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A general handle addition theorem

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0 Introduction

This paper extends results of [Przytycki, [9], [10]] and [Jaco, [4]]. Namely the following fact was proved by Przytycki in [9] in 1983.

Theorem 0.1 *Let H be a handlebody with genus k (> 0) and J a 2-sided simple closed curve in ∂H. If $\partial H - J$ is incompressible, then the manifold H_J obtained from H by adding a 2-handle along J has incompressible boundary, or it is equal to D^3.*

The proof in [9] is algebraic. In [4], Theorem 0.1 is generalized by Jaco using a geometric approach in the following version:

Theorem 0.2 *Let M be a 3-manifold with compressible boundary and J a 2-sided simple closed curve in ∂M. If $\partial M - J$ is incompressible, then ∂M_J is incompressible, or it is equal to D^3.*

Several alternative proofs of Theorem 0.2 have been published (see [1], [5], [11]), and it has been applied very successfully in dealing with incompressible surfaces, surgeries, and other related topics. (see for example [4], [5], [2].)

There exist examples in [9] which show that a direct generalization of Theorem 0.1 and 0.2 is not possible. It seems to be difficult to find the appropriate conditions for the conclusions of Theorem 0.1 and 0.2 to hold if more than one 2-handle is added. However, a generalization of Theorem 0.2 is given by Przytycki in [10].

Theorem 0.3 *Let $\Gamma = \{J_1, \ldots, J_n\}$ be a family of 2-sided, pairwise disjoint, simple closed curves in the boundary of a 3-manifold M. Let the following*

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conditions be satisfied:

(1) $\partial M - \Gamma$ *is incompressible in* M,

(2) *for each* j, $\partial M - (\Gamma - J_j)$ *is compressible in* M,

(3) *a compressing disk from* (2), *say* D, *can be chosen in such a way that* ∂D *is not coplanar with* $\Gamma - J_j$.

Then M_Γ *has incompressible boundary or it is equal to* D^3.

When M is a handlebody H, Przytycki in [10] proposed a set of conditions which he conjectured to be sufficient for an n-relator 3-manifold with incompressible boundary, that is,

Przytycki's Conjecture. Let $\Gamma = \{J_1, \ldots, J_n\}$ be a family of 2-sided, pairwise disjoint, simple closed curves in the boundary of a handlebody H (with genus $k > 0$). Assume the following conditions are satisfied (refer to Def. 1.4 for the definition of "binding"):

(0) $\partial H - \Gamma$ is incompressible in H,

(1) for each j, $\partial H - (\Gamma - J_j)$ is compressible in H,

(2) for each pair j, s ($j \neq s$), $\Gamma - \{J_j, J_s\}$ does not bind any free factor F_{k-1} of $F_k = F_{k-1} \times F_1$,

......

(p) no $(n-p)$-element subfamily of Γ binds a free factor F_{k-p+1} of $F_k = F_{k-p+1} \times F_{p-1}$,

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$(n-1)$ no curve J_j of Γ binds a free factor F_{k-n+2} of $F_k = F_{k-n+2} \times F_{n-2}$.

Then the n-relator 3-manifold H_Γ has incompressible boundary, or it is equal to D^3.

By using Theorem 0.3, Przytycki proved his conjecture true for $n = 2, 3$. (For $n = 1$, it is Theorem 0.1). When $n > 3$, he had examples to show that all the assumptions in his conjecture are necessary. It is remarkable that it is always possible, for a given family Γ of curves in ∂H, to verify whether the conditions in Przytycki's conjecture are satisfied.

In this paper we give a generalization of Theorem 0.3 (therefore it is also a generalization of Theorem 0.2, the Handle Addition Theorem) as follows:

Theorem 0.4 *Let* $\Gamma = \{J_1, \ldots, J_n\}$ *be a family of* 2*-sided, pairwise disjoint, simple closed curves in the boundary of a* 3*-manifold* M. *Assume the following conditions are satisfied:*

(1) $\partial M - \Gamma$ *is incompressible in* M,

(2) *for each* j, $\partial M - (\Gamma - J_j)$ *is compressible in* M,

(3) *for each* j, *a compressing disk from* (2), *say* D, *can be chosen in such a way that either* ∂D *is not coplanar with* $\Gamma - J_j$ *or* D *does not separate* M.

Then M_Γ *has incompressible boundary, or it is equal to* D^3.

By using Theorem 0.4, we give out an affirmative answer to Przytycki's conjecture by verifying the conditions in Theorem 0.4 are satisfied in the situation.

Part 1 of this paper consists of necessary preliminaries and part 2 consists of the proofs of Theorem 0.4 and Przytycki's Conjecture.

1 Preliminaries

We work in the PL-category. We use M to denote a connected 3-manifold with connected boundary. Submanifolds in M are assumed to be in general position. For convenience, we use s.c.c. as an abbreviation of simple closed carve(s), and refer to disk with holdes as "planar surface". We also abuse notation slightly by using the symbol Γ, which represents a family of oriented s.c.c. in M, to represent the corresponding elements of $\pi_1(M)$ when this causes no confusion.

Definition 1.1 *Let M be a 3-manifold and S a surface which is either properly embedded in M or contained in ∂M. We say that S is compressible (in M) if one of the following conditions is satisfied:*

(1) *S is a 2-sphere which bounds a 3-cell in M, or*

(2) *S is a 2-cell and either $S \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset S \cup \partial M$, or*

(3) *there is a 2-cell $D \subset M$ with $D \cap S = \partial D$ and ∂D is not contractible in S. In the case (3), D is also called a compressing disk for S (in M).*

We say that S is incompressible if each component of S is not compressible.

Definition 1.2 *Let M be a 3-manifold and J a 2-sided s.c.c. in ∂M. Let A_J be a regular neighbourhood of J in ∂M, (D^3, A) a 3-cell with an annulus $A \subset \partial D^3$, and h a homeomorphism from A_J to A. Then the 3-manifold $(M, A_J) \cup_h (D^3, A)$ is denoted by M_J. We also say M_J is obtained from M by adding a 2-handle along J. If $\Gamma = \{J_1, \ldots, J_n\}$ is a family of pairwise disjoint, 2-sided s.c.c. in ∂M, then we define $M_\Gamma := (\cdots((M_{J_1})_{J_2}\cdots)_{J_n}$. When M is a handlebody H, H_Γ is called an n-relator 3-manifold.*

Clearly, the definition of M_Γ does not depend on the order of J_i.

Definition 1.3 *Let $\Gamma = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint, 2-sided s.c.c. in a surface S. We say that a s.c.c. J contained in $S - \Gamma$ is coplanar with Γ if J cuts out a planar surface from $S - \Gamma$.*

Definition 1.4 *Let $W \subset F_k$ be a set of cyclic words in the free group F_k with a basis X. The incidence graph $J(W)$ is the graph whose vertices are in 1–1 correspondence with the non-trivial words in W, with an edge joining vertices w_1 and w_2 if there exists $x \in X$ such that x or x^{-1} lies in w_1 and x or x^{-1} lies in w_2. W is connected with respect to the basis X if $J(W)$ is connected, and is connected if it is connected with respect to each basis of F_k. If the set*

W of cyclic elements is not contained in any proper free factor of F_k and if W is connected, we say that W binds F_k.

Remark 1.5 It is shown in [8] Corollary 1 that if Γ is a family of s.c.c. in the boundary of a handlebody H, then $\partial H - \Gamma$ is compressible if and only if the family of elements of $\pi_1(H)$ represented by Γ does not bind the free group $\pi_1(H)$.

Note that Theorem 0.2 is valid for a noncompact 3-manifold, so as an immediate consequence we have

Lemma 1.6 (Lemma 2.3 in [10]) *Let $\Gamma = \{J_1, \ldots, J_n\}$ be a family of 2-sided, pairwise disjoint s.c.c. in the boundary of a 3-manifold M. Let $\partial M - \Gamma$ be incompressible in M and $\partial M - (\Gamma - J_n)$ compressible in M. Then $\partial M_{J_n} - (\Gamma - J_n)$ is incompressible in M_{J_n}.*

2 The proofs of the main results

First we give a proof of Theorem 0.4.

Proof of Theorem 0.4 By Theorem 0.2, we know the theorem holds for $n = 1$. We only need to consider the case of $n \geqq 2$.

Let $\mathscr{X}_i$ denote the set which consists of all the i-element subsets of $\Gamma = \{J_j, 1 \leqq j \leqq n\}$, $1 \leqq i \leqq n$, and for $K \in \mathscr{X}_i$, denote $\overline{K} = \Gamma - K$, thus $\overline{K} \in \mathscr{X}_{n-i}$.

We know, by the assumption conditions (1), (2) in Theorem 0.4 and Lemma 1.6, that $\partial M_{J_i} - (\Gamma - J_i)$ is incompressible in M_{J_i}, for $i = 1, 2, \ldots, n$. That is, for any $K \in \mathscr{X}_1$, $\partial M_K - \overline{K}$ is incompressible in M_K.

Now we suppose that for some $m < n$ $(m \geqq 1)$ and each $i \leqq m$ $(i \geqq 1)$, $\partial M_K - \overline{K}$ is incompressible in M_K for each $K \in \mathscr{X}_i$. We will prove that for any $K \in \mathscr{X}_{m+1}$, $\partial M_K - \overline{K}$ is incompressible in M_K. Thus the proof will be finished by a finite induction on m.

Let $K \in \mathscr{X}_{m+1}$, say, $K = \{J_1, \ldots, J_{m+1}\}$. Write $L = K - J_{m+1}$, then $L \in \mathscr{X}_m$. From the inductive assumption we know that $\partial M_L - \overline{L}$ is incompressible in M_L and we need $\partial M_L - \overline{K}$ to be compressible (then we can use Lemma 1.6 if $m < n - 1$ or Theorem 0.2 if $m = n - 1$). From fhe conditions (2) and (3) we know that there exists a compressing disk D of $\partial M - (L \cup \overline{K})$ in M such that either ∂D is not coplanar with $L \cup \overline{K}$ (therefore L) or D does not separate M (i.e. $M - D$ is connected). If ∂D is not coplanar with L, then ∂D is not trivial in $\partial M_L - \overline{K}$, so $\partial M_l - \overline{K}$ is compressible in M_L, thus Lemma 1.6 or Theorem 0.2 implies that $\partial M_K - \overline{K}$ is incompressible in M_K or $\overline{K} = \phi$ and $M_K = M_\Gamma = D^3$. If D does not separate M (therefore ∂D does not separate ∂M), we assert that ∂D is not coplanar with L (but possibly, ∂D may be coplanar with $\overline{K}$, therefore $L \cup \overline{K}$!). In fact, if ∂D is coplanar with L, ∂D cuts out a planar surface S from ∂M cut open along L. Let J_i^+ and J_i^- denote the two cutting sections of J_i, then ∂S consists of ∂D and a subset A (with at least two elements, by condition (1)) of the cutting section set of L. Since ∂D does not separate ∂M, there exists an element, say J_m^+, in A such that J_m^- is not contained in A. Denote $L_1 = L - \{J_m\}$, then by the property of S we know that J_m is trivial in ∂M_{L_1}, so $\partial M_{L_1} - \overline{L}_1$ is compressible. But this contradicts the inductive assumption (since $|L_1| = m - 1$). This contradiction proves our assertion, and again by Lemma 1.6 or Theorem 0.2 we have $\partial M_K - \overline{K}$ is incompressible in M_K or $K = \Gamma$ and $M_K = D^3$.

This finishes the proof of Theorem 0.4.

As an application of Theorem 0.4, we give an affirmative answer to Przytycki's Conjecture.

Proofs of Przytycki's Conjecture. By Theorem 0.4, we only need to verify the condition (3) in Theorem 0.4 is satisfied in this situation.

Without losing generality, we consider the compressing disks of $\partial H - K$ in H, $K = \{J_1, \ldots, J_{n-1}\}$. If there exists a compressing disk D of $\partial H - K$ in H such that ∂D is not coplanar with K in ∂H, the proof has already been finished. In the following we assume that each compressing disk of $\partial H - K$ in H is coplanar with K, then we have

Assertion. There exists a compressing disk of $\partial H - K$ in H which does not separate H.

Let D_1 be a compressing disk of $\partial H - K$ in H. If D_1 does not separate H, the Assertion already holds. If D_1 separates H into two handlebodies H_1 and H_2, then $g(H_i)$ (=genus of H_i) > 0, $i = 1, 2$, and $H = H_1 \Delta H_2$, where Δ denotes the boundary connected sum realized by D_1. Since ∂D_1 is coplanar with K, so ∂D_1 cuts out a planar surface S from ∂H cut open along K. Let J_i^+ and J_i^- denote the two cutting sections of J_i for $J_i \in K$. Since D_1 separates H, we have that $\partial S - \partial D_1$ are all paired (i.e. if one of J_i^+ and J_i^- is in ∂S then so is the other one). Without loss of generality, say, $L_1 = \{J_{p+1}, J_{p+2}, \ldots, J_{n-1}\}$ $(0 \leqq p \leqq n - 1)$ is the subset of K which consists of all the curves whose cutting sections are in ∂S, and $L_1 \subset \partial H_2$, then $K_1 = K - L_1 \subset \partial H_1$ and $g(H_2) = n - p - 1$, $g(H_1) = k - n + p + 1$. By the assumption condition $(n - p)$, K_1 does not bind $F_{k-n+p+1} = \pi_1(H_1)$. By Corollary 1 in [8], $\partial H_1 - K_1$ is compressible, and each compressing disk (missing D_1) of $\partial H_1 - K_1$ in H_1 is a compressing disk of $\partial H - K$ in H and therefore coplanar with K in ∂H (so with K_1 in ∂H_1). If D_2 is a compressing disk of $\partial H_1 - K_1$ in H_1 (missing D_1), and D_2 does not separate H_1, then D_2 does not separate H, and the proof of the Assertion is done. Otherwise, after repeating a finite number of the same steps (if necessary) we can reduce the situation to that (say):

$$H = H' \Delta H'', \qquad g(H') = k - n + 2, \qquad g(H'') = n - 2,$$
$$J_1 \subset \partial H', \qquad K - J_1 \subset \partial H''.$$

By the assumption $(n-1)$, J_1 does not bind $F_{k-n+2} = \pi_1(H')$, so $\partial H' - J_1$ is compressible in H'. If D' is a compressing disk (missing $H' \cap H''$) of $\partial H' - J_1$ and D' does not separate H', the proof is done. If D' separates H' into two handle body with positive genus, say H_1' and H_2' with $J_1 \subset H_1'$, then we can choose a nonseparating compressing disk D (missing D') of $\partial H_2'$ in H_2' such that D is a nonseparating compressing disk of $\partial H - K$ in H.

This completes the proof.

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