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(1,1)-geodesic maps into Grassmann manifolds

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1 Introduction

Let M be a Kähler manifold of complex dimension m. We study smooth maps $\varphi : M \to N$ into some Riemannian manifold N. The Hessian $Dd\varphi$ is a symmetric 2-form on M with values in the pull back bundle φ^*TN. After complexification, we may decompose $Dd\varphi$ into its $(2,0)$, $(1,1)$ and $(0,2)$ parts. The map φ is called *pluriharmonic* if $Dd\varphi^{(1,1)} = 0$, or equivalently, if $\varphi|C$ is harmonic for any complex one-dimensional submanifold $C \subset M$. The second definition shows that pluriharmonicity does not depend on the metric on M.

If φ is an isometric immersion, then $Dd\varphi = \alpha$ is its second fundamental form. Since an isometric immersion is totally geodesic if $\alpha = 0$, a pluriharmonic isometric immersion is called *(1,1)-geodesic*. In particular, a (1,1)-geodesic immersion is minimal; in fact, $\varphi|C$ is a minimal surface for any complex curve $C \subset M$. More generally, an immersion $\varphi : M \to N$ will be called *(1,1)-geodesic* if it is pluriharmonic and the induced metric on M is a compatible Kähler metric. If N is also a Kähler manifold, then holomorphic and antiholomorphic ("$\pm$-holomorphic") immersions are (1,1)-geodesic. Clearly, for immersions of Riemann surfaces ($m = 1$), (1,1)-geodesic is the same as minimal.

Are there (1,1)-geodesic immersions of higher dimension which are not $\pm$-holomorphic? Sampson [S] has shown that harmonic already implies pluriharmonic provided that M is compact and N has nonpositive curvature operator. Ferreira, Rigoli and Tribuzy [FRT] obtained a similar result, but without compactness assumption, for minimal immersions: If N has nonpositive curvature operator, then minimal immersions of Kähler manifolds are already (1,1)-geodesic (cf. also [DR] if $N = \mathbb{R}^n$, and [U] if N is hermitean symmetric). On the other hand, if N is symmetric, there are dimension restrictions: Udagawa (cf. [U], Theorem 4) has shown that a pluriharmonic immersion into N which is not $\pm$-holomorphic has dimension $m \le p(N)$ where $p(N)$ is an invariant of N introduced by Siu [Si] (cf.

also [DT] for complex space forms and [FRT] for complex Grassmannians). For a complex Grassmann manifold $N = G_p(\mathbb{C}^{p+q})$, this number is $(p-1)(q-1)+1$.

In the present paper, we construct non-holomorphic (1,1)-geodesic immersions in complex and real Grassmannians. In particular, we recieve compact examples in the limit dimension $m = (p-1)(q-1)+1$ in the case $q = 2$. These are obtained by extending the constructions of Eells and Wood [EW], Erdem and Wood [ErW], Burstall and Salamon [BS] and Wood [W] to higher dimension.

2 Pluriharmonic and (1,1)-geodesic maps

Let M be complex manifold and J its almost complex structure. J has eigenspaces $T'M = \pi'(TM)$, $T''M = \pi''(TM) \subset TM \otimes \mathbb{C}$ where

$$\pi'(X) = \frac{1}{2}(X - iJX),\ \pi''(X) = \frac{1}{2}(X + iJX).$$

Any r-form on M with values in a vector bundle E over M can be complex linearly extended to $TM \otimes \mathbb{C}$ with values in $E \otimes \mathbb{C}$ and then decomposed into its (p,q)-components $(p+q=r)$ which are tensor products of p 1-forms vanishing on $T''M$ and q 1-forms vanishing on $T'M$. The (1,0) and (0,1) components of a 1-form θ, i.e. its restrictions to $T'M$ and $T''M$, will be denoted by θ' and θ''. E.g. if $\varphi : M \to N$ is a smooth map, we have $d\varphi = d'\varphi + d''\varphi$, and if s is a section in a vector bundle E with connection D over M, we get $Ds = D's + D''s$. For a symmetric 2-form α we have

$$\alpha^{(1,1)}(X,Y) = \frac{1}{2}(\alpha(X,Y) + \alpha(JX,JY)).$$

So $\alpha^{(1,1)} = 0$ iff $\alpha(JX,JY) = -\alpha(X,Y)$. On the other hand, α is called a *(1,1)-form* if its (2,0) and (0,2) components vanish, i.e. if $\alpha(JX,JY) = \alpha(X,Y)$.

Let $\langle , \rangle$ be a J-invariant Riemannian metric on M. Then $\langle , \rangle$ is a (1,1)-form which means that $T'M$ and $T''M$ become isotropic subspaces. Sometimes, we will also use the hermitean inner product

$$(X,Y) = \langle X, \bar{Y} \rangle$$

on $TM \otimes \mathbb{C}$. The J-invariance of the metric says that $T'M \perp T''M$ with respect to (,).

The metric is Kähler if J is parallel or if the 2-form $\omega(X,Y) = \langle X, JY \rangle$ is closed. In local coordinates $z_a = x_a + iy_a$ $(a = 1, \dots, m)$, we have

$$\omega = \frac{1}{i} \sum \langle Z_a, \bar{Z}_b \rangle dz_a \wedge d\bar{z}_b$$

where $Z_a = \partial/\partial z_a = \frac{1}{2}(X_a - iY_a)$ with $X_a = \partial/\partial x_a$, $Y_a = \partial/\partial y_a$. The form ω is closed if

$$D_{Z_a}\bar{Z}_b = 0 \tag{1}$$

where D denotes the Levi-Civita connection, extended complex linearly to $TM \otimes \mathbb{C}$. So (1) implies the Kähler condition. On the other hand, the subbundles $T'M$ and $T''M$ of $TM \otimes \mathbb{C}$ are parallel if J is parallel, and consequently,

$$D_{Z_a}\bar{Z}_b = D_{\bar{Z}_b}Z_a \in T'M \cap T''M = 0.$$

Therefore, (1) is equivalent to the Kähler condition.

Now let $(N, \langle,\rangle)$ be a Riemannian manifold and $\varphi : M \to N$ an immersion. Then the induced metric $\varphi^*\langle,\rangle = \langle d\varphi, d\varphi\rangle$ is Kähler if and only if $\langle d\varphi, d\varphi\rangle$ is a (1,1)-form and $D_{Z_a}(d\varphi(\bar{Z}_b))$ is perpendicular to the image of $d\varphi$. Suppose that M comes with a possibly different Kähler metric $\langle,\rangle$. Then by (1),

$$D_{Z_a}(d\varphi(\bar{Z}_b)) = Dd\varphi(Z_a, \bar{Z}_b) \tag{2}$$

Hence, an immersion $\varphi : M \to N$ induces a compatible Kähler metric iff $\langle d\varphi, d\varphi\rangle$ is a (1,1)-form and $Dd\varphi^{(1,1)}$ is normal.

Recall that the mapping $\varphi : M \to N$ is called *pluriharmonic* if $Dd\varphi^{(1,1)} = 0$, i.e. if $D''d'\varphi = 0$. This condition does not depend on the choice of the Kähler metric on M since the left hand side of (2) is independent of this metric. The mapping φ is called *pluriconformal* if J is isometric with respect to $\langle d\varphi, d\varphi\rangle$, in other words if $\langle d\varphi, d\varphi\rangle$ is a (1,1)-form. An isometric immersion $\varphi : M \to N$ is called *(1,1)-geodesic* if $\alpha^{(1,1)} = 0$ where $\alpha : TM \otimes TM \to \nu M$ denotes the second fundamental form of φ. Now we have proved:

Theorem 1 *Let M be a complex manifold admitting a Kähler metric, $(N, \langle,\rangle)$ a Riemannian manifold and $\varphi : M \to N$ an immersion. The induced metric $\langle d\varphi, d\varphi\rangle$ on M is a compatible Kähler metric and φ a (1,1)-geodesic immersion with respect to this metric if and only if φ is pluriharmonic and pluriconformal.*

More generally, an arbitrary smooth map $\varphi : M \to N$ will be called *(1,1)-geodesic* if φ is pluriconformal and pluriharmonic.

Remarks

1. Pluriconformality implies that the kernel of $d\varphi$ is J-invariant. Thus on the open subset M_0 of M where ker $d\varphi$ has minimal dimension, the levels $\varphi^{-1}(y)$ form a foliation of complex submanifolds of M_0, and a transversal complex submanifold M' inherits a Kähler metric such that $\varphi|M'$ is a (1,1) geodesic immersion on M'. Hence on an open dense subset, a (1,1)-geodesic map locally is the composition of a (1,1)-geodesic immersion and a holomorphic submersion.
2. If M is compact with $c_1(M) > 0$ then any pluriharmonic map $\varphi : M \to N$ is also pluriconformal (cf. [OU], p.374).

Now let N be also a Kähler manifold. The mapping φ is called *holomorphic* if $d\varphi$ preserves T' and T'', and *antiholomorphic* if $d\varphi$ interchanges T' and T''. Clearly, a holomorphic or antiholomorphic map φ is (1,1)-geodesic: In fact, since $T'N$ is isotropic, $\langle d'\varphi, d'\varphi\rangle = 0$, so φ is pluriconformal, and since $T'N$ and $T''N$ are parallel subbundles, the values of $D''d'\varphi = D'd''\varphi$ are lying in $T'N \cap T''N = 0$.

3 Maps into complex Grassmannians

Let $G = G_p = G_p(\mathbb{C}^n)$ denote the Grassmann manifold of complex p-dimensional subspaces of $\mathbb{C}^n$. Identifying a subspace $\underline{\xi} \subset \mathbb{C}^n$ with the orthogonal projection ξ onto $\underline{\xi}$, we embed G_p into the real vector space $H(n)$ of hermitean $n \times n$-matrices with its usual trace inner product (*standard embedding*). Then the tangent space $T_\xi G_p$ becomes the space of hermitean matrices mapping $\underline{\xi}$ into $\underline{\xi}^\perp$ and vice versa. There is an isomorphism between $T_\xi G_p$ and $\mathrm{Hom}_{\mathbb{C}}(\underline{\xi}, \underline{\xi}^\perp)$ mapping $A \in \mathrm{Hom}(\underline{\xi}, \underline{\xi}^\perp)$ onto the matrix

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}^n = \underline{\xi} + \underline{\xi}^\perp$. The multiplication by $i = \sqrt{-1}$ in $\mathrm{Hom}(\underline{\xi}, \underline{\xi}^\perp)$ gives the complex structure in $T_\xi G$. The complexification $T_\xi G \otimes \mathbb{C} \subset H(n) \otimes \mathbb{C} = \mathbb{C}^{n\times n}$ is the space of *all* complex $n \times n$-matrices mapping $\underline{\xi}$ into $\underline{\xi}^\perp$ and vice versa, with its usual hermitean inner product $(\ ,\ )$. Note that e.g.

$$\pi'' \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} + i \begin{pmatrix} 0 & (iA)^* \\ iA & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}.$$

Now let M be a Kähler manifold and $\varphi : M \to G$ a smooth map. This can also be viewed as a p-dimensional subbundle $\underline{\varphi}$ of the trivial bundle $M \times \mathbb{C}^n$ with fibre $\underline{\varphi_x} = \mathrm{im}\,(\varphi(x))$. Let $v \in T_x M$. Then $\partial_v \varphi = d\varphi.v \in T_{\varphi(x)} G$ maps $\underline{\varphi}_x$ onto a subspace of $\underline{\varphi}_x^\perp$. Let $\underline{\delta\varphi}_x$ be the (complex linear) span of all these subspaces, i.e.

$$\underline{\delta\varphi}_x = Span \bigcup_{v \in T_x M} \partial_v \varphi(\underline{\varphi}_x) = Span \bigcup_{v \in T_x M} \mathrm{im}\,(\partial_v \varphi \cdot \varphi(x)).$$

If $f : M \to \mathbb{C}^n$ is a section of $\underline{\varphi}$, i.e. $f = \varphi \cdot f$, we have $df = d\varphi \cdot f + \varphi \cdot df$, hence

$$d\varphi \cdot f = (1 - \varphi) df = \varphi^\perp \cdot df.$$

So for local bases $f_1, \ldots, f_p$ of $\underline{\varphi}$ and $V_1, \ldots, V_{2m}$ of TM we have

$$\underline{\delta\varphi} = Span\{\varphi^\perp \cdot \partial_{V_j} f_i\ ;\ i = 1, \ldots, p, j = 1, \ldots, 2m\}.$$

If $\underline{\delta\varphi}$ has constant rank q, it defines a smooth map $\delta\varphi$ into another Grassmannian G_q. Similarly, by restriction to $T'M$ and $T''M$, we define $\underline{\delta'\varphi}$ and $\underline{\delta''\varphi}$. Clearly, $\underline{\delta\varphi} = \underline{\delta'\varphi} + \underline{\delta''\varphi}$. The map $\varphi : M \to G_p$ is holomorphic iff $d''\varphi$ takes values in $T''G_p$, i.e. iff $d''\varphi \cdot \varphi = 0$.

Lemma 1 *Let $\varphi : M \to G$ be smooth. The following statements are equivalent:*
(a) φ is holomorphic, i.e. $d''\varphi \cdot \varphi = 0$,
(b) $\underline{\delta''\varphi} = 0$,
(c) $\mathrm{im}\, d''f \subset \underline{\varphi}$ for any $f \in \Gamma\underline{\varphi}$,
(d) $\underline{\varphi}$ is a holomorphic subbundle of $M \times \mathbb{C}^n$.

Proof. For any $f \in \Gamma\varphi$ we have $d''\varphi \cdot f = \varphi^\perp \cdot d''f$ which shows $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. If (c) holds, choose a local basis $f_1, ..., f_p : U \to \mathbb{C}^n$ of $\underline{\varphi}|U$ on some open subset $U \subset M$. Then $d''f_j \in \underline{\varphi}$, i.e.

$$d''f_j = \sum_i f_i a_{ij}$$

or in matrix notation,

$$(*) \qquad d''f = f \cdot a$$

where $f = (f_1, ..., f_p)$, $a = ((a_{ij}))$. After a unitary transformation of $\mathbb{C}^n$, we may assume that $\underline{\varphi}_x$ is close to the subspace $\mathbb{C}^p \subset \mathbb{C}^n$ for all $x \in U$, so that $\underline{\varphi}_x \cap (\mathbb{C}^p)^\perp = 0$. Thus the projection $\pi : \mathbb{C}^n \to \mathbb{C}^p$ is an isomorphism on $\underline{\varphi}_x$, and $g := \pi f$ is an invertible $p \times p$-matrix. Let $\pi^\perp : \mathbb{C}^n \to (\mathbb{C}^p)^\perp$ be the complementary projection and put $h = \pi^\perp f$. Then $F = hg^{-1} : U \to \mathbb{C}^{p\times(n-p)}$ is holomorphic: by $(*)$ we have $d''g = ga$, $d''h = ha$, hence

$$\begin{aligned} d''F &= d''h \cdot g^{-1} - hg^{-1} \cdot d''g \cdot g^{-1} \\ &= hag^{-1} - hg^{-1}gag^{-1} \\ &= 0. \end{aligned}$$

Thus, $\tilde{f}_i := e_i + F \cdot e_i$ $(i = 1, ..., p)$ is a holomorphic basis of $\underline{\varphi}$ which shows (d). The converse $(d) \Rightarrow (c)$ is clear. □

Lemma 2 *$\varphi : M \to G$ is pluriharmonic iff $\partial_{Z''}\partial_{Z'}\varphi$ leaves $\underline{\varphi}$ and $\underline{\varphi}^\perp$ invariant for any two commuting (1,0) and (0,1)-vector fields Z', Z''.*

Proof. Let D denote the Levi-Civita connection on $G \subset H(n)$. The map φ is pluriharmonic iff $D_{Z''}\partial_{Z'}\varphi = 0$, i.e. iff $\partial_{Z''}\partial_{Z'}\varphi$ is in the normal bundle of G. But the normal vectors of G at φ are precisely the matrices which leave φ and $\varphi^\perp$ invariant. □

4 Strongly isotropic (1,1)-geodesic immersions

The following construction goes back to Eells and Wood [EW] and Erdem and Wood [ErW] in the case $m = 1$. It was extended to higher dimension by Ohnita and Udagawa (cf. [OU]). We will use it to construct (1,1)-geodesic immersions. Let $p + q + r = n$. Consider the flag manifold

$$H = \{(\xi, \eta) \in G_p \times G_q |\ \underline{\xi} \perp \underline{\eta}\}$$

with its induced metric, and the map $\pi : H \to G_r$,

$$\pi(\xi, \eta) = (\xi + \eta)^\perp.$$

Lemma 3 *(cf. [EW])* $\pi : H \longrightarrow G_r$ *is a Riemannian submersion, and a smooth map* $(\xi, \eta) : M \longrightarrow H$ *is horizontal with respect to* π *iff* $\delta\underline{\xi}\perp\underline{\eta}$.

Proof. The orthogonality relation $\underline{\xi}\perp\underline{\eta}$, in other words $\xi\eta = 0$, shows that the tangent space $T_{(\xi,\eta)}H$ contains those pairs $(\xi', \eta') \in T_\xi G_p \times T_\eta G_q$ such that

$$\xi'\eta = -\xi\eta'.$$

So we have the following matrix representation with respect to the decomposition $\mathbb{C}^n = \underline{\xi} + \underline{\eta} + \underline{\varphi}$ where $\varphi = \pi(\xi, \eta)$:

$$\xi' = \begin{pmatrix} 0 & A^* & B^* \\ A & 0 & 0 \\ B & 0 & 0 \end{pmatrix}, \quad \eta' = \begin{pmatrix} 0 & -A^* & 0 \\ -A & 0 & C^* \\ 0 & C & 0 \end{pmatrix}$$

with $A \in \mathrm{Hom}(\underline{\xi}, \underline{\eta})$, $B \in \mathrm{Hom}(\underline{\xi}, \underline{\varphi})$, $C \in \mathrm{Hom}(\underline{\eta}, \underline{\varphi})$. For $\varphi' := d\pi.(\xi', \eta')$ we get the orthogonality relations

$$\varphi'\xi = -\varphi\xi', \quad \varphi'\eta = -\varphi\eta'$$

and therefore

$$\varphi' = \begin{pmatrix} 0 & 0 & -B^* \\ 0 & 0 & -C^* \\ -B & -C & 0 \end{pmatrix}.$$

Thus we have

$$\|(\xi', \eta')\|^2 = \|\xi'\|^2 + \|\eta'\|^2 = 2 \cdot (2\|A\|^2 + \|B\|^2 + \|C\|^2) \geq \|\varphi'\|^2$$

with equality iff $A = 0$. Therefore, π is a Riemannian submersion, and $(\xi', \eta') \in T_{(\xi,\eta)}H$ is horizontal iff $\eta\xi' = 0$, i.e. iff im $\xi'\perp\underline{\eta}$. Thus a smooth map $(\xi, \eta) : M \rightarrow H$ is horizontal iff im $d\xi\perp\underline{\eta}$ which means that $\delta\underline{\xi}\perp\underline{\eta}$. □

Theorem 2 *Let* M *be a Kähler manifold,* $\xi : M \longrightarrow G_p$ *a holomorphic immersion and* $\eta : M \longrightarrow G_q$ *an antiholomorphic map such that* $\underline{\eta}\perp\underline{\xi}, \underline{\delta'\xi}$. *Then* $\varphi := (\xi+\eta)^\perp : M \longrightarrow G_r$ *with* $r := n - (p + q)$ *is a (1,1)-geodesic immersion.*

Proof. Clearly, $(\xi, \eta) : M \longrightarrow G_p \times G_q$ is a (1,1)-geodesic immersion since the two components are $\pm$-holomorphic and the first one is an immersion. On the other hand, the map (ξ, η) takes values in $H \subset G_p \times G_q$, so it is (1,1)-geodesic in H, and it is horizontal by Lemma 3 since $\underline{\delta''\xi} = 0$ (cf. Lemma 1) and $\underline{\delta'\xi}\perp\underline{\eta}$. Since the second fundamental form of a horizontal immersion remains unchanged under Riemannian submersion, $\varphi = \pi \circ (\xi, \eta)$ is a (1,1)-geodesic immersion. □

Remarks

1. Since $\underline{fy}^\perp = \underline{\xi} + \underline{\eta}$, the induced bundle $\varphi^*(TG_r) = T_\varphi G_r$ splits as $T_\xi + T_\eta$ where

$$T_\eta = \{\varphi' \in T_\varphi G_r;\ \varphi'(\underline{\varphi}) \subset \underline{\eta}\}$$

and T_ξ similar. *Claim:* $\underline{\delta\xi}\perp\eta$ iff T_ξ and T_η are parallel subbundles of φ^*TG_r. In fact, let φ' be a section of T_η and f a section of $\underline{\varphi}$. Since $D\varphi'$ is the projection of $d\varphi'$ onto $T_\varphi G_r$, we have

$$D\varphi' \cdot f = \varphi^\perp \cdot d\varphi' \cdot f = \varphi^\perp(d(\varphi' \cdot f) - \varphi'(\varphi \cdot df) - \varphi'(\varphi^\perp \cdot df)).$$

The last term at the right hand side vanishes; recall that $\varphi^\perp \cdot \varphi' \cdot \varphi^\perp = 0$ for all $\varphi' \in T_\varphi G_r$. The middle term takes values in $\underline{\eta}$, by definition of T_η. The first term takes values in $\varphi^\perp(\underline{\xi}) = \underline{\eta}$ iff $\underline{\delta\eta \perp \xi}$. Thus $D\varphi'$ maps $\underline{\varphi}$ to $\underline{\xi}$, and so the covariant derivative D leaves T_η invariant.

2. It is easy to see that $d'\varphi$ takes values in T_η. In fact, let f and x be local sections of $\underline{\varphi}$ and $\underline{\xi}$. Since ξ is holomorphic, $d''x$ takes values in $\underline{\xi}$ and therefore $(d'f, x) = (f, d''x) = 0$. So the values of $d'\varphi \cdot f = \varphi^\perp \cdot d'f$ are perpendicular to $\underline{\xi}$ and $\underline{\varphi}$, hence they lie in $\underline{\eta}$. Similar, $d''\varphi \in T_\xi$. Thus the pluriharmonicity also follows from Remark 1 since $D''d'\varphi = D'd''\varphi$ takes values in $T_\eta \cup T_\xi = 0$. Moreover, $(D')^j d'\varphi \in T_\eta$ and $(D'')^j d''\varphi \in T_\xi$ for all $j \geq 0$. Thus φ is *strongly isotropic* in the sense of [ErW]. In particular, φ is pluriconformal since the values of $d'\varphi$ and $d''\varphi$ are perpendicular.

Example. Let M be a complex (possibly immersed) submanifold of $\mathbb{C}P^{n-1} = G_1$, i.e. there exists a holomorphic immersion $\xi : M \to \mathbb{C}P^{n-1}$. Then $\underline{\delta'\xi}$ defines a map $\varphi = \delta'\xi : M \to G_m$ where $m = \dim_{\mathbb{C}} M$. By Theorem 2, this is a (1,1)-geodesic immersion: In fact, $\underline{\xi} + \underline{\delta'\xi}$ is a holomorphic vector bundle (spanned by a holomorphic section x of ξ and its partial derivatives with respect to a holomorphic chart, cf. Lemma 1), so $\eta = (\xi + \delta'\xi)^\perp$ is antiholomorphic and $\varphi = (\xi + \eta)^\perp$. If M is a hypersurface, i.e. $m = n-2$, then φ is a non-holomorphic (1,1)-geodesic immersion of maximal dimension in $G_m(\mathbb{C}^{m+2})$ (cf. [U],[FRT]).

Remarks

1. If we replace G_1 by G_p in the above example, the rank of $\underline{\delta'\xi}$ could drop essentially on a singular set $S \subset M$ of complex codimension ≥ 2, cf. [OU]. So the construction works only on $M \setminus S$.

2. Recently we learned that the case where M is a hypersurface of $\mathbb{C}P^{n-1}$ in the above example has also been considered by P.Kobak ([K], p.57). In this case, the Grassmannian $G_m = G_{n-2}(\mathbb{C}^n)$ is a quaternionic symmetric space and the flag manifold H its quaternionic twistor space (the canonical S^2-bundle over a quaternionic symmetric space). Using this example and the birational equivalence of quaternionic twistor spaces proved by F.Burstall [B], Kobak is able to construct (1,1)-geodesic submanifolds of the same dimension in the other quaternionic symmetric spaces. However, in general these submanifolds are not compact since the birational map may have singularities.

3. It is an open question whether there are non-holomorphic (1,1)-geodesic immersions of maximal dimension $m = (p-1)(q-1)+1$ into $G_p(\mathbb{C}^{p+q})$ for $p, q \geq 3$.

The construction of Theorem 2 is a special case of the so called *replacement* (cf. [BS], [W], [OU]). Let $\varphi : M \to G_p$ be a smooth map. Suppose that we have a decomposition of $\underline{\varphi}$ and $\underline{\varphi^\perp}$ into subbundles

$$\underline{\varphi} = \underline{\varphi_1} + \underline{\varphi_2}, \ \underline{\varphi^\perp} = \underline{\varphi_3} + \underline{\varphi_4}$$

such that

$$\varphi_i \cdot d'\varphi_{i+1} = 0$$

for $i \equiv 1,2,3,4 \bmod 4$. It is known [OU] that φ is pluriharmonic if and only if $\psi = \varphi_2 + \varphi_3$ is pluriharmonic. We extend this result to (1,1)-geodesic maps:

Theorem 3 *ψ is pluriconformal if and only if so is φ. Moreover, both φ and ψ are immersions if the 1-form $\varphi_3 \cdot d\varphi_2 + \varphi_4 \cdot d\varphi_1$ has zero kernel.*

Proof. Put $A_{ij} = \varphi_i \cdot d\varphi_j = -d\varphi_i \cdot \varphi_j$ and define A'_{ij}, A''_{ij} accordingly. By assumption, $A'_{i,i+1} = 0$. Note that $A''_{ij}.Y = (A'_{ji}.\overline{Y})^*$ for $Y \in T''M$ since for $f_i \in \Gamma\varphi_i$ and $i \neq j$ we have

$$((A''_{ij}Y)f_j, f_i) = (\partial_Y f_j, f_i) = -(f_j, \partial_{\overline{Y}} f_i) = -(f_j, (A'_{ji}\overline{Y})f_i).$$

Therefore, $A''_{i+1,i} = 0$. Thus

$$\varphi^\perp d'\varphi = A'_{31} + A'_{32} + A'_{42}, \quad \varphi^\perp d''\varphi = A''_{31} + A''_{41} + A''_{42}.$$

Since for any $X \in TM \otimes \mathbb{C}$, we have $\partial_X\varphi = \varphi^\perp \partial_X \varphi + \varphi \partial_X \varphi$ with

$$\varphi\partial_X\varphi = \partial_X\varphi \cdot \varphi^\perp = (\varphi^\perp \partial_{\overline{X}}\varphi)^*$$

and since $(A^*, B^*) = (B, A)$ for any $A, B \in \mathbb{C}^{n\times n}$, we get for all $X, Y \in TM \otimes \mathbb{C}$

$$(\partial_X\varphi, \partial_Y\varphi) = (\varphi^\perp\partial_X\varphi, \varphi^\perp\partial_Y\varphi) + (\varphi^\perp\partial_{\overline{Y}}\varphi, (\varphi^\perp\partial_{\overline{X}}\varphi).$$

Hence, if $X \in T'M$ and $Y \in T''M$, we have

$$(\partial_X\varphi, \partial_Y\varphi) = (A'_{31}.X, A''_{31}.Y) + (A'_{42}.X, A''_{42}.Y) + (A'_{31}.\overline{Y}, A''_{31}.\overline{X}) + (A'_{42}.\overline{Y}, A''_{42}.\overline{X}).$$

Similar we get

$$(\partial_X\psi, \partial_Y\psi) = (A'_{42}.X, A''_{42}.Y) + (A'_{13}.X, A''_{13}.Y) + (A'_{42}.\overline{Y}, A''_{42}.\overline{X}) + (A'_{13}.\overline{Y}, A''_{13}.\overline{X}).$$

Since

$$(A'_{31}.X, A''_{31}.Y) = (A'_{13}.\overline{Y}, A''_{13}.\overline{X}),$$
$$(A'_{31}.\overline{Y}, A''_{31}.\overline{X}) = (A'_{13}.X, A''_{13}.Y),$$

we get $(d'\varphi, d''\varphi) = (d'\psi, d''\psi)$. So φ is pluriconformal iff.so is ψ. Since $d\varphi$ and $d\psi$ have $A_{32} + A_{41}$ and its adjoint in common, the "moreover" statement follows.

Example. Let $\varphi = \varphi_1 : M \to G_p$ be a (1,1)-geodesic immersion such that $d'\varphi \cdot \varphi$ has image of constant rank q. Then $\varphi_3 := \delta'\varphi : M \to G_q$ defines a subbundle of $\underline{\varphi}^\perp$. Let $\underline{\varphi}_4 := \underline{\varphi}^\perp \cap \underline{\varphi}_3^\perp$. Then $\varphi_4 \cdot d'\varphi = 0$. Moreover we claim that $\varphi_4 \cdot d''\varphi_3 = 0$ which is equivalent to $\varphi_3 \cdot d'\varphi_4 = 0$. In fact, let $s = \partial_{Z'}\varphi \cdot f$ be a section of $\underline{\varphi}_3$, where $Z' \in T'M$ and f as section of $\underline{\varphi}$. Then for $Z'' \in T''M$ with $[Z', Z''] = 0$,

$$\partial_{Z''}s = \partial_{Z''}(\partial_{Z'}\varphi \cdot f) = (\partial_{Z''}\partial_{Z'}\varphi)f + \partial_{Z'}\varphi \cdot \varphi \cdot \partial_{Z''}f + \partial_{Z'}\varphi \cdot \varphi^\perp \cdot \partial_{Z''}f.$$

The first and the third term at the right hand side take values in $\underline{\varphi}$ (cf. Lemma 2) while the second term lies in $\underline{\varphi}_3 = \underline{\delta'\varphi}$. Thus $\partial_{Z''}s \in \underline{\varphi} + \underline{\varphi}_3$ which proves the claim. Now $\varphi_1 = \varphi$, $\varphi_2 = 0$, φ_3 and φ_4 satisfy the assumptions of Theorem 3 which shows that $\psi = \varphi_3 = \delta'\varphi$ is a (1,1)-geodesic map. This is the generalization of the *∂-transform* introduced by Wolfson [Ws].

5 (1,1)-geodesic maps into real Grassmannians

Let $G^{\mathbb{R}} = G_p(\mathbb{R}^n)$ be the Grassmannian of real p-dimensional subspaces of $\mathbb{R}^n$. We consider $G^{\mathbb{R}}$ as a subset of $G = G_p(\mathbb{C}^n)$ in the following way:

$$G^{\mathbb{R}} = \{\mu \in G \mid \bar{\mu} = \mu\}.$$

When do the (1,1)-geodesic immersions φ constructed in Theorem 2 actually lie in $G^{\mathbb{R}}$? Recall that $\varphi = (\xi + \eta)^{\perp}$ where ξ is holomorphic and η antiholomorphic. Apparently $\varphi \in G^{\mathbb{R}}$ if $\eta = \bar{\xi}$. So our problem is to find holomorphic immersions $\xi : M \to G$ with $\underline{\xi}, \underline{\delta'\xi} \perp \underline{\bar{\xi}}$. We give two such constructions.

Let $F : M \to \mathbb{R}^n$ be an isometric immersion. Let $\xi = \tau : M \to G_m(\mathbb{C}^n)$ denote the (1,0) Gauß map, i.e.

$$\underline{\tau}_p = dF(T_p'M)$$

for any $p \in M$. Since F is isometric, $\underline{\tau}_p$ is an isotropic subspace of $\mathbb{C}^n$, i.e. the bundles $\underline{\tau}$ and $\underline{\bar{\tau}}$ are perpendicular with respect to the hermitean inner product of $\mathbb{C}^n$. Moreover, $\underline{\delta'\tau} \perp \underline{\bar{\tau}}$, since for arbitrary (1,0) vector fields W' and Z' we have

$$\partial_{W'}\partial_{Z'}F = dF(D_{W'}Z') + \alpha(W', Z') \in \underline{\tau} + \underline{\nu}$$

where $\underline{\nu}$ is the real normal bundle of F. Note that $\varphi = (\tau + \bar{\tau})^{\perp} : M \to G_{n-2m}(\mathbb{R}^n)$ is the usual normal Gauß map of F. There are two cases where $\xi = \tau$ is holomorphic:

1. If F itself is (1,1)-geodesic then τ is holomorphic [RT], i.e. $d''t \in \underline{\tau}$ for any section $t \in \Gamma\underline{\tau}$ (cf. Lemma 1). In fact, if $t = \partial_{Z'}F$ for some (1,0) vector field Z', then

$$\partial_{Z''}t = \partial_{Z''}\partial_{Z'}F = 0$$

for any (0,1) vector field Z'' with $[Z', Z''] = 0$. It is known [DR] that any isometric minimal immersion of a Kähler manifold M into $\mathbb{R}^n$ is (1,1)-geodesic. Examples are considered by Dacjzer and Gromoll [DG].

2. If F is (2,0)-geodesic, i.e. $\alpha^{(2,0)} = 0$, then τ is antiholomorphic since for $t = \partial_{Z'}F \in \Gamma\underline{\tau}$ we have

$$\partial_{W'}t = \partial_{W'}\partial_{Z'}F = \alpha(W', Z') + dF(D_{W'}Z') \in \underline{\tau}$$

since $\alpha^{(2,0)} = 0$ and $D_{W'}Z' \in T'M$. It is known (cf. [F], [ET]) that an immersed Kähler manifold in $\mathbb{R}^n$ is (2,0)-geodesic if and only if it is an extrinsic hermitean symmetric space.

We have shown:

Theorem 4 *The normal Gauß map of a minimally immersed Kähler manifold or an extrinsic hermitean symmetric space in $\mathbb{R}^n$ is a (1,1)-geodesic map into a real Grassmannian.*

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