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Lie group theoretical construction of period mapping

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0 Introduction

A relation between simple singularities and Coxeter–Dynkin diagrams first established by Du Val ([Du]). Later, A. Grothendieck, E. Brieskorn and P. Slodowy ([Br], [Sl1], [Sl2]) clarified this mysterious relation. Let us explain the outline of their theory. Let $\mathcal{G}$ be a simple Lie algebra over $\mathbf{C}$ of type A, D, E and G the adjoint group of $\mathcal{G}$, and let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ and W the Weyl group with respect to $\mathcal{H}$. In their theory, they constructed the simple singularities and their semi-universal deformation, based on the results of B. Kostant ([Ko2]) on the adjoint quotient $\chi: \mathcal{G} \to \mathcal{H}/W$ and T.A. Springer, J. Tits and R. Steinberg ([St]) on the geometry of nilpotent variety $\mathcal{N}(\mathcal{G})$. Namely,

(1) The intersection of the nilpotent variety $\mathcal{N}(\mathcal{G})$ with a transversal slice S to the subregular nilpotent orbit is a 2-dimensional surface with a simple singularity of type $\mathcal{G}$.

(2) The restriction of the adjoint quotient $\chi: \mathcal{G} \to \mathcal{H}/W$ to the slice S, $\chi_S: S \to \mathcal{H}/W$, is a realization of a semi-universal deformation of the simple singularity.

Moreover, Grothendieck constructed a simultaneous resolution of the adjoint quotient $\chi: \mathcal{G} \to \mathcal{H}/W$. Let $\mathcal{B}$ be a Borel subalgebra which contains $\mathcal{H}$ and B the corresponding Borel subgroup of G. Then Grothendieck's simultaneous resolution is the following commutative diagram (see [St], [Sl1], [Sl2]):

$$\begin{array}{ccc} G \times_B \mathcal{B} & \xrightarrow{\psi} & \mathcal{G} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \chi} \\ \mathcal{H} & \xrightarrow[\pi_W]{} & \mathcal{H}/W\,, \end{array} \tag{0.1}$$

where $\psi([g,b]_B) = \mathrm{Ad}(g)b$ and $\theta([g,b]_B) = \mathscr{H}$-component of $b \in \mathscr{B}$. This induces a simultaneous resolution of $\chi_S: S \to \mathscr{H}/W$ (see [Sl1], [Sl2]):

$$\begin{array}{ccc} \psi^{-1}(S) & \xrightarrow{\psi} & S \\ {\scriptstyle \theta_S}\downarrow & & \downarrow{\scriptstyle \chi_S} \\ \mathscr{H} & \xrightarrow[\pi_W]{} & \mathscr{H}/W\,. \end{array} \tag{0.2}$$

On the other hand, as a generalization of the Abel differentials of the first kind on elliptic curves, K. Saito ([Sa1], [Sa2], [Sa3]) introduced the notion of the primitive forms to construct the period mapping for the universal deformation of a holomorphic function with isolated singularities. Since the primitive form is defined by a certain system of infinitely many bilinear differential equations, the global existence of it is known except for the cases of simple and simply elliptic singularities (see [Sa3]). Therefore, it is natural to wish to construct the primitive forms as some geometric structure.

The purpose of this paper is to show how the theory of the symplectic reductions with symmetry can be applied to the construction of the primitive forms and their period mappings associated with the simple singularities.

Let $\mathscr{N} = [\mathscr{B}, \mathscr{B}]$ be the nilradical and N the corresponding Lie group, and let H be the Cartan subgroup corresponding to $\mathscr{H}$. Then, on the holomorphic cotangent bundle $T^*(G/N)$, the Lie groups G and H act from the left and the right, respectively. These actions are Hamiltonian (see section 1). Hence, we have two momentum mappings $J_G: T^*(G/N) \to \mathscr{G}^*$ and $J_H: T^*(G/N) \to \mathscr{H}^*$. Considering the symplectic reduction with respect to H, we shall obtain the commutative diagram:

$$\begin{array}{ccccc} & & T^*(G/N) & & \\ & \swarrow^{J_H} & \downarrow{\scriptstyle \tilde{\pi}_H} & \searrow^{J_G} & \\ & & T^*(G/N)/H & \xrightarrow{\hat{J}_G} & \mathscr{G}^* \\ & & {\scriptstyle \hat{J}_H}\downarrow & & \downarrow{\scriptstyle J} \\ & & \mathscr{H}^* & \xrightarrow[\pi_W]{} & \mathscr{H}^*/W\,. \end{array} \tag{0.3}$$

This commutative diagram will give Grothendieck's simultaneous resolution (0.1). Moreover, since the cannonical symplectic 2-form ω on $T^*(G/N)$ is H-invariant and $i(\tilde{h})\omega \equiv 0 \bmod J_H^* \Omega^{(1)}_{\mathscr{H}^*}$, there exists a unique holomorphic relative 2-form $\hat{\omega} \in \Gamma(T^*(G/N)/H, \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*})$ which gives the reduced symplectic 2-form $\hat{\omega}(\lambda)$ on each fibre $\hat{J}_H^{-1}(\lambda)(= \theta^{-1}(\lambda))$. Here $\tilde{h}$ is a Hamiltonian vector field corresponding to $h \in \mathscr{H}$ (see Theorem 2.5). We would like to comment that F. Knop ([Kn]) already studied the properties of the momentum mappings on the cotangent bundle over a class of homogeneous spaces.

Let X_f be a special transversal slice, a so called Harish–Chandra slice (see (4.4)) and $\tilde{X}_f = J_G^{-1}(X_f)$, the preimage of X_f under the momentum mapping J_G. Let $\tilde{\omega}_f$ be the holomorphic 2-form on $\tilde{X}_f$ induced by the holomorphic canonical symplectic 2-form on $T^*(G/N)$. We now state our main results.

Theorem 0.1 (*see Theorem* 4.6 *and Corollary* 4.7)

$(\tilde{X}_f, \tilde{\omega}_f)$ is a complex symplectic manifold which satisfies the following properties:

(1) *The action of H on $\tilde{X}_f$ is a Hamiltonian action which admits the momentum mapping $J_H^f : \tilde{X}_f \to \mathscr{H}^*$.*

(2) *The commutative diagram*

$$\begin{array}{ccc} \tilde{X}_f/H & \xrightarrow{j_G^f} & X_f \\ {\scriptstyle j_H^f}\downarrow & & \downarrow{\scriptstyle J^f} \\ \mathscr{H}^* & \xrightarrow[\pi_W]{} & \mathscr{H}^*/W \end{array}$$

is a simultaneous resolution of the semi-universal deformation $J^f : X_f \to \mathscr{H}^*/W$ (see (0.2)). *Furthermore, there exists a unique holomorphic relative* 2*-form $\hat{\omega}_f \in \Gamma(\hat{X}_f, \Omega^{(2)}_{\hat{X}_f/\mathscr{H}^*})$ which gives the reduced symplectic* 2*-form $\hat{\omega}_f(\lambda)$ on each fibre $(\hat{J}_H^f)^{-1}(\lambda)$.*

(3) *The holomorphic relative* 2*-form $\hat{\omega}_f$ is just the primitive form associated with the simple singularity.*

Based on this result, we shall obtain the period mapping which has already been studied by K. Saito ([Sa3]), E.J.N. Looijenga ([Lo1], [Lo2]) and P. Slodowy ([Sl3]).

Theorem 0.2 (*see Theorem* 5.4)

Let $\lambda_0 \in \mathscr{H}^*$ be a regular element and denote by $[\lambda_0]_W$ the class of λ_0 in $\mathscr{H}^*/W$. The period mapping

$$P: \mathscr{H}^* \to H^2((J^f)^{-1}([\lambda_0]_W), \mathbf{C})$$

is a W-equivariant linear isomorphism.

Using the Lie–Poisson structure on $\mathscr{G}$, in [Sl3], P. Slodowy has already constructed a period mapping for simple singularities (see Remark 2 and (5.1)). Our construction is closely related to his and we make essential use of some of his results. However, our general approach provides a more detailed and comprehensive picture. In particular, we obtain a natural extension of the period mapping to the singular elements of $\mathscr{H}^*$ (see Remark 2 and 4(2)). Also, in [Kr], P.B. Kronheimer constructed the gravitational instantons from hyper-Kähler geometric and polyhedral group theoretic view point. As a by-product, he constructed the primitive forms associated with the simple singularities.

The present paper is organized as follows. In Sect. 1, we give a brief review of momentum mappings and symplectic reductions with symmetry. In Sect. 2, we will reconstruct Grothendieck's simultaneous resolutions from the view point of symplectic geometry (see Theorem 2.4). In Sect. 3, we will make a detailed investigation of the structure of the reduced symplectic 2-form $\hat{\omega}_f(\lambda)$ on $\hat{J}_H^{-1}(\lambda)$ (see Proposition 3.2). In Sect. 4, we shall reconstruct the simultaneous resolution of a simple singularity in terms of symplectic geometry. Then we shall explain that the holomorphic relative 2-form $\hat{\omega}_f$ is just the primitive form (see Theorem 4.6 and Corollary 4.7). In Sect. 5, we shall study the period mapping associated with the primitive form, and then we shall obtain an extension of the domain for the period mapping (see Theorem 5.3).

1 Review of the notion of symplectic reduction

In this section, we shall give a brief review of the complex version of momentum mappings and symplectic reductions. For details and proofs, we refer to the books [A-M], [Ar], [G-S], [L-M].

Let (M, ω) be a complex symplectic manifold and $\mathcal{O}_M$ the sheaf of holomorphic functions of M. Then the space of holomorphic functions $\Gamma(M, \mathcal{O}_M)$ is a Lie algebra under the Poisson bracket. Let A be a complex connected Lie group and $\Phi: A \times M \to M$ an action of A and M. Let $\mathcal{A}$ be the Lie algebra of A and $\mathcal{A}^*$ the dual space. To each $a \in \mathcal{A}$ corresponds the fundamental vector field $\tilde{a}$ on M, which is the infinitesimal generator of the 1-parameter subgroup generated by a. The action Φ is said to be symplectic if Φ preserves ω. Then the fundamental vector field $\tilde{a}$ is a symplectic vector field on M. The action Φ is said to be Hamiltonian if Φ satisfies the following properties:

(1) the action Φ is symplectic

(2) The fundamental vector field $\tilde{a}$ is a Hamiltonian vector field on M. Namely there exists a function $J_a \in \Gamma(M, \mathcal{O}_M)$ such that

$$i(\tilde{a})\omega = -dJ_a$$

where $i(\tilde{a})$ is the interior product with respect to $\tilde{a}$.

(3) The mapping

$$\mathcal{A} \to \Gamma(M, \mathcal{O}_M), \quad a \to J_a$$

is a Lie algebra homomorphism.

Then we define the momentum mapping of the Hamiltonian action Φ by

$$J_A: M \to \mathcal{A}^*, \qquad \langle J_A(x), a \rangle = J_a(x)$$

for any $a \in \mathcal{A}$.

Proposition 1.1 *The momentum mapping $J_A: M \to \mathcal{A}^*$ is A-equivariant with respect to the coadjoint action* Ad^* *of A on $\mathcal{A}^*$, i.e.,*

$$J_A(\Phi_g(x)) = \mathrm{Ad}^*(g^{-1})J_A(x).$$

for any $g \in A$ and $x \in M$.

Let x be a point in M and $O_A(x)$ the A-orbit of x. Denote the symplectic orthogonal of the tangent space $T_x(O_A(x))$ by $T_x(O_A(x))^\perp$. Let $\omega_{O_A}(x)$ be the 2-form on $O_A(x)$ induced by the symplectic 2-form ω. Define the characteristic subspace of $\omega_{O_A(x)}$ at x by

$$\mathrm{Ch}_x(O_A(x)) = \{u \in T_x(O_A(x)) | (\omega_{O_A(x)})_x(u, v) = 0 \text{ for any } v \in T_x(O_A(x))\} .$$

Then we have the following proposition (see [L-M] Chap. 4 Propositions 2.2 and 5.5):

Proposition 1.2 *Let $Z_A(x)$ be the isotropy subgroup of $x \in M$ and $Z_A(x)$ the Lie algebra of this group.*

(1) *The transpose of the derivative $J_{A*x}: T_x(M) \to \mathscr{A}^*$ of J_A in x is the linear mapping from $\mathscr{A}$ into $T_x^*(M)$ given by*

$$a \to -i(\tilde{a}_x)\omega_x .$$

(2) *Let $\mathrm{Ann}(Z_{\mathscr{A}}(x))$ be the annihilator of $Z_{\mathscr{A}}(x)$ in $\mathscr{A}^*$, i.e.,*

$$\mathrm{Ann}(Z_{\mathscr{A}}(x)) = \{\alpha \in \mathscr{A}^* | \langle \alpha, a \rangle = 0 \text{ for any } a \in Z_{\mathscr{A}}(x)\} .$$

Then we have

$$J_{A*x}(T_x(M)) = \mathrm{Ann}(Z_{\mathscr{A}}(x)) .$$

(3) *The kernel of J_{A*x} is the symplectic orthogonal subspace of $T_x(O_A(x))$, i.e.,*

$$\mathrm{Ker} J_{A*x} = T_x(O_A(x))^\perp .$$

(4) *The characteristic subspace $\mathrm{Ch}_x(O_A(x))$ is the tangent space of the $Z_A(J_A(x))$-orbit of x, $O_{Z_{A(J_A(x))}}(x)$, i.e.,*

$$\mathrm{Ch}_x(O_A(x)) = T_x(O_{Z_A(J_A(x))}(x)) .$$

Now let α be a point in $J_A(M) \subset \mathscr{A}^*$. The subset $M_\alpha = J_A^{-1}(\alpha)$ is invariant under the subgroup $Z_A(\alpha)$ of A by Proposition 1.1. If the point α is a regular value of J_A and $Z_A(\alpha)$ acts on M_α without fixed-points and properly, the orbit space

$$R_\alpha = M_\alpha / Z_A(\alpha)$$

is a complex manifold. Let $\omega(\alpha)$ be the 2-form on M_α induced by 2-form ω. Then, by Proposition 1.2, for any $x \in M_\alpha$,

$$\begin{aligned} \mathrm{Ch}_x(M_\alpha) &= T_x(M_\alpha) \cap T_x(M_\alpha)^\perp \\ &= T_x(O_A(x))^\perp \cap T_x(O_A(x)) \\ &= \mathrm{Ch}_x(O_A(x)) \\ &= T_x(O_{Z_A(\alpha)}(x)) . \end{aligned}$$

Hence the characteristic subspace of $(\omega(\alpha))_x$ is $T_x(O_{Z_A(\alpha)}(x))$. Therefore, there exists a unique holomorphic symplectic 2-form $\hat{\omega}(\alpha)$ on R_α such that

$$\pi_\alpha^* \hat{\omega}(\alpha) = \omega(\alpha) .$$

where $\pi_\alpha : M_\alpha \to R_\alpha$ is the projection. Summarizing we have the following theorem:

Theorem 1.3 (*Marsden–Weinstein* [*M-W*]) *Let $\alpha \in J_A(M)$ be a regular value of J_A. The 2-form $\omega(\alpha)$ induced by ω on M_α is of constant rank, and the leaves of its characteristic foliation are the orbits of the action of $Z_A(\alpha)$ on M_α. Furthermore, if we assume that this $Z_A(\alpha)$-action on M_α is fix-point free and proper, there exists a unique holomorphic symplectic 2-form $\hat{\omega}(\alpha)$ on the quotient space R_α. Then we call $(R_\alpha, \hat{\omega}(\alpha))$ as above the complex reduced symplectic manifold associated with $\alpha \in \mathscr{A}^*$.*

2 Symplectic reduction and Grothendieck's simultaneous resolution

In this section, we reconstruct Grothendieck's simultaneous resolutions for adjoint quotients in terms of symplectic reductions. For details on Grothendieck's simultaneous resolutions for adjoint quotients, see [Sl1], [Sl2].

Let $\mathscr{G}$ be a simple Lie algebra over $\mathbf{C}$ of rank l, $(\cdot \mid \cdot)$ the Killing form on $\mathscr{G}$, and G the adjoint group of $\mathscr{G}$. We fix a Cartan group H and a Borel subgroup B which contains the Cartan subgroup H once and for all. Denote the Lie algebras of H and B by $\mathscr{H}$ and $\mathscr{B}$, respectively. Let $\mathscr{N} = [\mathscr{B}, \mathscr{B}]$ be the nilradical of $\mathscr{B}$ and N the Lie subgroup of B with Lie algebra $\mathscr{N}$. Note that $\mathscr{B}$ and B have the following decomposition:

$$\mathscr{B} = \mathscr{H} \oplus \mathscr{N} \tag{2.1}$$

$$B = H \ltimes N . \tag{2.2}$$

Let $T^*(G/N)$ be the holomorphic cotangent bundle over the homogeneous space G/N and

$$\tilde{\pi}_N : T^*(G/N) \to G/N$$

the projection. The holomorphic Liouville 1-form θ_N on $T^*(G/N)$ is defined by

$$(\theta_N)_\xi(v) = \langle \xi, \tilde{\pi}_{N*\xi}(v) \rangle \tag{2.3}$$

for any $\xi \in T^*(G/N)$ and $v \in T_\xi(T^*(G/N))$. Then $(T^*(G/N), \omega_N = d\theta_N)$ is a complex symplectic manifold.

Let $g \in G$ and $h \in H$. The left action on G/N by g and the right action on G/N by h will be denoted by L_g and R_h, respectively. Their respective canonical lifting to the holomorphic cotangent bundle $T^*(G/N)$ will be denoted by $\tilde{L}_g$

and $\tilde{R}_h$. Then they satisfy

$$\tilde{L}_g^* \theta_N = \theta_N \tag{2.4}$$

$$\tilde{R}_h^* \theta_N = \theta_N \,. \tag{2.5}$$

From these properties, we obtain the following proposition (cf. [L-M] Chap. IV 1.19):

Proposition 2.1 *The action L and $\tilde{R}$ on $T^*(G/N)$ are Hamiltonian actions which admit the momentum mappings*

$$J_G : T^*(G/N) \to \mathcal{G}^*, \qquad \langle J_G(\xi), a \rangle = (\theta_N)_\xi(\tilde{a}_\xi) \tag{2.6}$$

$$J_H : T^*(G/N) \to \mathcal{H}^*, \qquad \langle J_H(\xi), h \rangle = (\theta_N)_\xi(\tilde{h}_\xi) \tag{2.7}$$

where $\xi \in T^*(G/N)$, $a \in \mathcal{G}$ and $h \in \mathcal{H}$.

Now the adjoint group G acts on the dual space of $\mathcal{G}, \mathcal{G}^*$, by the coadjoint action Ad^*. Let $\mathcal{G}^*/G$ be the variety whose $\mathbf{C}$-algebra is the algebra of G-invariant polynomials on $\mathcal{G}^*$ and

$$J : \mathcal{G}^* \to \mathcal{G}^*/G$$

the quotient morphism. Let W be the Weyl group of $\mathcal{G}$ associated with the Cartan subalgebra $\mathcal{H}$. By Chevalley's theorem, the space $\mathcal{G}^*/G$ is holomorphically isomorphic to $\mathcal{H}^*/W$, an affine space of dimension $l = \operatorname{rank} \mathcal{G}$. Identifying $\mathcal{G}^*/G$ and $\mathcal{H}^*/W$, we obtain the mapping

$$J : \mathcal{G}^* \to \mathcal{H}^*/W \,.$$

This mapping is called the adjoint quotient (cf. [Sl1], [Sl2]). By Proposition 2.1, we obtain the following diagram:

$$\begin{array}{ccc} T^*(G/N) & \xrightarrow{J_G} & \mathcal{G}^* \\ {\scriptstyle J_H}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{H}^* & \xrightarrow[\pi_W]{} & \mathcal{H}^*/W \,, \end{array}$$

where π_W is the canonical projection.

Since H acts on G/N without fixed points and properly, the action $\tilde{R}$ on $T^*(G/N)$ is also fix-point free and proper. Hence the isotropy subalgebra of $\xi \in T^*(G/N)$ is a 0-dimensional subspace of $\mathcal{H}$. Thus, by Proposition 1.2 (2), the momentum mapping J_H is a submersion. Then we have the following proposition:

Proposition 2.2 *The momentum mappings $J_G : T^*(G/N) \to \mathcal{G}^*$ and $J_H : T^*(G/N) \to \mathcal{H}^*$ are invariant under the action $\tilde{R}$ of H. In particular, there*

exist unique mappings as follows:

$$\hat{J}_G: T^*(G/N)/H \to \mathcal{G}^*, \qquad \hat{J}_G \circ \tilde{\pi}_H = J_G$$

$$\hat{J}_H: T^*(G/N)/H \to \mathcal{H}^*, \qquad \hat{J}_H \circ \tilde{\pi}_H = J_H$$

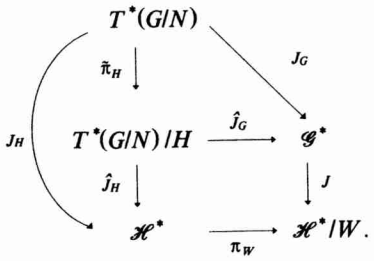
Proof. Since the actions $\tilde{L}$ and $\tilde{R}$ commute, the Hamiltonian vector field $\tilde{a}$ associated with $a \in \mathcal{G}$ in $\tilde{R}$-invariant, i.e.,

$$\tilde{R}_{h*}(\tilde{a}) = \tilde{a} \tag{2.8}$$

for every $h \in H$. This implies that the momentum mapping J_G is H-invariant. Therefore, we obtain a unique mapping $\hat{J}_G: T^*(G/N)/H \to \mathcal{G}^*$ such that $\hat{J}_G \circ \tilde{\pi}_H = J_G$.

Since H is an Abelian group, by Proposition 1.1, the momentum mapping J_H is H-invariant. Therefore, we obtain all statements. □

By Proposition 2.2, we obtain the following diagram:



(2.9)

The commutativity of this diagram will be proved later. Since the isotropy subgroup of $\lambda \in \mathcal{H}^*$ is just H, and since J_H is a submersion, by Theorem 1.3, each fibre of $\hat{J}_H$ is a complex symplectic manifold $(\hat{J}_H^{-1}(\lambda), \hat{\omega}(\lambda))$.

Let $\Omega^{(p)}_{T^*(G/N)/\mathcal{H}^*}$ (resp. $\Omega^{(p)}_{T^*(G/N)/H/\mathcal{H}^*}$) be the sheaf of holomorphic relative p-forms associated with the momentum mapping J_H (resp. $\hat{J}_H$). Let ω' be the image of the symplectic 2-form ω under the natural homomorphism $\Omega^{(2)}_{T^*(G/N)} \to \Omega^{(2)}_{T^*(G/N)/\mathcal{H}^*}$.

Lemma 2.3* *There exists an unique holomorphic relative 2-form* $\hat{\omega} \in \Gamma(T^*(G/N)/H, \Omega^{(2)}_{T^*(G/N)/H/\mathcal{H}^*})$ *such that*

$$\tilde{\pi}_H^* \hat{\omega} = \omega' . \tag{2.10}$$

This holomorphic relative 2-form $\hat{\omega}$ *gives the reduced symplectic 2-form* $\hat{\omega}(\lambda)$ *on each fibre* $\hat{J}_H^{-1}(\lambda)$.

* The author owes to the referee the sheaf theoretical description in this proof.

Proof. Since the second statement follows from (2.10), we prove the first statement. Since $\tilde{\pi}_H^*$ is a smooth morphism, there is an exact sequence

$$0 \to \tilde{\pi}_H^* \Omega^{(1)}_{T^*(G/N)/H/\mathscr{H}^*} \to \Omega^{(1)}_{T^*(G/N)/\mathscr{H}^*} \to \Omega^{(1)}_{T^*(G/N)/T^*(G/N)/H} \to 0 .$$

In particular $\tilde{\pi}_H^* \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*}$ is a subsheaf of $\Omega^{(2)}_{T^*(G/N)/\mathscr{H}^*}$ and a section φ of $\Omega^{(2)}_{T^*(G/N)/\mathscr{H}^*}$ is a section of $\tilde{\pi}_H^* \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*}$ if and only if $\varphi(\xi, \cdot\ ) = 0$ for all local vector fields ξ on $T^*(G/N)$ which are tangent to the fibres of $\tilde{\pi}$. Since

$$i(\tilde{h})\omega \equiv 0 \bmod J_H^* \Omega^{(1)}_{\mathscr{H}^*} ,$$

one can consider ω' as a section of $\tilde{\pi}_h^* \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*}$. But it is invariant under the action of H and vector space of H-invariant sections of $\tilde{\pi}_H^* \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*}$ is isomorphic to the vector space of sections of $\Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*}$. By this identification, there exists an unique holomorphic relative 2-form $\hat{\omega} \in \Gamma(T^*(G/N), \Omega^{(2)}_{T^*(G/N)/\mathscr{H}^*})$ such that

$$\tilde{\pi}_H^* \hat{\omega} = \omega' . \qquad \square$$

Remark 1 Let H_f be the Hamiltonian vector field on $T^*(G/N)$ associated with a Hamiltonian $f \in \Gamma(T^*(G/N), \mathcal{O}_{T^*(G/N)})$. On $T^*(G/N)/H$, we define a Poisson structure as follows:

$$\{f, g\} = \omega(H_{\pi_H^* f}, H_{\pi_H^* g})$$

for any $f, g \in \Gamma(T^*(G/N)/H, \mathcal{O}_{T^*(G/N)/H})$. Note that, for any $z \in \Gamma(\mathscr{H}^*, \mathcal{O}_{\mathscr{H}^*})$, we have

$$\{f, \hat{J}_H^* z\} = 0 .$$

Then the mapping

$$\hat{J}_H : T^*(G/N)/H \to \mathscr{H}^*$$

is a Poisson morphism. Let $(z_1, \ldots, z_l)$ be a coordinate system of $\mathscr{H}^*$ and set $\hat{z}_j = \hat{J}_H^* z_j (j = 1, \ldots, l)$. Since the Lie–Poisson structure on $\mathscr{H}^*$ is trivial, and since $\hat{J}_H$ is a submersion, for any point $\hat{\xi} \in T^*(G/N)/H$, there exists a local coordinate system $(U, (\xi_1, \ldots, \xi_m, \hat{z}_1, \ldots, \hat{z}_l))$, $m + l = \dim T^*(G/N)/H$, such that

$$\{f, g\} = \sum_{i<j} \{\xi_i, \xi_j\} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j}$$

Note that the matrix $(\{\xi_i, \xi_j\})$ is non-degenerate on U (cf. [L-M] chap. III). We define a holomorphic 2-form on U by

$$\hat{\omega}_U = \sum_{i<j} \omega_U^{i,j} d\xi_i \wedge d\xi_j ,$$

where $(\omega_U^{i,j}) = (\{\xi_i, \xi_j\})^{-1}$. By the definition of $\hat{\omega}_U$, if $U \cap U' \neq \emptyset$, we have

$$\hat{\omega}_U \equiv \hat{\omega}_{U'} \bmod \hat{J}_H^* \Omega^{(1)}_{\mathscr{H}^*} \wedge \Omega^{(1)}_{T^*(G/N)/H} .$$

Therefore, $\{\hat{\omega}_U, U\}$ determines a unique holomorphic relative 2-form. This relative 2-form coincides to $\hat{\omega}$.

Now we still show that the diagram (2.9) gives Grothendieck's simultaneous resolution for the adjoint quotient. Since $\mathscr{B}$ is stable under the adjoint action of N, we can define the associated vector bundle over G/N

$$G \times_N \mathscr{B} \to G/N$$

as quotient of $G \times \mathscr{B}$ by the following N-action from the right:

$$(G \times \mathscr{B}) \times N \to G \times \mathscr{B}, \quad ((\mathrm{g}, b), \mathrm{n}) \to (\mathrm{gn}, \mathrm{Ad}(\mathrm{n}^{-1})b). \tag{2.11}$$

We denote the class of $(\mathrm{g}, b) \in G \times \mathscr{B}$ in $G \times_N \mathscr{B}$ and $\mathrm{g} \in G$ in G/N by $[\mathrm{g}, b]_N$ and $[\mathrm{g}]_N$, respectively. By (2.11), the following mapping

$$T^*(G/N) \to G \times_N \mathscr{B}, \qquad \xi_{[\mathrm{g}]_N} \to [\mathrm{g}, L_{\mathrm{g}}^* \xi_{[\mathrm{g}]_N}]_N \tag{2.12}$$

is well-defined and a holomorphic vector bundle isomorphism. Here we identify $\mathscr{B}$ with $T^*_{[\mathrm{e}]_N}(G/N)$ using the Killing form on $\mathscr{G}$. Thus, we identify $T^*(G/N)$ and $G \times_N \mathscr{B}$. Then the Liouville 1-form θ_N on $G \times_N \mathscr{B}$ is given by the following formula:

$$(\theta_N)_{[\mathrm{g}, b]_N}(v) = (b | L_{\mathrm{g}^{-1}*}(\tilde{\pi}_{N*}(v))) \tag{2.13}$$

for any $[\mathrm{g}, b]_N \in G \times_N \mathscr{B}$ and $v \in T_{[\mathrm{g}, b]_N}(G \times_N \mathscr{B})$. Furthermore, the actions $\tilde{L}$ and $\tilde{R}$ on $T^*(G/N)$ are represented on $G \times_N \mathscr{B}$ by the following formulae:

$$\tilde{L}_{\mathrm{g}'}: G \times_N \mathscr{B} \to G \times_N \mathscr{B}, \qquad \tilde{L}_{\mathrm{g}'}([\mathrm{g}, b]_N) = [\mathrm{g}'\mathrm{g}, b]_N \tag{2.14}$$

$$\tilde{R}_{\mathrm{h}}: G \times_N \mathscr{B} \to G \times_N \mathscr{B}, \qquad \tilde{R}_{\mathrm{h}}([\mathrm{g}, b]_N) = [\mathrm{gh}, \mathrm{Ad}(\mathrm{h}^{-1})b]_N \tag{2.15}$$

for every $\mathrm{g} \in G$ and $\mathrm{h} \in H$. Using the Killing form of $\mathscr{G}$, we identify $\mathscr{G}$ and $\mathscr{H}$ with the dual spaces $\mathscr{G}^*$ and $\mathscr{H}^*$, respectively. Then we have the following lemma:

Lemma 2.4 *The momentum mappings $J_G: G \times_N \mathscr{B} \to \mathscr{G}$ and $J_H: G \times_N \mathscr{B} \to \mathscr{H}$ are represented by the following:*

$$J_G([\mathrm{g}, b]_N) = \mathrm{Ad}(\mathrm{g})b \tag{2.16}$$

$$J_H([\mathrm{g}, b]_N) = b_{\mathscr{H}} \tag{2.17}$$

where $b_{\mathscr{H}}$ is the $\mathscr{H}$-component of $b \in \mathscr{B}$ (cf. (2.1)).

Proof. Let $\hat{a}$ be a fundamental vector field on G/N corresponding to $a \in \mathscr{G}$. By (2.6) and (2.13), for any $[\mathrm{g}, b]_N \in G \times_N \mathscr{B}$,

$$\begin{aligned}(J_G([\mathrm{g}, b]_N) | a) &= (\theta_N)_{[\mathrm{g}, b]_N}(\tilde{a}_{[\mathrm{g}, b]_N}) \\ &= (b | L_{\mathrm{g}^{-1}*}(\tilde{\pi}_{N*}(\tilde{a}_{[\mathrm{g}, b]_N}))) \\ &= (b | L_{\mathrm{g}^{-1}*}(\hat{a}_{[\mathrm{g}]_N})).\end{aligned}$$

Since the vector field $\hat{a}$ on G/N is induced by the left-action of G, $\hat{a}$ satisfies

$$L_{g^{-1}*}\hat{a} = \widehat{\mathrm{Ad}(g^{-1})a}\,.$$

Hence we have

$$\begin{aligned}(b|L_{g^{-1}*}(\hat{a}_{[g]_N})) &= (b|\mathrm{Ad}(g^{-1})a)\\ &= (\mathrm{Ad}(g)b|a)\,.\end{aligned}$$

Therefore we obtain (2.16).

By (2.7) and (2.13),

$$\begin{aligned}(J_H([g,b]_N)|h) &= (\theta_N)_{[g,b]_N}(\tilde{h}_{[g,b]_N})\\ &= (b|L_{g^{-1}*}(\hat{h}_{[g]_N}))\,.\end{aligned}$$

Since the vector field $\tilde{h}$ is induced by the right action of H on G/N, $\tilde{h}$ is a left-invariant vector field on G/N. Hence we have

$$(b|L_{g^{-1}*}(\hat{h}_{[g]_N})) = (b|h)\,.$$

Therefore, we have

$$J_H([g,b]_N) = b_{\mathcal{H}}\,,$$

where $b_{\mathcal{H}}$ is $\mathcal{H}$-component of b of $\mathcal{B}$. □

By (2.15), $(G \times_N \mathcal{B})/H$ is holomorphically isomorphic to the associated vector bundle $G \times_B \mathcal{B}$. Hence we can identify $T^*(G/N)/H$ and $G \times_B \mathcal{B}$. Under this identification, by Proposition 2.2, (2.16) and (2.17), $\hat{J}_G: G \times_B \mathcal{B} \to \mathcal{G}$ and $\hat{J}_H: G \times_B \mathcal{B} \to \mathcal{H}$ are represented by

$$\hat{J}_G([g,b]_B) = \mathrm{Ad}(g)b \tag{2.18}$$

$$\hat{J}_H([g,b]_B) = b_{\mathcal{H}} \tag{2.19}$$

where $[g,b]_B$ is the class of $(g,b) \in G \times \mathcal{B}$ in $G \times_B \mathcal{B}$.

Note that, by (2.17) and (2.19), $J_H: T^*(G/N) \to \mathcal{H}^*$ and $\hat{J}_H: T^*(G/N)/H \to \mathcal{H}^*$ are surjective submersions. Since the semi-simple part of $\mathrm{Ad}(g)(h+n)$ is conjugate to h, by (2.16), (2.17), (2.18) and (2.19), the diagram (2.9) is commutative (cf. [Sl2] Sect. 3.3 Theorem). As a consequence, we obtain the following theorem:

Theorem 2.5

(1) *The commutative diagram*

$$\begin{array}{ccc} T^*(G/N)/H & \xrightarrow{\hat{J}_G} & \mathcal{G}^* \\ {\scriptstyle \hat{J}_H}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{H}^* & \xrightarrow[\pi_W]{} & \mathcal{H}^*/W \end{array} \tag{2.20}$$

is Grothendieck's simultaneous resolution for the adjoint quotient.

(2) *There exists a unique holomorphic relative* 2*-form* $\hat{\omega} \in \Gamma(T^*(G/N)/H, \Omega^{(2)}_{T^*(G/N)/H/\mathscr{H}^*})$ *which gives the reduced symplectic* 2*-form* $\hat{\omega}(\lambda)$ *on each fibre* $\hat{J}_H^{-1}(\lambda)$ *for any* $\lambda \in \mathscr{H}^*$. *Furthermore, the Hamiltonian G-action on* $T^*(G/N)$ *induces a Hamiltonian G-action on* $(\hat{J}_H^{-1}(\lambda), \hat{\omega}(\lambda))$, *which admits the momentum mapping* $\hat{J}_{G,\lambda}: \hat{J}_H^{-1}(\lambda) \to \mathscr{G}^*$ *such that*

$$\hat{J}_{G,\lambda} = \hat{J}_G|_{\hat{J}_H^{-1}(\lambda)}.$$

Proof. (1) Using the Killing form on $\mathscr{G}$, we identify $\mathscr{G}^*$, $\mathscr{H}^*$ and $T^*(G/N)/H$ with $\mathscr{G}$, $\mathscr{H}$ and $G \times_B \mathscr{B}$, respectively. Then the commutative diagram (2.20) is identified with the following diagram:

$$\begin{array}{ccc} G \times_B \mathscr{B} & \xrightarrow{\hat{J}_G} & \mathscr{G} \\ {\scriptstyle j_H}\downarrow & & \downarrow{\scriptstyle J} \\ \mathscr{H} & \xrightarrow{\pi_W} & \mathscr{H}_W . \end{array}$$

Hence, by (2.18) and (2.19),

$$\hat{J}_G([g, b]_B) = \mathrm{Ad}(g)b, \qquad \hat{J}_H([\hat{g}, b]) = b_{\mathscr{H}}$$

for every $[g, b] \in G \times_B \mathscr{B}$. Therefore, the commutative diagram (2.20) is Grothendieck's simultaneous resolution for the adjoint quotient (cf. [Sl1] Chap. 4).

(2) The first statement follows from Theorem 1.3 and Lemma 2.3. The second statement is already proved in [L-M] Chap. 4 Theorem 6.5 in a more general setting. □

3 Cohomology classes of the reduced symplectic 2-forms

In this section, following an idea of M. Kummer [Ku] and J.J. Duistermaat and G.J. Heckman [D-H], we investigate the structure of the reduced symplectic 2-form $\hat{\omega}(\lambda)$ on $\hat{J}_H^{-1}(\lambda)$. Then we will construct a W-equivariant linear isomorphism $P_{\hat{\omega}}: \mathscr{H}^* \to H^2(T^*(G/B), \mathbf{C})$.

Let K be a maximal compact Lie group of G such that $T = K \cap H$ is a maximal torus of K. We use the C^∞-identification

$$K/T \simeq G/B$$

induced by the inclusion $K \hookrightarrow G$, and the identifications

$$T^*(G/N)/H \simeq G \times_B \mathscr{B} \qquad \text{and} \qquad T^*(G/B) \simeq G \times_B \mathscr{N}$$

$$\searrow \quad \swarrow \qquad\qquad \searrow \quad \swarrow$$

$$G/B \qquad\qquad G/B$$

(cf. (2.12) and (2.15)). Since $G \times_B \mathscr{B}$ and $G \times_B \mathscr{N}$ are K-equivariant vector bundles over $G/B = K/T$, we can write them in the forms $K \times_T \mathscr{B}$ and $K \times_T \mathscr{N}$, respectively. Here the T-actions on $\mathscr{B}$ and $\mathscr{N}$ are the ones obtained by restriction from B via $T \hookrightarrow H \hookrightarrow B$. We denote by $[\mathrm{k}, b]_T$ and $[\mathrm{k}, n]_T$ the class of $(\mathrm{k}, b) \in K \times \mathscr{B}$ in $K \times_T \mathscr{B}$ and $(\mathrm{k}, n) \in K \times \mathscr{N}$ in $K \times_T \mathscr{N}$, respectively. The C^∞-mapping

$$\hat{\pi}_0 : K \times_T \mathscr{B} \to K \times_T \mathscr{N}, \qquad \hat{\pi}_0([\mathrm{k}, b]_T) = [\mathrm{k}, b - \hat{J}_H([\mathrm{k}, b]_T)]_T \tag{3.1}$$

is well-defined. Then we define the C^∞-mapping

$$\hat{\varphi} : K \times_T \mathscr{B} \to \mathscr{H} \times (K \times_T \mathscr{N})$$

by

$$\hat{\varphi}([\mathrm{k}, b]_T) = (\hat{J}_H([\mathrm{k}, b]_T), \hat{\pi}_0([\mathrm{k}, b]_T)) . \tag{3.2}$$

It is well-known that $\hat{\varphi}$ is a diffeomorphism and the restriction of $\hat{\varphi}$ to each fibre of $\hat{J}_H$ induces a diffeomorphism

$$\hat{\varphi}_\lambda : \hat{J}_H^{-1}(\lambda) \to K \times_T \mathscr{N} = T^*(G/B)$$

for any $\lambda \in \mathscr{H}^*$. In particular, $\hat{\varphi} : \hat{J}_H(0) \to T^*(G/B)$ is a holomorphic isomorphism. Moreover, by (3.2), this diffeomorphism $\hat{\varphi}$ is also C^∞-bundle isomorphism over $\mathscr{H}^* = \mathscr{H}$:

$$\begin{array}{ccccc} T^*(G/N)/H & = & K \times_T \mathscr{B} & \xrightarrow{\hat{\varphi}} & \mathscr{H} \times (K \times_T \mathscr{N}) = \mathscr{H}^* \times T^*(G/B) \\ & & \searrow^{\hat{J}_H} & & \swarrow_{\pi_1} \\ & & & \mathscr{H} = \mathscr{H}^* & \end{array}$$

where π_1 is the canonical projection to the first factor.

Now we make a comparison between the reduced symplectic 2-forms on the fibres of $\hat{J}_H$ and the holomorphic canonical symplectic 2-form on $T^*(G/B) = K \times_T \mathscr{N}$. To this end, we recall a result of Borel–Hirzebruch ([B-H] section 14) on the cohomology of the flag manifold $G/B = K/T$.

Let Δ (resp. Δ_+) be the set of roots (resp. positive roots) associated with $\mathscr{H}$ and $\mathscr{B}$, i.e.,

$$\mathscr{G} = \mathscr{H} \oplus \left(\sum_{\alpha \in \Delta} \mathscr{G}_\alpha \right) : \text{root space decomposition}, \qquad \mathscr{B} = \mathscr{H} \oplus \left(\sum_{\alpha \in \Delta_+} \mathscr{G}_\alpha \right) .$$

For any $\alpha \in \Delta$, let e_α be a basis of $\mathscr{G}_\alpha$ such that

$$[e_\alpha, e_{-\alpha}] = h_\alpha \qquad (e_\alpha | e_{-\alpha}) = 1$$

where h_α is the element of $\mathscr{H}$ defined by $\langle h_\alpha, \beta \rangle = (\alpha | \beta)$ for any $\beta \in \mathscr{H}^*$. Let θ_α be the left-invariant 1-form on G whose restriction to $\mathscr{G}$ is annihilated by

$\mathscr{H}$ and which is such that

$$\langle (\theta_\alpha)_e, e_\beta \rangle = \delta_{\alpha,\beta} .$$

Let $\tilde{\lambda}$ be the left-invariant 1-form on G whose restriction to $\mathscr{G}$ is zero on $\sum_{\alpha\in\Delta} \mathscr{G}_\alpha$ and which induces the linear form λ on $\mathscr{H}$. By the Maurer–Cartan equation, we have

$$d\tilde{\lambda} = \sum_{\alpha\in\Delta_+} (\lambda|\alpha)\theta_{-\alpha} \wedge \theta_\alpha . \tag{3.4}$$

The restriction $\tilde{\lambda}_K$ of $\tilde{\lambda}$ to K is a complex left-invariant 1-form. By (3.4), we have

$$d\tilde{\lambda}_K = \sum_{\alpha\in\Delta_+} (\lambda|\alpha)\theta_{-\alpha} \wedge \tilde{\theta}_{-\alpha} . \tag{3.5}$$

Here $\tilde{\theta}_{-\alpha}$ is the complex conjugate of $\theta_{-\alpha}$. Since $d\tilde{\lambda}_K$ is K-invariant and, by (3.5), is zero on the Lie algebra $\mathscr{T}$ of T, there exists a unique (1,1)-form on $G/B = K/T$ such that

$$d\tilde{\lambda}_K = \pi_T^* \Omega(\lambda) , \tag{3.6}$$

where $\pi_T : K \to K/T$ is the projection. Furthermore, the action of the Weyl group W on K/T induces an action of W on the cohomology of $G/B = K/T$.

Proposition 3.1 (*Borel–Hirzebruch* [*B-H*] *Proposition* 14.6)
The linear mapping

$$\mathscr{H}^* \to H^2(G/B, \mathbf{C}), \qquad \lambda \to [\Omega(\lambda)] ,$$

where $[\Omega(\lambda)]$ *is the cohomology class of* $\Omega(\lambda)$, *is a W-equivariant linear isomorphism.*

Let $\iota_K : K \times \mathscr{B} \hookrightarrow G \times_N \mathscr{B} = T^*(G/N)$ be the inclusion which is induced by the inclusion $K \hookrightarrow G$. We denote by J_H^K and π_H^K the restrictions of J_H and π_H to $K \times \mathscr{B}$, respectively. We denote by $\tilde{\pi}_0$ the composition of the mappings π_H^K and $\hat{\pi}_0$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
G/N \supset K & \xleftarrow{\tilde{\pi}} & K \times \mathscr{B} & \subset & T^*(G/N) \\
\downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \tilde{\pi}_0} & \overset{\pi_H^K}{\searrow} & K \times_T \mathscr{B} = T^*(G/N)/H \\
 & & & \underset{\hat{\pi}_0}{\swarrow} & \\
G/B = K/T & \xleftarrow[\pi]{} & K \times_T \mathscr{N} = T^*(G/B) & &
\end{array}$$

where $\tilde{\pi}$ and π are the projections.

Let θ_B be the holomorphic Liouville 1-form on $T^*(G/B)$ (cf. (2.3)). We define the 1-forms on $K \times \mathscr{B}$ by

$$\theta_K = \iota_K^* \theta_N, \qquad \theta_0 = \tilde{\pi}_0^* \theta_B .$$

Then, for any $(\mathrm{k}, b) \in K \times \mathscr{B}$ and $v \in T_{(\mathrm{k},b)}(K \times \mathscr{B})$, we have

$$\begin{aligned}(\theta_0)_{(\mathrm{k},b)}(v) &= (b - J_H^K(\mathrm{k},b)|\pi_{T*}(L_{\mathrm{k}^{-1}*}(\tilde{\pi}_*(v))))\\ &= (\theta_K)_{(\mathrm{k},b)}(v) - \tilde{\pi}^* L_{\mathrm{k}^{-1}}^* J_H^K(\mathrm{k},b)(v)\,.\end{aligned}$$

Restricting $\tilde{\pi}_0$ to the fibre $(J_H^K)^{-1}(\lambda)(\lambda \in \mathscr{H}^*)$, we have

$$\tilde{\pi}_{0,\lambda}^* \theta_B = (\imath^* \theta_N - \tilde{\pi}^* L_{\mathrm{k}^{-1}}^* \lambda)|_{(J_H^K)^{-1}(\lambda)}\,, \tag{3.7}$$

where $\tilde{\pi}_{0,\lambda}$ is the restriction of $\tilde{\pi}_0$ to $(J_H^K)^{-1}(\lambda)$. Since $L_{\mathrm{k}^{-1}}^* \lambda$ is a complex-valued left-invariant 1-form on K which induces the linear form λ on $\mathscr{T}$, by (3.6), we have

$$d(L_{\mathrm{k}^{-1}}^* \lambda) = \pi_T^* \Omega(\lambda)\,. \tag{3.8}$$

By exterior differentiation of (3.7) and by making use of (3.8), we have

$$\tilde{\pi}_{0,\lambda}^*(d\theta_B + \pi^* \Omega(\lambda)) = \imath^* \omega|_{(J_H^K)^{-1}(\lambda)}\,.$$

Since $\tilde{\pi}_{0,\lambda} = \hat{\varphi}_\lambda \circ \pi_H^K, \pi_H^{K*} \hat{\omega}(\lambda) = \imath_K^* \omega|_{(J_H^K)^{-1}(\lambda)}$ and π_H^K is a submersion, we obtain

$$\hat{\omega}(\lambda) = \hat{\varphi}_\lambda^*(d\theta_B + \pi^* \Omega(\lambda))\,.$$

Summarizing the above discussion, we obtain the following proposition (cf. [Ku]) Theorem 1 and [D-H] Theorem 1.1):

Proposition 3.2 *Let $\omega_B = d\theta_B$ be the holomorphic canonical symplectic 2-form on $T^*(G/B)$. Then, for any $\lambda \in \mathscr{H}^*$, we have*

$$\hat{\omega}(\lambda) = \hat{\varphi}_\lambda^*(\omega_B + \pi^* \Omega(\lambda))\,.$$

Furthermore,

$$\hat{\varphi}_0 : (\hat{J}_H^{-1}(0), \hat{\omega}(0)) \to (T^*(G/B), \omega_B)$$

is a complex symplectic isomorphism.

Now, since $\hat{J}_H : T^*(G/N)/H \to \mathscr{H}^*$ is a C^∞-trivial fibre bundle (see (3.3)) and $\hat{J}_H^{-1}(0) = T^*(G/B)$, the C^∞-mapping $\hat{\varphi}$ induces a sheaf isomorphism

$$(\hat{\varphi}^*)^{-1} : R^2 \hat{J}_{H*} \mathbf{C}_{T^*(G/N)/H} \to \mathscr{H}^* \times (R^2 \hat{J}_{H*} \mathbf{C}_{T^*(G/N)/H})_0$$

and we get an isomorphism of stalks

$$\Phi_\lambda = (\hat{\varphi}_\lambda^*)^{-1} : (R^2 \hat{J}_{H*} \mathbf{C}_{T^*(G/N)/H})_\lambda \to (R^2 \hat{J}_{H*} \mathbf{C}_{T^*(G/N)/H})_0\,.$$

Then we define the following mapping

$$\begin{aligned}P_{\hat{\omega}} &: \mathscr{H}^* \to H^2(T^*(G/B), \mathbf{C})\\ P_{\hat{\omega}}(\lambda) &= \Phi_\lambda([\hat{\omega}(\lambda)])\,.\end{aligned} \tag{3.9}$$

Since $T^*(G/B)$ is homotopy equivalent to G/B, the mapping

$$H^2(G/B,\mathbf{C}) \to H^2(T^*(G/B),\mathbf{C}), \qquad [\Omega(\lambda)] \to [\pi^*\Omega(\lambda)]$$

is a linear isomorphism. This induces an W-action on $H^2(T^*(G/B),\mathbf{C})$. By the proposition of Borel–Hirzebruch and Proposition 3.2, we get

Corollary 3.3 *The linear mapping*

$$P_{\hat{\omega}}: \mathscr{H}^* \to H^2(T^*(G/B),\mathbf{C}) \qquad P_{\hat{\omega}}(\lambda) = \Phi_\lambda([\hat{\omega}(\lambda)])$$

is a W-equivariant linear isomorphism.

Proof. Since the direct image $R^2\hat{J}_{H*}\mathbf{C}_{T^*(G/N)/H}$ is a constant sheaf, by Proposition 3.2,

$$P_{\hat{\omega}}(\lambda) = \Phi_\lambda([\varphi_\lambda^*\pi^*\Omega(\lambda)]) = [\pi^*\Omega(\lambda)]\,.$$

By the proposition of Borel–Hirzebruch, $P_{\hat{\omega}}$ is a W-equivariant linear isomorphism. □

This corollary will be important in our construction of the period mapping (cf. [Sl3], chap. 2 Lemma).

4 Simultaneous resolutions of simple singularities and primitive forms

In this section, we construct simultaneous resolutions of simple singularities from the view point of symplectic geometry. See [Sl1], [Sl2], for details on simultaneous resolutions of simple singularities. Then we will explain that reduced symplectic 2-forms are just primitive forms associated with simple singularities. For details on primitive forms, we refer to [Sa1], [Sa2], [Sa3].

First, we recall Jacobson–Morozov's lemma (cf. [Ko1]).

Theorem 4.1 (*Jacobson–Morozov*) *Let a be a nilpotent element of $\mathscr{G}$. Then there exist elements t and b of $\mathscr{G}$ such that the triplet $\{a,t,b\}$ generates a three dimensional simple Lie subalgebra in $\mathscr{G}$, i.e.,*

$$[t,a] = 2a, \qquad [t,b] = -2b, \qquad [a,b] = t\,..$$

Furthermore, all such triplets are conjugate under the centralizer $Z_G(a)$ of a.

We call such a triplet $\{a,t,b\}$ an $sl(2,\mathbf{C})$-triplet for a. Let e be a subregular nilpotent element of $\mathscr{G}$, i.e., $\dim Z_{\mathscr{G}}(e) = l+2$ and $\{e,h,f\}$ an $sl(s,\mathbf{C})$-triplet for e. Define a complex 1-parameter transformation $\tilde{\varphi}_t(t \in \mathbf{C})$ on $T^*(G/N)$ by

$$\tilde{\varphi}_t(\xi) = \mathrm{e}^{2t}\tilde{L}_{\exp(th)}(\xi) \tag{4.1}$$

for every $\xi \in T^*(G/N)$. This complex 1-parameter transformation on $T^*(G/N)$ induces complex 1-parameter transformations on $T^*(G/N)/H$, $\mathscr{G}^*$, $\mathscr{H}^*$ and $\mathscr{H}^*/W$. Then, all arrows in the commutative diagram (2.9) are equivariant under these complex 1-parameter transformations (cf. [Sl2] Sect. 4.2).

Furthermore $\tilde{\varphi}_t$ induces a complex 1-parameter transformation φ_t on G/N such that

$$\varphi_t([g]_N) = L_{\exp(th)}([g]_N)\,.$$

for every $[g]_N \in G/N$.

Lemma 4.2 *The holomorphic canonical symplectic 2-form ω on $T^*(G/N)$ has the following property*:

$$\tilde{\varphi}_t^* \omega = e^{2t}\omega\,. \tag{4.2}$$

Proof. For any $\xi \in T^*(G/N)$ and $v \in T_\xi(T^*(G/N))$, we have

$$\begin{aligned}(\tilde{\varphi}_t^* \theta_N)_\xi(v) &= (\theta_N)_{\tilde{\varphi}_t(\xi)}(\tilde{\varphi}_{t*}(v)) \\ &= \langle e^{2t} L^*_{\exp(th)}(\xi), L_{\exp(th)}(\tilde{\pi}_{N*}(v))\rangle \\ &= e^{2t}\langle \xi, \pi_{N*}(v)\rangle \\ &= e^{2t}(\theta_N)_\xi(v)\,.\end{aligned}$$

Hence we have

$$\tilde{\varphi}_t^* \theta_N = e^{2t}\theta_N\,.$$

Since $\omega = d\theta_N$, we obtain the statement. □

From now on, using the Killing form on $\mathcal{G}$, we identify $T^*(G/N)$, $T^*(G/N)/H, \mathcal{G}^*$ and $\mathcal{H}^*$ with $G \times_N \mathcal{B}, G \times_B \mathcal{B}, \mathcal{G}$ and $\mathcal{H}$, respectively.

Let $\mathcal{A}$ be the $sl(2,\mathbf{C})$-triplet $\{e,h,f\}$ for e. Then $\mathcal{A}$ acts on $\mathcal{G}$ by the adjoint action. Hence there is a decomposition

$$\mathcal{G} = \bigoplus_{j=1}^{l+2} \mathcal{G}_j$$

where $\mathcal{G}_j$ is an irreducible $\mathcal{A}$-module of dimension $n_j + 1$. The highest weight space $\mathcal{G}_j(n_j)$ is the centralizer of e in $\mathcal{G}_j$. Thus we have

$$Z_{\mathcal{G}}(e) = \bigoplus_{j=1}^{l+2} \mathcal{G}_j(n_j) \quad \text{and} \quad \operatorname{ad}(f)\mathcal{G} \oplus Z_{\mathcal{G}}(e) = \mathcal{G}\,. \tag{4.3}$$

Let X_f be the affine subspace such that

$$X_f = f + Z_{\mathcal{G}}(e) \subset \mathcal{G}\,. \tag{4.4}$$

By (4.3) and (4.4), this affine subspace X_f is a transversal slice to the G-orbit of f at f. (cf. [Sl1], [Sl2])

Let $\varphi_t^{\mathcal{G}}$ be the complex 1-parameter transformation on $\mathcal{G}$ induced by $\tilde{\varphi}_t$. By (2.16) and (4.1), we have

$$\varphi_t^{\mathcal{G}}(a) = e^{2t}\operatorname{Ad}(\exp(th))a \tag{4.5}$$

for any $a \in \mathcal{G}$. Let $e_j (j = 1,\ldots,l+2)$ be a basis of $\mathcal{G}_j(n_j)$. Since e_j is a highest weight vector of weight n_j, we have

$$[h, e_j] = n_j e_j \quad (j = 1,\ldots,l+2)\,. \tag{4.6}$$

Here n_j is a non-negative integer (see [Sl2] Sect. 2.5 and [Sl1] Sect. 7.4 Proposition 2). By (4.5) and (4.6), for every $x_f = f + \sum_{j=1}^{l+2} c_j e_j \in X_f$, we have

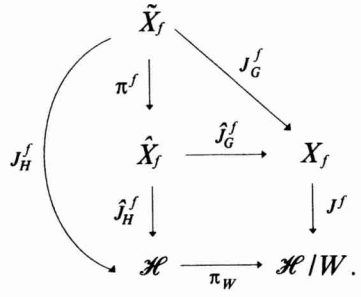
$$\varphi_t^{\mathscr{G}}(x_f) = f + \sum_{j=1}^{l+2} c_j e^{t(n_j+2)} e_j . \tag{4.7}$$

Therefore, the affine subspace X_f is invariant under the complex 1-parameter transformation $\varphi_t^{\mathscr{G}}$.

Let $\tilde{X}_f = J_G^{-1}(X_f)$ and $\hat{X}_f = \hat{J}_G^{-1}(X_f)$, the preimages of X_f under J_G in $G \times_N \mathscr{B}$ and $\hat{J}_G$ in $G \times_B \mathscr{B}$, respectively. Since X_f is transversal to the orbits in $\mathscr{G}$, $\tilde{X}_f$ and $\hat{X}_f$ are submanifolds in $G \times_N \mathscr{B}$ and $G \times_B \mathscr{B}$, respectively. Moreover, since X_f is invariant under $\varphi_t^{\mathscr{G}}$, $\tilde{X}_f$ and $\hat{X}_f$ are also invariant under $\tilde{\varphi}_t$ and $\hat{\varphi}_t$, respectively. Here $\hat{\varphi}_t$ is the complex 1-parameter transformation on $G \times_B \mathscr{B}$ induced by $\tilde{\varphi}_t$. Since the momentum mapping J_G is H-invariant, $\tilde{X}_f$ is a H-invariant submanifold in $G \times_N \mathscr{B}$. By the definition of $\hat{X}_f$, we have

$$\hat{X}_f = \tilde{X}_f / H .$$

Thus we obtain the following commutative diagram:



(4.8)

Here J_G^f, J_H^f, etc. are the restriction of the mappings J_G, J_H etc. on the spaces $\tilde{X}_f$, etc.

Let $\tilde{\imath}: \tilde{X}_f \to G \times_N \mathscr{B}$ be the inclusion. Set

$$\tilde{\omega}_f = \tilde{\imath}^* \omega .$$

We will prove that $(\tilde{X}_f, \tilde{\omega}_f)$ is a complex symplectic submanifold. Let p be a point in $\tilde{X}_f$. By the definition of $\tilde{X}_f$, we have

$$\mathrm{Ker} J_{G*p} \subset T_p(\tilde{X}_f) . \tag{4.9}$$

By Proposition 1.2 (3),(4) and (4.9), We have two subspaces of $T_p(O_G(p))$ as follows:

$$T_p(\tilde{X}_f)^{\perp}, \qquad \mathrm{Ch}_p(O_G(p)) \subset T_p(O_G(p)) .$$

Then we obtain the following proposition:

Proposition 4.3 *Let p be a point of $J_G^{-1}(f) \subset \tilde{X}_f$. Then we have*

$$T_p(O_G(p)) = \mathrm{Ch}_p(O_G(p)) \oplus T_p(\tilde{X}_f)^{\perp} .$$

In particular, $T_p(\tilde{X}_f)^{\perp}$ and $T_p(\tilde{X}_f)$ are symplectic vector spaces.

Proof. Let $\mathrm{Ann}(Z_{\mathcal{G}}(e))$ be the annihilator of $Z_{\mathcal{G}}(e)$ in $\mathcal{G}$ with respect to the killing form $(\cdot\,|\,\cdot)$. For any $x \in Z_{\mathcal{G}}(e)$ and $[e, y] \in \mathrm{ad}(e)\mathcal{G}$, we have

$$(x|[e, y]) = ([e,x]|y) = 0 .$$

Since $\dim \mathrm{ad}(e)\mathcal{G} = \dim \mathrm{Ann}(Z_{\mathcal{G}}(e))$, we have $\mathrm{ad}(e)\mathcal{G} = \mathrm{Ann}(Z_{\mathcal{G}}(e))$. Hence, by (4.3), we have

$$\mathcal{G} = Z_{\mathcal{G}}(f) \oplus \mathrm{Ann}(Z_{\mathcal{G}}(e)) . \tag{4.10}$$

Let $\varphi: G \to O_G(p) \subset T^*(G/N)$ be the mapping induced by the left-action of G on $T^*(G/N)$, i.e.,

$$\varphi(\mathrm{g}) = \tilde{L}_{\mathrm{g}}(p) .$$

For any $a \in \mathrm{Ann}(Z_{\mathcal{G}}(e))$ and $v \in T_p(\tilde{X}_f)$, since $J_{G*p}(v) \in Z_{\mathcal{G}}(e)$,

$$\omega_p(v, \varphi_{*\mathrm{e}}(a)) = \omega_p(v, \tilde{a}_p) = (J_{G*p}(v)|a) = 0 .$$

Hence we have

$$\varphi_{*\mathrm{e}}(\mathrm{Ann}(Z_{\mathcal{G}}(e))) \subset T_p(\tilde{X}_f)^{\perp} .$$

Moreover, since $\mathrm{Ker}\varphi_{*\mathrm{e}} = Z_{\mathcal{G}}(p) \subset Z_{\mathcal{G}}(f)$, by (4.10), we have

$$\varphi_{*\mathrm{e}}(Z_{\mathcal{G}}(f)) \cap \varphi_{*\mathrm{e}}(\mathrm{Ann}(Z_{\mathcal{G}}(e))) = \{0\} .$$

Thus,

$$\varphi_{*\mathrm{e}}|_{\mathrm{Ann}(Z_{\mathcal{G}}(e))}: \mathrm{Ann}(Z_{\mathcal{G}}(e)) \to T_p(\tilde{X}_f)^{\perp}$$

is injective. In particular, by (4.10), $\dim \varphi_{*\mathrm{e}}(\mathrm{Ann}(Z_{\mathcal{G}}(e))) = \dim \mathcal{G}\text{-}(1+2)$. On the other hand, $\dim T_p(\tilde{X}_f)^{\perp} = \dim T^*(G/N) - \dim T_p(\tilde{X}_f) = \dim \mathcal{G}\text{-}(1+2)$. Therefore, we have

$$\varphi_{*\mathrm{e}}(\mathrm{Ann}(Z_{\mathcal{G}}(e))) = T_p(\tilde{X}_f)^{\perp} .$$

By Proposition 1.2(4) and (4.10), we obtain the statement. □

By this proposition, there exists some neighborhood U for $p \in J_G^{-1}(f)$ in $\tilde{X}_f$ such that $\tilde{\omega}_f$ is non-degenerate on U. Since $\hat{J}_G^f: \hat{X}_f \to X_f$ is a simultaneous resolution of J^f, the mapping $\hat{J}_G^f$ is proper. Using the properness of the mapping $\hat{J}_G^f$, we obtain the following stronger result:

Lemma 4.4 *There exists a neighborhood U_f of f in X_f such that $\tilde{\omega}_f$ is non-degenerate on $(J_G^f)^{-1}(U_f)$.*

Proof. By Proposition 4.3, there exists a neighborhood $\tilde{V}(p)$ of any point $p \in J_G^{-1}(f)$ in $\tilde{X}_f$ such that $\tilde{\omega}_f$ is non-degenerate on $\tilde{V}(p)$. Set

$$\tilde{U}(p) = \bigcup_{h \in H} \tilde{R}_h(\tilde{V}(p)) .$$

Since $\pi^f :\to \hat{X}_f$ is a principal H-bundle, π^f is open mapping. Hence $\hat{U}(\hat{p}) = \pi^f(\tilde{U}(p))$ is an open neighborhood of $\hat{p} = \pi^f(p)$ in $\hat{X}^f$. Since $\hat{J}_G^f$ is proper, $(\hat{J}_G^f)^{-1}(f)$ is compact. Therefore, there exist points $p_1, \ldots, p_n$ of $(\hat{J}_G^f)^{-1}(f)$ such that

$$(\hat{J}_G^f)^{-1}(f) \subset \hat{U}(\hat{p}_1) \cup \cdots \cup \hat{U}(\hat{p}_n) .$$

Moreover, by the continuity of $\hat{J}_G^f$, there exists a neighborhood U_f of f in X_f such that

$$(\hat{J}_G^f)^{-1}(U_f) \subset \hat{U}(\hat{p}_1) \cup \cdots \cup \hat{U}(\hat{p}_n) .$$

Hence we obtain

$$(J_G^f)^{-1}(U_f) \subset \tilde{U}(p_1) \cup \cdots \cup \tilde{U}(p_n) .$$

By the choice of $\tilde{U}(p_j)$, $\tilde{\omega}_f$ is non-degenerate on $(J_G^f)^{-1}(U_f)$. □

Theorem 4.5 *$(\tilde{X}_f, \tilde{\omega}_f)$ is a complex symplectic manifold which satisfies the following properties*:

(1) *The action of the Cartan subgroup H on $\tilde{X}_f$ is a Hamiltonian action which admits the momentum mapping $J_H^f : \tilde{X}_f \to \mathcal{H}^*$.*

(2) *The commutative diagram*

$$\begin{array}{ccc} \tilde{X}_f/H = \hat{X}_f & \xrightarrow{\hat{J}_G^f} & X_f \\ \Big\downarrow{\scriptstyle \hat{J}_H^f} & & \Big\downarrow{\scriptstyle J^f} \\ \mathcal{H}^* & \xrightarrow[\pi_W]{} & \mathcal{H}^*/W \end{array}$$

is a simultaneous resolution of the semi-universal deformation $J^f : X_f \to \mathcal{H}^*/W$. Furthermore, there exists a unique holomorphic relative 2-form $\hat{\omega}_f \in \Gamma(\hat{X}_f, \Omega^{(2)}_{\hat{X}_f/\mathcal{H}^*})$ which gives the reduced symplectic 2-form $\hat{\omega}_f(\lambda)$ on each fibre $(\hat{J}_H^f)^{-1}(\lambda)$.

Proof. To prove the statement, one only needs to prove that $(\tilde{X}_f, \tilde{\omega}_f)$ is a symplectic manifold. By Lemma 4.4, there exists a neighborhood U_f of f in X_f such that $\tilde{\omega}_f$ is non-degenerate on $(J_G^f)^{-1}(U_f)$. By (4.7), for any $\tilde{x}_f \in \tilde{X}_f$ and for sufficiently large $t_0 \in \mathbf{R}$, we have

$$\tilde{\varphi}_{-t_0}(\tilde{x}_f) \in (J_G^f)^{-1}(U_f) .$$

Since, by (4.2), $\tilde{\omega}_f$ is non-degenerate at $\tilde{\varphi}_{-t_0}(\tilde{x}_f)$, $\tilde{\omega}_f$ is non-degenerate at $\tilde{x}_f$, thus on all of $\tilde{X}_f$. Therefore, $(\tilde{X}_f, \tilde{\omega}_f)$ is a complex symplectic manifold. □

Now we explain the relation between the holomorphic relative 2-form $\hat{\omega}_f$ and the primitive form associated with simple singularity. First, following [Sa3], we give the characterization of the primitive form on the simultaneous resolution space $\hat{X}_f$ associated with a simple singularity.

Characterization of primitive form (*K. Saito* [*Sa*3]) *The primitive form Ξ is a holomorphic relative 2-form on the fibre of $\hat{J}_H^f : \hat{X}_f \to \mathscr{H}^*$ such that*

(1) *On each fibre $(\hat{J}_H^f)^{(-1)}(\lambda)$, Ξ gives a non-degenerate holomorphic 2-form.*

(2) *For any $t \in \mathbf{C}$,*

$$\hat{\varphi}_t^* \Xi = \mathrm{e}^{2t} \Xi .$$

Therefore, by Lemma 4.2 and Theorem 4.5, we obtain the following corollary:

Corollary 4.6 *The holomorphic relative 2-form $\hat{\omega}_f$ on $\hat{J}_H^f : \hat{X}_f \to \mathscr{H}^*$ is just the primitive form.*

Remark 2 In [Sl3], P.Slodowy already constructed a period mapping associated with simple singularities. In this remark, we explain the relation between his theory and ours. Let x be an element of $\mathscr{G}$. On the G-orbit of x, $O_G(x)$, there exists the following symplectic 2-form which is called Kostant–Kirillov–Souriau form:

$$\omega_{\mathscr{G}}(x)([x,a],[x,b]) = (x|[a,b])$$

for any $a, b \in \mathscr{G}$. Let $\lambda_0 \in \mathscr{H}^*$ be a regular element, i.e., $\dim Z_{\mathscr{G}}(\lambda_0) = l$. Then $J^{-1}([\lambda_0]_W)$ is the G-orbit $O_G(\lambda_0)$. Hence $(J^{-1}([\lambda_0]_W), \omega_{\mathscr{G}}(\lambda_0))$ is a complex symplectic manifold. By Theorem 2.5, the momentum mapping

$$\hat{J}_{G,\lambda_0} : \hat{J}_H^{-1}(\lambda_0) \to J^{-1}([\lambda_0]_W)$$

is a holomorphic symplectic isomorphism (cf. [L-M] IV. 5.7 Theorem). Let $\hat{J}_{G,\lambda_0}^f$ be the restriction of $\hat{J}_{G,\lambda_0}$ to $\hat{X}_f(\lambda_0)$ and $\omega_{\mathscr{G}}^f(\lambda_0)$ the holomorphic 2-form on $X_f([\lambda_0]_W) = (J^f)^{-1}([\lambda_0]_W)$ induced by $\omega_{\mathscr{G}}(\lambda_0)$. Then the mapping

$$\hat{J}_{G,\lambda_0}^f : (\hat{X}_f(\lambda_0), \hat{\omega}_f(\lambda_0)) \to (X_f([\lambda_0]_W), \omega_{\mathscr{G}}^f(\lambda_0))$$

is a holomorphic symplectic isomorphism. Using the symplectic 2-form $\omega_{\mathscr{G}}^f(\lambda_0)$, in [Sl3], P. Slodowy constructed a period mapping (see (5.1)). In his theory, he treats the generic case, i.e., λ_0 is a regular element of $\mathscr{H}^*$. In our approach, we are able to treat not only regular elements of $\mathscr{H}^*$ but also singular elements of $\mathscr{H}^*$.

5 Period mapping associated with the primitive form

In this section, we assume that $\mathcal{G}$ is a simple Lie algebra over $\mathbf{C}$ of type A_l, D_l and E_l.

Let C be the critical set of the adjoint quotient $J_f: X_f \to \mathcal{H}^*/W$ and $D = J_f(C)$ the discriminant set in $\mathcal{H}^*/W$. Denote the spaces $\mathcal{H}^* \setminus \pi_W^{-1}(D)$ and $X_f \setminus (J^f)^{-1}(D)$ by $\mathcal{H}_r^*$ and X_f^r, respectively. The following proposition was proved by P. Slodowy in a more general situation.

Proposition 5.1 (*P. Slodowy* [Sl2] 4.2 *Proposition*) (1) $J_H^f: \hat{X}_f \to \mathcal{H}^*$ *is a* C^∞*-global trivial fibre bundle.*

(2) *Let* J_r^f *be the restriction of* J^f *to the space* X_f^r. *Then* $J_r^f: X_f^r \to \mathcal{H}_r^*/W$ *is a* C^∞*-fibre bundle which becomes trivial after pull-back by* $\pi_W^r: \mathcal{H}_r^* \to \mathcal{H}_r^*/W$.

By Proposition 5.1(2), the higher direct image $R^2 J_{r*}^f \mathbf{C}_{X_f}$ of the constant sheaf $\mathbf{C}_{X_f}$ is a local system on $\mathcal{H}_r^*/W$, and thus, if we choose a base point $[\lambda_0]_W \in \mathcal{H}_r^*/W$, we get a monodromy action of $\pi_1(\mathcal{H}_r^*/W, [\lambda_0]_W)$ on the stalk

$$(R^2 J_{r*}^f \mathbf{C}_{X_f})_{[\lambda_0]_W} = H^2(X_f([\lambda_0]_W), \mathbf{C}).$$

Since the covering $\pi_W^r: \mathcal{H}_r^* \to \mathcal{H}_r^*/W$ is unramified and Galois with covering group W, and since the fibre bundle $J_r^f: X_f^r \to \mathcal{H}_r^*/W$ becomes trivial after pull-back by π_W^r, the covering space $\mathcal{H}_r^*$ is a monodromy covering of $\mathcal{H}_r^*/W$. Thus the inverse image of $R^2 J_{r*}^f \mathbf{C}_{X_f}$ on $\mathcal{H}_r^*$ is a constant sheaf isomorphic to $\mathcal{H}_r^* \times (R^2 J_{r*}^f \mathbf{C}_{X_f})_{[\lambda_0]_W}$ and we get an isomorphism of stalks

$$\Phi_{\lambda, \lambda_0} : (R^2 J_{r*}^f \mathbf{C}_{X_f})_{[\lambda]_W} \to (R^2 J_{r*}^f \mathbf{C}_{X_f})_{[\lambda_0]_W}$$

for any $\lambda \in \mathcal{H}_r^*$. Therefore, we can define the period mapping associated with the primitive form $\omega_{\mathcal{G}}^f$ (see Remark 2) as follows:

$$\begin{aligned} \hat{P}: \mathcal{H}_r^* &\to H^2(X_f([\lambda_0]_W), \mathbf{C}) \\ \hat{P}(\lambda) &= \Phi_{\lambda, \lambda_0}([\omega_{\mathcal{G}}^f(\lambda)]) \end{aligned} \tag{5.1}$$

(cf. [Sl3] Chap. 2 Satz).

Now we return to our situation. By Proposition 5.1(1), the higher direct image $R^2 \hat{J}_{H*}^f \mathbf{C}_{\hat{X}_f}$ is a constant sheaf isomorphic to $\mathcal{H}^* \times (R^2 \hat{J}_{H*}^f \mathbf{C}_{\hat{X}_f})_0$ and we get an isomorphism

$$\Phi_\lambda^f : (R^2 \hat{J}_{H*}^f \mathbf{C}_{\hat{X}_f})_\lambda \to (R^2 \hat{J}_{H*}^f \mathbf{C}_{\hat{X}_t})_0 .$$

Using the primitive form $\hat{\omega}_f$, we define the following mapping:

$$\begin{aligned} P_{\hat{\omega}_f}: \mathcal{H}^* &\to H^2(\hat{X}_f(0), \mathbf{C}) \\ P_{\hat{\omega}_f}(\lambda) &= \Phi_\lambda^f([\hat{\omega}_f(\lambda)]) \end{aligned} \tag{5.2}$$

Let $\hat{\imath}: \hat{X}_f \hookrightarrow T^*(G/N)/H$ be the inclusion. Then we get the sheaf homomorphism

$$\hat{\imath}^*: R^2\hat{J}_{H*}\mathbf{C}_{T^*(G/N)/H} \to R^2\hat{J}^f_{H*}\mathbf{C}_{\hat{X}_f}.$$

This sheaf homomorphism induces the homomorphism of stalks

$$\hat{\imath}_0^*: (R^2\hat{J}_{H*}\mathbf{C}_{T^*(G/N)/H})_0 \to (R^2\hat{J}^f_{H*}\mathbf{C}_{\hat{X}_f})_0.$$

By (3.9) and (5.2), we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{H}^* & \xrightarrow{P_{\hat{\omega}_f}} & H^2(\hat{X}_f(0), \mathbf{C}) \\ & \searrow^{P_{\hat{\omega}}} & \uparrow \hat{\imath}_0^* \\ & & H^2(T^*(G/B), \mathbf{C}). \end{array}$$

Now let $\mathbf{B}$ be the set of all Borel subalgebras of $\mathscr{G}$. Since all Borel subalgebras of $\mathscr{G}$ are conjugate under the adjoint action of G, we identify $\mathbf{B}$ and G/B. Define the subvariety $\mathbf{B}_f$ of $\hat{J}_H^{-1}(0) = T^*(G/B)$ by

$$\mathbf{B}_f = \hat{J}_H^{-1}(f).$$

Note that $\mathbf{B}_f$ is the exceptional divisor of the resolution $\hat{X}_f(0) \to X_f([0]_W)$. Moreover, $\mathbf{B}_f$ can be identified with the following subvariety of $\mathbf{B}$:

$$\mathbf{B}_f \cong \{\mathscr{B}' \in \mathbf{B} | f \in \mathscr{B}'\}$$

(cf. [St]). Steinberg and Tits showed that $\mathbf{B}_f$ is a non-empty union of projective curves and the configuration of the projective curves is described by a homogeneous Dynkin diagram (cf. [St]).

Theorem 5.2 (*P. Slodowy* [*Sl2*] *section* 4.3 *and* 4.5)

(1) *The fibre of* $\hat{J}_H^f$ (*equivalently* J_r^f) *is homotopy equivalent to* $\mathbf{B}_f$. *Therefore, the monodromy representation of* $\pi_1(\mathscr{H}_r^*/W)$ *on* $H^2(X_f([\lambda_0]_W, \mathbf{C})$ *induces a representation of* W *on* $H^2(\mathbf{B}_f, \mathbf{C})$.

(2) *Let* $\imath^*: H^2(\mathbf{B}, \mathbf{C}) \to H^2(\mathbf{B}_f, \mathbf{C})$ *be the homomorphism induced by the inclusion* $\imath: \mathbf{B}_f \to \mathbf{B}$. *Then* $\imath^*$ *is a* W*-equivariant linear isomorphism.*

Remark 3 In this section, we have assumed that $\mathscr{G}$ is a simple Lie algebra of type A_l, D_l and E_l. For the other cases, one needs to modify the statement of Theorem 5.4 (2) (see P. Slodowy [Sl2] Sect. 4.5).

By Corollary 3.3, Theorem 5.2 and the diagram (5.3), $P_{\hat{\omega}_f}$ is a W-equivariant linear isomorphism.

Let $\hat{J}^f_{G,\lambda}$ be the restriction of the momentum mapping $\hat{J}_{G,\lambda}: \hat{J}_H^{-1}(\lambda) \to \mathscr{G}^*$ to $\hat{X}_f(\lambda)$ for every $\lambda \in \mathscr{H}^*$ (cf. Theorem 2.5 (2)). Let λ_0 be a point of $\mathscr{H}_r^*$.

Since $\hat{J}_H^f : \hat{X}_f \to \mathscr{H}^*$ is a simultaneous resolution of $J^f : X_f \to \mathscr{H}^*/W$, the mapping

$$\hat{J}_{G,\lambda_0}^f : \hat{X}^f(\lambda_0) \to X_f(\lambda_0)$$

is a holomorphic isomorphism. Hence we have an isomorphism

$$(\hat{J}_{G,\lambda_0}^f)^{-1*} : H^2(\hat{X}_f(\lambda_0), \mathbf{C}) \to H^2(X_f([\lambda_0]_W), \mathbf{C}) .$$

Then we define the following mapping:

$$P : \mathscr{H}^* \to H^2(X_f([\lambda_0]_W), \mathbf{C})$$

$$P(\lambda) = (\hat{J}_{G,\lambda_0}^f)^{-1*} \circ (\Phi_{\lambda_0}^f)^{-1} \circ \Phi_\lambda^f([\hat{\omega}_f(\lambda)]) . \tag{5.4}$$

Since $P_{\hat{\omega}_f}$ is a W-equivariant linear isomorphism, the mapping P is also a W-equivariant linear isomorphism. Moreover, by (5.4) and (5.1), we obtain

$$P|_{\mathscr{H}_r^*} = \hat{P} .$$

Summarizing the above discussion, we obtain the following theorem which is already proved by K. Saito ([Sa3]), E.J. Looijenga (Lo1], [Lo2]) and P. Slodowy ([Sl3]):

Theorem 5.3 ([*Sa*3], [*Lo*1], [*Sl*3]) *The period mapping* $\hat{P} : \mathscr{H}_r^* \to H^2(X_f([\lambda_0]_W), \mathbf{C})$ *extends to the* W*-equivariant linear isomorphism*

$$P : \mathscr{H}^* \to H^2(X_f([\lambda_0]_W) .$$

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