(z) = -8a + O(r), \tag{29}$$

while if $m \geq 1$ or $l \geq 2$

$$\frac{\partial h}{\partial r}(z) = O(r^{\frac{\tau}{l}-1}). \tag{30}$$

Step 2: *Proof of the formula*

For the case of a conformal point ($k_f(p) = 0$) or a Teichmüller point ($0 < k_f(p) < 1$) the statements of the theorem follow from (16), (24) and (25). Now assume that $\mathscr{P}_q$ is a union of vertical arcs. We define a path to evaluate the period of ω as follows: For a small radius $r > 0$ and any $p \in \mathscr{P}_q$ with $m \geq 1$ or $l \geq 2$ in (26) we choose a natural parameter at p and delete the coordinate disk of radius r from the surface Σ. The remaining part of $\mathscr{P}_q$ consists of finitely many vertical arcs along which we can choose a natural parameter $z = x + iy$ with the arc contained in $x = 0$. For $0 < \varepsilon << r$ we can go along the lines $\{x = \pm\varepsilon\}$ from the boundary of one disk to the boundary of the next. Now choose an appropriate orientation to define the path $c_{r,\varepsilon}$ depending on r and ε. Using (29) and (30) we get

$$\begin{aligned}\operatorname{per}(\omega, q) &= \lim_{r\to 0}\lim_{\varepsilon\to 0}\oint_{c_{r,\varepsilon}}\omega\\ &= -\frac{1}{8\pi}\oint_{\mathscr{P}_q}\frac{\partial h}{\partial x}(iy)\,dy = \oint_{\mathscr{P}_q}\omega,\end{aligned}$$

where each vertical arc is traversed twice in opposite directions and $z = x + iy$ is the natural parameter with the positive y-axis defined by the direction of the path. By (28) the period is strictly positive. Defining $g = w^\alpha$ by continuous continuation along $c_{r,\varepsilon}$ one has

$$\frac{1}{2\pi}\oint_{c_{r,\varepsilon}} d\arg g = \alpha,$$

since w is diffeomorphic outside $\mathscr{P}_q$. One can now use the expansion (27) to compute the argument integral. We omit the straightforward calculation and state only the result. Let $\mathscr{M}$ be the set of points in $\mathscr{P}_q$ with $m \geq 1$ or $l \geq 2$. For $p \in \mathscr{M}$ the disk of radius r around p decomposes into $j_p \geq 1$ wedges as in

(26) with angles $l_p^j \frac{2\pi}{m_p+2}$, $1 \le j \le j_p$. By letting first $\varepsilon \to 0$ and then $r \to 0$ one obtains

$$\alpha = \oint_{\mathscr{P}_q} \omega + \sum_{p\in\mathscr{M}} \sum_{j=1}^{j_p} \frac{l_p^j - 1}{2}.$$

Now $\sum_{j=1}^{j_p} \frac{1}{2} l_p^j = \frac{m_p+2}{2}$ and Euler's formula for the graph $\mathscr{P}_q$ gives card$\mathscr{M} - \frac{1}{2}\sum_{p\in\mathscr{M}} j_p = 1$. Therefore the above double sum is equal to $\frac{1}{2}\left(\left(\sum_{p\in\mathscr{P}_q} m_p\right) + 2\right)$ and thus $\delta(q) = \oint_{\mathscr{P}_q} \omega$. This proves the theorem. ▮

4 Characterization of the Teichmüller map

In this section we continue to assume that γ is a degree one homotopy class and suppose additionally that it does not contain a holomorphic map. By *Teichmüller's theorem* there is a unique quasiconformal homeomorphism $f_0 \in \gamma$ having constant dilatation $K \in (1, \infty)$, cf. [1, 2]. Furthermore there are unique holomorphic quadratic differentials Φ on Σ and Ψ on S, normalized by the condition that the area $A_\Psi(S)$ of S with respect to the cone metric $|\Psi|$ equals one, such that the local expression of f_0 in the natural parameters is given by (9) where $k = \frac{K-1}{K+1} \in (0, 1)$.

The following generalizes and interpretes geometrically a result of Miyahara [13]:

Corollary (to theorem 3). *Let f be the minimizer in a degree one homotopy class for the cone metric ds^2. If f is univalent (or equivalently if f has no conformal points), then it is the Teichmüller map and ds^2 is (a constant multiple of) the Teichmüller metric $|\Psi|$.*

Proof. The Gauss–Bonnet theorem for cone metrics implies

$$\sum_{q\in S} \delta(q) = \sum_{q\in S} (\alpha(q) - 1) - \sum_{p\in\Sigma} \frac{m_p}{2} = 0.$$

By theorem 3 each of the two assumptions is equivalent to $\delta(q) = 0$ for all $q \in S$, that is all points are Teichmüller points. By (25) these are removable singularities for the harmonic function h so that by (19) $k_f(p) \equiv k \in (0, 1)$. This means that f has constant dilatation and therefore is the Teichmüller map. Assume w. l. o. g. that the area of S with respect to ds^2 is equal to 1. If $f = w(z)$ and $\Phi = \varphi(z)\,dz^2$ denotes the Hopf differential then $\frac{w_{\bar z}}{w_z} = k\,\frac{\bar\varphi}{|\varphi|}$, thus Φ is also the quadratic differential associated to f in Teichmüller's theorem. Now the differential Ψ on the target is *defined* by the condition that $\Psi = \left(\frac{m+2}{2}\right)^2 w^m\,dw^2$ whenever w is a parameter on S such that for a natural parameter z of Φ $f = w(z)$ is given by (9), see [2]. Therefore the Hopf differential of f with respect to $|\Psi|$ is also equal to Φ and this gives $ds^2 = |\Psi|$. ▮

The Teichmüller map is a harmonic map in the sense of this paper.

Theorem 4 *The Teichmüller map $f_0\colon \Sigma \to (S, |\Psi|)$ is the unique energyminimizing map in its homotopy class.*

Proof. By theorem 2 and lemma 2 it is enough to show that if $f = f_0$ and $g \in C^\infty(\Sigma, S)$ is homotopic to f, then $E(g) \geq E(f)$. Our proof follows the proof of uniqueness for f as given for example in [2]; we exploit that $f: (\Sigma, |\Phi|) \to (S, |\Psi|)$ is totally geodesic. Note that

$$E(f) = \frac{1}{4}\left(\frac{1}{k} + k\right) A_\Phi(\Sigma) = \frac{1}{2}\left(K + \frac{1}{K}\right).$$

Let $p \in \Sigma$ with $\Phi(p) \neq 0$. Denote by v^+, v^- the unit tangent vectors to the horizontal and vertical trajectories of Φ at p, and define

$$(\lambda^\pm g)(p) = |\Psi(dg \cdot v^\pm, dg \cdot v^\pm)|^{1/2}.$$

Now let $a > 0$ and

$$(\lambda_a^\pm g)(p) = \begin{cases} \int_\alpha (\lambda^\pm g)\, ds_\Phi, & \text{if } p \text{ is the midpoint of a horizontal} \\ & \text{(vertical) arc } \alpha \text{ of length } 2a \\ 0 & \text{else.} \end{cases} \tag{31}$$

Let $f: [0,1] \times \Sigma \to S$ be the geodesic homotopy from $f = f_0$ to $g = f_1$. By compactness reasons the lengths of the curves $f(\cdot, p)$, $0 \leq t \leq 1$, have a uniform upper bound $d < \infty$ independent of p. For α as in (31) the curve $f_0 \circ \alpha$ is a minimizing geodesic. But the curve $f(\cdot, p_1)^{-1} * (f_1 \circ \alpha) * f(\cdot, p_0)$ is homotopic to $f_0 \circ \alpha$, where p_0, p_1 are endpoints of α. Thus

$$(\lambda_a^\pm g)(p) \geq (\lambda_a^\pm f)(p) - 2d.$$

Applying Fubini's theorem as in [2] yields

$$\int_\Sigma (\lambda_a^\pm g)(p)\, dA_\Phi(p) = 2a \int_\Sigma (\lambda^\pm g)(p)\, dA_\Phi(p).$$

We obtain

$$\begin{aligned} \int_\Sigma (\lambda_a^\pm g)\, dA_\Phi &\geq \int_\Sigma (\lambda_a^\pm f)\, dA_\Phi - 2d\, A_\Phi(\Sigma) \\ &= 2a \int_\Sigma (\lambda^\pm f)\, dA_\Phi - 2d\, A_\Phi(\Sigma) \\ \text{(by (9))} \quad &= a\left\{\left(\frac{1}{\sqrt{k}} \pm \sqrt{k}\right) - \frac{2d}{a}\right\} A_\Phi(\Sigma). \end{aligned}$$

Therefore we can estimate for sufficiently large a

$$\begin{aligned} E(g) &= \frac{1}{2}\int_\Sigma (|\lambda^+ g|^2 + |\lambda^- g|^2)\, dA_\Phi \\ &\geq \frac{1}{2A_\Phi(\Sigma)}\left\{\left(\int_\Sigma \lambda^+ g\, dA_\Phi\right)^2 + \left(\int_\Sigma \lambda^- g\, dA_\Phi\right)^2\right\} \\ &= \frac{1}{8a^2 A_\Phi(\Sigma)}\left\{\left(\int_\Sigma \lambda_a^+ g\, dA_\Phi\right)^2 + \left(\int_\Sigma \lambda_a^- g\, dA_\Phi\right)^2\right\} \\ &\geq \frac{1}{8} A_\Phi(\Sigma)\left\{\left(\frac{1}{\sqrt{k}} + \sqrt{k} - \frac{2d}{a}\right)^2 + \left(\frac{1}{\sqrt{k}} - \sqrt{k} - \frac{2d}{a}\right)^2\right\}. \end{aligned}$$

Letting $a \to \infty$ we get

$$E(g) \geq \frac{1}{4}\left(\frac{1}{k}+k\right) A_{\Phi}(\Sigma) = E(f).$$

This proves the theorem. ∎

A result similar to theorem 4 but for quasiconformal solutions to a Dirichlet problem on the unit disk is proved in [15]. It is possible that the result was known for example to Bers, because the proof uses the same argument as in [2], but we could not find it in the literature. Miyahara [13] stated that the Teichmüller map is harmonic, but only in the sense that its Hopf differential is holomorphic.

We finally remark that theorem 4 implies a sup-inf characterization of the dilatation K of the Teichmüller map suggested in [6]. Namely we have

Corollary (to theorem 4). $\frac{1}{2}\left(K + 1/K\right) = \sup \left\{\inf_{f\in\gamma} E_{ds^2}(f)\text{: } ds^2 \textit{ is a conformal metric on } S \textit{ with } A_{ds^2}(S) = 1\right\}$.

Proof. Since $E_{\Psi}(f_0) = \frac{1}{2}\left(K + 1/K\right)$, theorem 4 shows that the right hand side is not smaller than the left hand side. On the other hand we can use f_0 as a competitor for a given metric ds^2. We compute in natural parameters for $f_0 = w(z)$ and $ds^2 = \rho^2(w)\,|dw|^2$

$$\begin{aligned} E_{ds^2}(f_0) &= \int_{\Sigma} \rho^2(w)\,(|w_z|^2 + |w_{\bar z}|^2)\,dx\,dy \\ &= \frac{1}{4}\left(\frac{1}{k}+k\right)\int_{\Sigma}\rho^2(w)\,dx\,dy \\ &= \frac{1/k+k}{1/k-k}\int_{S}\rho^2(w)\,du\,dv = \frac{1}{2}\left(K+\frac{1}{K}\right). \quad \blacksquare \end{aligned}$$

It would be interesting to solve the existence problem for the Teichmüller metric and the Teichmüller map via the above variational characterization, as was proposed in [6].

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