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Global existence of low regularity solutions of non-linear wave equations

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1 Introduction

In this work we consider the Cauchy problem for systems of hyperbolic non-linear equations of semilinear type in Minkowski space $\mathbb{R}^{n+1}$:

$$\Box\, u = F(u, \partial u) \tag{1.1}$$

$$u(0,x) = f(x)\ ,\, u_t(0,x) = g(x), \tag{1.2}$$

where $u = (u^I)_{I=1}^N$ stands for a $\mathbb{R}^N$-valued function defined in Minkowski space $\mathbb{R}^{n+1}$. The non-linear function $F(y,z)$ is a C^∞ function on a neighborhood of the origin satisfying $F(0,0) = 0$. Here $\mathbb{R}^{n+1}$ denotes Minkowski space , consisting of points $(x^0, x^1, ..., x^n)$, $x^0 = t$, equipped with the flat metric $\eta = \mathrm{diag}(-1, +1, ..., +1)$, $\Box = \partial_0^2 - \partial_1^2 - ... - \partial_n^2$ is the wave operator, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$,α=0,1,...,n . We write $\partial u = (\partial_0 u, \partial_1 u, ..., \partial_n u)$ for the full space-time gradient of u . The initial data f, g are taken in the weighted Sobolev spaces $(f, g) \in H^{s,s-1}(\mathbb{R}^n) \times H^{s-1,s}(\mathbb{R}^n)$, where the norm $|| \cdot ||_{s,\delta}$, $s \in Z^+$, $\delta > 0$, is defined by (cf. [3] , [5]):

$$||u||_{s,\delta} = (\sum_{i=0}^{s} \int_{\mathbb{R}^n} (1 + |x|^2)^{\delta+i} |\nabla^i u(x)|^2 dx)^{1/2}. \tag{1.3}$$

Here one is interested in the existence of global solutions to (1.1)-(1.2) in case the initial data norms are sufficiently small. It is known ([9] , [5]) that if $n > 3$ global small solutions exist. In the critical case n=3 , it is also known that global solutions cannot exist unless the nonlinearity F satisfies a certain structural

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*** Work supported by Sonderforschungsbereich 256 of the Deutsche Forschungsgemeinschaft condition called the null-condition ([9] , [5]). This means that for every null vector $\xi \in \mathbb{R}^{3+1}$ (that is to say, $\xi \in \mathbb{R}^{3+1}$, $\eta^{\mu\nu}\xi_\mu\xi_\nu = 0$) we have:

$$\frac{\partial^2 F}{\partial z_\mu \partial z_\nu}(0,0)\xi^\mu\xi^\nu = 0. \tag{1.4}$$

Roughly speaking , it means that the quadratic part of $F(u, \partial u)$ is a linear combination with constant coefficients of the basic null forms:

$$Q_0(\partial u^I, \partial u^J) = \eta^{\alpha\beta}\partial_\alpha u^I \partial_\beta u^J, \tag{1.5}$$

$$Q_{\alpha\beta}(\partial u^I, \partial u^J) = \partial_\alpha u^I \partial_\beta u^J - \partial_\beta u^I \partial_\alpha u^J. \tag{1.6}$$

For the sake of simplicity let us just consider here the case when the non-linearity F is of type (1.5)-(1.6). The more general case can be dealt with in a similar way by appealing to the Sobolev embedding theorem.

Exploiting the special decay properties of the null forms (1.5)- (1.6) it is possible to establish the existence of small global solutions([5]) in case the initial data satisfy the regularity requirements :

$$f \in H^{3,2}(\mathbb{R}^3) \ , \ g \in H^{2,3}(\mathbb{R}^3). \tag{1.7}$$

Recently, Klainerman and Machedon ([10] , check also [2]) were able to improve the regularity requirements on the initial data (1.2) and obtain a sharp local existence theorem for equations of type (1.1) in case the nonlinearity $F(u, \partial u)$ satisfies the null condition. The main idea relies on estimating space-time norms of the whole quadractic forms:

$$\int\int_{\mathbb{R}^3 \times [0,t_*]} |Q(\partial u, \partial u)|^2 dtdx \tag{1.8}$$

in terms of the initial data norms. In particular , one is able to obtain local solutions in H^2 provided the initial data are in $H^2 \times H^1$. For general nonlinearities and three dimensional wave equation Ponce and Sideris ([11]) showed that the local solution is in H^s for $s > 2$, by using the Strichartz estimate in place of the Sobolev inequality.

The aim of this paper is to investigate if those low-regularity local solutions can exist globally in time. The affirmative answer to this question can be summarized in the following result:

Theorem 1.1 *Let $F(z)$ be a C^∞ function defined on a neighborhood of the origin, with $F(0) = 0$, satisfy the null-condition (1.5)-(1.6). Suppose also that the Cauchy data lie in the low-regularity space $(f, g) \in H^{2,1}(\mathbb{R}^3) \times H^{1,2}(\mathbb{R}^3)$. Then, there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, if*

$$||f||_{H^{2,1}} + ||g||_{H^{1,2}} \leq \epsilon, \tag{1.9}$$

then there exists a unique global solution $u \in L^\infty([0, +\infty), H^{2,1}(\mathbb{R}^3))$ to problem (1.1)-(1.2) taking at time $t = 0$ the data (f, g) and verifying :

$$\int\int_{\mathbb{R}^{3+1}} (1+|t+|x||)^2(1+|t-|x||)^2|Q(\partial u,\partial u)|^2 dtdx +$$
$$\int\int_{\mathbb{R}^{3+1}} (1+|t+|x||)^4(1+|t-|x||)^4|\partial Q(\partial u,\partial u)|^2 dtdx < +\infty \quad (1.10)$$

for any of the null forms (1.5)- (1.6) .

Moreover the solution has the following decay property:

$$|u(t,x)| \leq \frac{C}{(1+|t+|x||)(1+|t-|x||)}, \quad (1.11)$$

where C depends only on the norms $||f||_{2,1}$ *and* $||g||_{1,2}$.

The idea of the proof consists of using the conformal compactification method of Penrose and reducing the global problem to a local setting, where estimates of the Klainerman-Machedon type could in principle be proved. Due to the curvature of the Einstein manifold $E = \mathbb{R} \times S^3$, one obtains a hyperbolic operator with non-constant coefficients, so that the task of proving such an estimate reduces to establishing some kind of L^2-estimates for Fourier integral operators arising in the parametrix construction of the solution. Here one does not need to exploit the exact structure of the manifold S^3, the proof being absolutely identical for any compact manifold . We rely only on the finite dependence domain argument for solutions of hyperbolic equations, which allows a localization of the estimates . The basic ingredient consists of a Paley-Littlewood decomposition of the initial data on S^3 in order to isolate the terms in the quadratic forms carrying the main singularities causing the loss of derivatives in the Strichartz estimate . These terms are very sensitive to the fact that the quadractic forms satisfy the null condition (1.4) and their estimation is reduced to establishing the L^2-continuity of an oscillatory integral operator:

$$\mathcal{T} : L^2(\mathbb{R}^6) \longrightarrow L^2(\mathbb{R}^4).$$

A suitable parameterization reduces the problem to proving the L^2-continuity of a local Fourier integral operator

$$\mathcal{T} : L^2(\mathbb{R}^4) \longrightarrow L^2(\mathbb{R}^4)$$

for which we can apply a criterium due to E. Stein ([13]).

The plan of the work is the following. In section 2 we review some basic facts concerning the conformal compactification method and the statement of the main theorems in terms of the compactified setting. In section 3 we introduce the parametrix construction of solutions to the wave equation and isolate the terms carrying the main singularities . Finally, in section 4 we consider the estimates for the oscillatory integrals. In order to apply the L^2-continuity criterium to our oscillatory integrals one is forced to consider a change of variables of the same type used by Klainerman and Machedon .

After this work was completed, we learned that earlier C. Sogge obtained independently similar estimates for quadractic forms on compact manifolds ([12]), by using also the theory of local Fourier integral operators.

2 The conformal transformation

In this section we review conformal compactification method (cf. [4], [5]). For general n, $n \geq 3$ one considers the Penrose compactification map:

$$P : \mathbb{R}^{n+1} \longrightarrow \mathbb{E}^{n+1} \tag{2.1}$$

which maps Minkowski space onto a bounded region of the so-called Einstein cylinder $(\mathbb{E}^{n+1}, g)$. This is the space-time manifold $\mathbb{E}^{n+1} = \mathbb{R} \times S^n$ equipped with the time coordinate $T \in \mathbb{R}$ and $0 < R < \pi$, $\omega \in S^{n-1}$ parameterizing $(\cos R, \sin R\omega) \in S^n$. The metric g is the product metric:

$$\begin{aligned} g &= -dT^2 + d\omega_n^2 \\ &= -dT^2 + dR^2 + \sin^2 R d\omega_{n-1}^2, \end{aligned} \tag{2.2}$$

where $d\omega_{n-1}^2$ is the usual line element of S^{n-1}. Penrose's compactification map is given in local coordinates as:

$$\begin{aligned} P : \mathbb{R}^{n+1} &\longrightarrow \mathbb{E}^{n+1} \\ (t, r, \omega) &\longmapsto (T, R, \omega), \end{aligned} \tag{2.3}$$

where $r = |x|$, $\omega = \frac{x}{|x|} \in S^{n-1}$, and T,R are defined by:

$$T = \arctan(t + r) + \arctan(t - r), \tag{2.4}$$

$$R = \arctan(t + r) - \arctan(t - r). \tag{2.5}$$

The image $P(\mathbb{R}^{n+1}) \subset \mathbb{E}^{n+1}$ is the bounded region of the cylinder characterized by the conditions $-\pi < T + R < \pi$, $-\pi < T - R < \pi$, $0 \leq R < \pi$ and its boundary $\partial P(\mathbb{R}^{n+1})$ is called the infinity of Minkowski space. Observe that in the region of the cylinder $\mathbb{E}^{n+1}$ covering Minkowski space-time the new time runs only in the bounded interval $-\pi < T < \pi$. Also, the weighted Sobolev spaces $H^{s,s-1}(\mathbb{R}^n)$ are mapped isomorphically onto $H^s(S^n)$. The crucial property satisfied by the Penrose mapping P is that it is a conformal isometry:

$$\eta = \Omega^2 g, \tag{2.6}$$

where Ω is the function :

$$\Omega = \frac{2}{(1 + |t + |x||^2)^{1/2} . (1 + |t - |x||^2)^{1/2}} \tag{2.7}$$

In particular one can exploit the conformal invariance of the wave operator under conformal changes of metric as follows. Assume that the function $\tilde{u}$ solves (1.1) in $\mathbb{R}^{n+1}$ with nonlinearity $\tilde{F}$. Define a new function u on $\mathbb{E}^{n+1}$ according to the relation $\tilde{u} = \Omega^{\frac{n-1}{2}} u$. It follows that the new function u will solve the non-linear wave equation on the cylinder $\mathbb{E}^{n+1}$:

$$\Box_g u - \frac{(n-1)^2}{4} u = F(u, \nabla u) \tag{2.8}$$

with F defined through

$$\tilde{F}(\tilde{y}, \tilde{z}) = \Omega^{\frac{n+3}{2}} F(y, z). \tag{2.9}$$

The relation between $(\tilde{y}, \tilde{z})$ and (y, z) is given in terms of an analytic bundle map $E : T^*\mathbf{E}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ defined solely in terms of the Penrose map (2.1). More precisely, for every $\alpha' = 0, 1, ..., n$ one has:

$$E^{\mu}_{\alpha'} = \Omega I^{\mu}_{\alpha'} + K^{\mu} \Lambda_{\alpha'}, \tag{2.10}$$

where $I_{\alpha'}$ and K are C^{∞} vector fields on $\mathbf{E}^{n+1}$ and $\Lambda_{\alpha'}$ are C^{∞} functions on $\mathbf{E}^{n+1}$ (cf. [5] for an explicit description). Also:

$$E^{\mu}_{\alpha'} \nabla_{\mu} \Omega = \Omega Y_{\alpha'} \tag{2.11}$$

for some C^{∞} functions $Y_{\alpha'}$ on $\mathbf{E}^{n+1}$. The new dependent variables on the cylinder are defined through the equations:

$$\tilde{y} = \Omega^{\frac{n-1}{2}} y, \tag{2.12}$$

$$\tilde{z}_{\alpha'} = \Omega^{\frac{n-1}{2}} [E^{\mu}_{\alpha'} z_{\mu} + \frac{n-1}{2} y Y_{\alpha'}]. \tag{2.13}$$

From now on we shall consider only the case $n = 3$. One has to show now that in terms of the new coordinates our nonlinear forms Q will be a product of Ω^3 with functions which are analytic on the cylinder $\mathbf{E}^{3+1}$. Let us consider first the form Q_0. In local coordinates on $\mathbf{E}^{3+1}$ we have $\Omega = cosT + cosR$. By straightforward differentiation one obtains $\nabla_{\mu}\Omega\nabla^{\mu}\Omega = -\Omega_T^2 + \Omega_R^2 = \Omega(cosT - cosR)$ and then it is possible to show that

$$\begin{aligned} Q_0(\tilde{u}, \tilde{v}) &= \Omega^4 g^{\mu\nu} \nabla_{\mu} u \nabla_{\nu} v \\ &+ \Omega^3 [a_{\alpha}(T, R, \omega) u \nabla_{\alpha} v + b_{\alpha}(T, R, \omega) \nabla_{\alpha} u v + c(T, R, \omega) u v] \\ &= \Omega^4 Q_0(u, v) \\ &+ \Omega^3 [a_{\alpha} u \nabla_{\alpha} v + b_{\alpha} \nabla_{\alpha} u v + c.uv] \end{aligned} \tag{2.14}$$

where a, b, c are C^{∞} functions on $\mathbf{E}^{3+1}$. In case of the forms $Q_{\alpha'\beta'}$ the situation is not so manifest. We have:

$$\begin{aligned} Q_{\alpha'\beta'}(\tilde{u}, \tilde{v}) &= \Omega^2 [(E^{\mu}_{\alpha'} u_{\mu} + u Y_{\alpha'})(E^{\nu}_{\beta'} v_{\nu} + v Y_{\beta'}) \\ &- (E^{\mu}_{\alpha'} v_{\mu} + v Y_{\alpha'}).(E^{\nu}_{\beta'} u_{\nu} + u Y_{\beta'})] \\ &= \Omega^2 [E^{\mu}_{\alpha'} E^{\nu}_{\beta'} (u_{\mu} v_{\nu} - v_{\mu} u_{\nu}) \\ &+ E^{\mu}_{\alpha'} Y_{\beta'} (u_{\mu} v - u v_{\mu}) + E^{\nu}_{\beta'} Y_{\alpha'} (v_{\nu} u - v u_{\nu})]. \end{aligned} \tag{2.15}$$

Inserting the expression (2.10) for E in the above expression we see that all terms appearing in the first summand will be manifestly the product of Ω with a function which is analytic on the cylinder, with the possible exception of the term

$$K^{\mu} K^{\nu} \Lambda_{\alpha'} \Lambda_{\beta'} \tag{2.16}$$

In this case though, the contraction with the null form will vanish:

$$K^{\mu}K^{\nu}\Lambda_{\alpha'}\Lambda_{\beta'}(u_{\mu}v_{\nu} - u_{\nu}v_{\mu}) = 0. \qquad (2.17)$$

The last summands are considered in the same fashion and one verifies that the troublesome terms (which are not manifestly the product of Ω with a function which is analytic on the cylinder) can be grouped as:

$$\begin{aligned} K^{\mu}\Lambda_{\alpha'}Y_{\beta'}(u_{\mu}v - uv_{\mu}) &+ K^{\nu}\Lambda_{\beta'}Y_{\alpha'}(v_{\nu}u - vu_{\nu}) \\ &= (\Lambda_{\alpha'}Y_{\beta'} - \Lambda_{\beta'}Y_{\alpha'})(Ku.v - u.Kv) \\ &= (\Lambda \wedge Y)_{\alpha'\beta'}(Ku.v - u.Kv). \qquad (2.18) \end{aligned}$$

It is possible to show (cf. [5]) that for all choices $\alpha', \beta' = 0, 1, \ldots, 3$, the functions $(\Lambda \wedge Y)_{\alpha'\beta'}$ are C^{∞} functions on $\mathbf{E}^{3+1}$ so that eventually we obtain:

$$\begin{aligned} Q_{\alpha'\beta'}(\tilde{u}, \tilde{v}) &= \Omega^3[k^{\mu\nu}_{\alpha'\beta'}(\nabla_{\mu}u\nabla_{\nu}v - \nabla_{\nu}u\nabla_{\mu}v) \\ &+ a^{\mu}_{\alpha'}\nabla_{\mu}uv + b^{\mu}_{\alpha'}\nabla_{\mu}v.u + c.uv] \\ &= \Omega^3[k^{\mu\nu}_{\alpha'\beta'}Q_{\mu\nu}(u, v) \\ &+ a^{\mu}_{\alpha'}\nabla_{\mu}uv + b^{\mu}_{\alpha'}\nabla_{\mu}v.u + c.uv], \qquad (2.19) \end{aligned}$$

where $k^{\mu\nu}_{\alpha'\beta'}$, $a^{\mu}_{\alpha'}$, $b^{\mu}_{\alpha'}$ and c are C^{∞} functions on $\mathbf{E}^{3+1}$. This reduces our task to estimating the quadractic forms:

$$Q_0(u, v) = g^{\mu\nu}\nabla_{\mu}u\nabla_{\nu}v \qquad (2.20)$$

$$Q_{\mu\nu}(u, v) = \nabla_{\mu}u\nabla_{\nu}v - \nabla_{\nu}u\nabla_{\mu}v \qquad (2.21)$$

It turns out that the original problem of finding global solutions to (1.1)- (1.2) reduces to finding local solutions [1] in $L^{\infty}([-\pi, \pi], H^2(S^3))$ of the problem:

$$(\Box_g - 1)u = Q(\nabla u, \nabla u) \qquad (2.22)$$

with Cauchy data:

$$u(0, x) = u_0(x) \in H^2(S^3)\ ,\ u_t(0, x) = u_1(x) \in H^1(S^3) \qquad (2.23)$$

and with non-linearity Q of type (2.20)-(2.21). This is exactly the problem considered in [10] in Minkowski space. The whole approach hinges now upon the possibility of establishing space-time estimates of Klainerman-Machedon type for the manifold S^3 . Our main Theorem 1.1 shall be then a simple consequence of the following:

Theorem 2.1 *Let (u, v) be the corresponding solutions to the following linear inhomogeneous wave equations on S^3:*

$$(\Box_g - 1)u = F \qquad (2.24)$$

with Cauchy data $u(0, x) = u_0(x) \in H^2(S^3), u_t(0, x) = u_1(x) \in H^1(S^3)$ and:

$$(\Box_g - 1)v = G \qquad (2.25)$$

[1] In the rest of this work we rename our coordinates on the cylinder by lowercase letters. The new time is just renamed $t \in [-\pi, \pi]$ for example.

with data $v(0,x) = v_0(x) \in H^2(S^3)$,$v_t(0,x) = v_1(x) \in H^1(S^3)$, *where* F *and* G *are given functions* $F,G \in H^1(S^3)$. *It follows that there exists a time interval* $0 \leq t \leq t_*$ *and a constant* $c > 0$, *independent of* u *and* v *such that for any null forms (2.20)-(2.21) the following estimate is verified:*

$$\int\int_{S^3\times[0,t_*]} |\nabla Q(u,v)|^2 dtdvol_{S^3} \leq c(||u_0||_{H^2} + ||u_1||_{H^1} + \int_0^{t_*} ||\nabla F(s,\cdot)||_{L^2} ds) \cdot(||v_0||_{H^2} + ||v_1||_{H^1} + \int_0^{t_*} ||\nabla G(s,\cdot)||_{L^2} ds). \tag{2.26}$$

Once this estimate is proved one can set a Picard iteration scheme, as in [10] and obtain a local solution to problem (2.22)-(2.23) in the time interval $[0,t_*]$. In particular, the solution will verify the estimate:

$$||u(t_*,\cdot)||_{H^2(S^3)} \leq c||u(0,\cdot)||_{H^2(S^3)}. \tag{2.27}$$

Since the Einstein metric on $S^3 \times \mathbb{R}$ is time-independent, it is possible to iterate this estimate N times, with $N > \frac{\pi}{t_*}$, by taking the initial data $||u(0,\cdot)||_{H^2(S^3)}$ so small that estimate (2.1) can be applied in every step. Property (1.10) in Minkowski space-time will now be a simple consequence of the main estimate (2.26) on the Einstein cylinder . This follows from the transformation properties (2.12)-(2.13) of the Penrose transform, the relation between the volume elements $dtdx = \Omega^{-4} dTd\mathrm{vol}_{S^3}$ and the transformation relations (2.14)-(2.19) for the null forms. From the boundness of the H^2-norm on S^3,the Sobolev embedding and relation (2.12) we deduce in a similar vein the decay property (1.11).

In order to prove Theorem 2.1 we appeal to Duhamel's principle and the following:

Theorem 2.2 *Let* (u,v) *denote, as in Theorem 2.1, the respective solutions to the wave equations (2.24)-(2.25) with vanishing inhomogeneous terms* $F = G = 0$ *and with Cauchy data* $u_0(x) = v_0(x) = 0$ *and* $u_1(x) = f(x)$, $v_1(x) = h(x)$, $(f,h) \in H^1(S^3)$. *If* Q *denotes any of the null forms (2.20)-(2.21), then the following estimate will be verified:*

$$\int\int_{S^3\times[0,t_*]} |Q(u,v)|^2 dtdvol_{S^3} \leq C||f||_{L^2(S^3)} \cdot ||h||_{H^1(S^3)} \tag{2.28}$$

for some $t_* > 0$ *and constant* $C > 0$ *independent of the initial data.*

By appealing to the finite dependence domain argument for solutions to the wave equation, it is possible to localize our estimates, so that Theorem 2.2 will actually follow from the proposition:

Proposition 2.1 *Let* (u,v) *be the solutions to the wave equations (2.24)-(2.25) as in Theorem 2.2. Assume in addition that the initial data* $(f,h) \in H^1(S^3)$ *is supported in a fixed geodesic ball* $B(\mathbf{1},\epsilon)$ *of radius* ϵ *centered at a fixed north pole* $\mathbf{1} = (1,0,0,0) \in S^3$, *where* $0 < \epsilon < \frac{\pi}{2}$ *is a sufficiently small real number.*

It follows that there exists a time interval $0 \leq t \leq t_*$ and a constant $c > 0$, independent of u and v such that for any null form (2.20)-(2.21) the following estimate is verified:

$$\int\int_{S^3\times[0,t_*]} |Q(u,v)|^2 dt dvol_{S^3} \leq C\|f\|_{L^2(S^3)} \cdot \|h\|_{H^1(S^3)}. \tag{2.29}$$

Proof of Theorem 2.2. Let $0 < \epsilon < \frac{\pi}{2}$ and t_* the universal constants given in Proposition 2.1. These numbers do not depend on the size of the initial data. Consider now a partition of unity $\{\varphi_k\}_{k=1}^N \in C_0^\infty(S^3)$, $N = N(\epsilon)$ such that each φ_k is supported in a geodesic ball of very small radius , chosen in such a way that for any k, the supports of the functions φ_j having non-empty intersection with suppφ_k are contained in a geodesic ball of radius less than ϵ. Let then δ denote the infimum of the geodesic distance between any two disjoint suppports of the members of the covering. Localize the initial data (f, h):

$$f = \sum_{k=1}^N \varphi_k f = \sum_{k=1}^N f_k \;, \; h = \sum_{k=1}^N \varphi_k h = \sum_{k=1}^N h_k \tag{2.30}$$

and denote the corresponding solutions of the Cauchy problem (2.24)-(2.25) by u_k and v_k respectively. By bilinearity we can write:

$$Q(u,v) = \sum_{k,l=1}^N Q(u_k, v_l). \tag{2.31}$$

There are now two possibilities. Either both f_k and h_l are supported in a geodesic ball of radius at most ϵ, and in this case the estimate will follow from Proposition 2.1, or their supports are distant at least δ from each other. In particular, by taking t_* even smaller and recalling the finite dependence domain property of solutions to the wave equation, we conclude that u_k and v_l have disjoint supports so that $Q(u_k, v_l) = 0$ and there is nothing to be estimated in this case. This concludes the proof of Theorem 2.2.

In the next section we shall develop the necessary Fourier analysis tools and prove Proposition 2.1.

3 Fourier representation of the quadratic forms

The first step to represent the quadratic forms (2.20)-(2.21) consists of deriving an approximation of the solution $u(t,x)$ of the wave equation on the n-dimensional sphere S^n, $n \geq 3$. We consider the problem on $\mathbb{E}^{n+1}$:

$$\Box_g u - \frac{(n-1)^2}{4} u = 0 \tag{3.1}$$

subject to the initial conditions:

$$u(0,x) = 0 \;, u_t(0,x) = f(x) \tag{3.2}$$

when $t = x^0 = 0$. Here $f \in H^1(S^n)$. Initially we shall restrict our attention to functions which are supported in a neighborhood $B(\mathbf{1}, \delta)$ of the north pole $\mathbf{1} = (1, 0, ..., 0)$ of S^n. We shall consider a local chart $x : B(\mathbf{1}, \delta) \longrightarrow \mathbb{R}^n$ and denote the coordinates of $p \in B(\mathbf{1}, \delta)$ by $x(p) = (x^1, ..., x^n)$. In local coordinates the metric read as:

$$g_{ij}(x) = \delta_{ij} + \frac{x^i x^j}{\sqrt{1 - |x|^2}} \tag{3.3}$$

and the wave operator as:

$$\Box_g u = \partial_t^2 u - \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u). \tag{3.4}$$

According to the general theory[2] the solution to problem (3.1) can be approximated by its parametrix. More precisely, for any $N \in Z^+$ we can write $u(t, x)$ solving equation (3.1) as:

$$u(t, x) = U(f)(t, x) + R(f)(t, x), \tag{3.5}$$

where U is a sum of local Fourier integral operators of type:

$$I_\pm(f)(t, x) = \int_{\mathbb{R}^n_\xi} \exp i\phi_\pm(t, x, \xi) a_{-1}(t, x, \xi) \hat{f}(\xi) d\xi \tag{3.6}$$

and the remainder R is a smoothing operator of infinite order mapping continuosly for any s and any N:

$$R : H^s \longrightarrow H^{s+N}. \tag{3.7}$$

The structure of the operators (3.6) can be totally described in terms of the metric g. Here $a_{-1}(t, x, \xi)$ is a classical symbol $a_{-1} \in S^{-1}(\mathbb{R}^{3+1} \times \mathbb{R}^3)$ homogeneous of order -1 obtained by solving the corresponding transport equations [3]. The phase functions $\phi_\pm(t, x, \xi)$ are homogeneous of order one obtained by solving the eikonal equation [4]:

$$\partial_t \phi_\pm(t, x, \xi) = \pm [g^{ij}(x) \partial_i \phi_\pm(t, x, \xi) . \partial_j \phi_\pm(t, x, \xi)]^{1/2} \tag{3.8}$$

subject to the initial data:

$$\phi_\pm(0, x, \xi) = x \cdot \xi. \tag{3.9}$$

It is sufficient to our purposes to know that the solution to the eikonal equation (3.8) will exist for some time $t \in [0, t_*]$. The same remark applies to the solution of the transport equations defining $a_{-1}(t, x, \xi)$. Moreover, by considering a cut-off function $\kappa(\xi)$ near $\xi = 0$, we can always write

$$a_{-1}(t, x, \xi) = \kappa(\xi) a_{-1}(t, x, \xi) + (1 - \kappa(\xi)) a_{-1}(t, x, \xi)$$

and absorb the last term in the remainder R so that we can always assume that $a_{-1}(t, x, \xi) = 0$ for all $|\xi| \leq 1$, say.

[2] We shall follow here [15], chapter VI.1, for example.
[3] Equations (1.49)-(1.50) in [15], vol.2, pages 311-312.
[4] Equation (1.41) in [15], vol.2, page 310

We turn now our attention to the proof of Proposition 2.1. Let u and v be the corresponding solutions to problem (3.1)- (3.2) with respective Cauchy data $f, h \in H^1(S^n)$. The quadractic forms $Q(u,v)$ will be written as a linear combination of terms involving the remainders R and terms of oscillatory type:

$$\begin{aligned} Q(u,v) &= Q(I_\pm(f), I_\pm(h)) \\ &+ Q(I_\pm(f), R(h)) + Q(R(f), I_\pm(h)) \\ &+ Q(R(f), R(h)) \\ &= I + II + III. \end{aligned} \tag{3.10}$$

For simplicity of notation we shall consider in detail only the case when both phase functions in (3.8) are chosen with the + sign and simply drop this subscript. The other cases can be treated in a totally analogous fashion.

The terms of type III in (3.10) involving the remainders can be estimated in a straightforward way, simply by using the regularizing property of the operators R. For the mixed terms of type II it is sufficient to apply,for fixed time t,the usual L^2-continuity theory of Fourier integral operators by appealing to the inequalities:

$$\|Q(I(f), R(h))(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq c\|\nabla I(f)(t,\cdot)\|_{L^2}.|\nabla R(h)(t,\cdot)|_{L^\infty}, \tag{3.11}$$

$$\|Q(R(f), I(h))(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq c|\nabla R(f)(t,\cdot)|_{L^\infty} \cdot \|\nabla I(h)(t,\cdot)\|_{L^2}. \tag{3.12}$$

The estimates follow then from the Sobolev embedding theorem. For example, the first term in (3.11) is bounded by noticing that $\nabla I(f)$ is a Fourier integral operator with symbol of type $a_0 \in S^0((\mathbb{R}^3 \times \mathbb{R}^3)$ and non-degenerate phase function (for fixed t) so that the L^2-continuity theory applies and we get:

$$\begin{aligned} \|Q(I(f), R(h))\|_{L^2(\mathbb{R}^3)} &\leq c\|f\|_{L^2(\mathbb{R}^3)}.|\nabla R(h)|_{L^\infty(\mathbb{R}^3)} \\ &\leq c\|f\|_{L^2(\mathbb{R}^3)}.\|\nabla^3 R(h)\|_{L^2(\mathbb{R}^3)} \\ &\leq c\|f\|_{L^2(\mathbb{R}^3)}.\|h\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

by the Sobolev embedding theorem and the regularizing property (3.7) of the remainder . In a similar way we can estimate:

$$\|Q(R(f), I(h))\|_{L^2(\mathbb{R}^3)} \tag{3.13}$$

by using (3.12) instead of (3.11). In order to estimate the main terms I in (3.10) we have to use the special structure of the null forms (2.20)-(2.21). These are the main terms which survive in the flat space theory.

In general, for any quadractic form :

$$Q(u,v) = q^{\alpha\beta}\partial_\alpha u \partial_\beta v + q^\alpha_{(1)}\partial_\alpha u.v + q^\alpha_{(2)}u.\partial_\alpha v + quv \tag{3.14}$$

it is not difficult to see that $Q(I(f), I(h))$ can be written in the following form as an oscillatory integral:

$$Q(f,h) = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i\Phi(t,x,\xi,\eta)} q(t,x,\xi,\eta)\hat{f}(\xi)\hat{h}(\eta)d\xi d\eta. \tag{3.15}$$

Here:

$$\Phi(t,x,\xi,\eta) = \phi(t,x,\xi)+\phi(t,x,\eta), \quad (3.16)$$
$$q(t,x,\xi,\eta) = a(t,x,\xi).b(t,x,\eta), \quad (3.17)$$

where $a,b \in S^0(\mathbb{R}^{3+1}\times\mathbb{R}^3)$ are classical symbols of zero order. It is possible to expand the symbols a,b as a sum of principal symbols a_0,b_0 and remainders in the symbol class $S^{-1}(\mathbb{R}^{3+1}\times\mathbb{R}^3)$. The principal symbols a_0,b_0 lead to the following representation of the main amplitude term $q_0(t,x,\xi,\eta)$:

$$\begin{aligned} q_0(t,x,\xi,\eta) &:= a_0(t,x,\xi)\cdot b_0(t,x,\eta) \\ &= q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi a_{-1}(t,x,\xi)b_{-1}(t,x,\eta), \end{aligned} \quad (3.18)$$

where we recall that $a_{-1},b_{-1} \in S^{-1}(\mathbb{R}^{3+1}\times\mathbb{R}^3)$. In the particular case we consider the quadratic null forms (2.20)-(2.21) we obtain for main symbols:

$$q_0(t,x,\xi,\eta) = g^{\alpha\beta}\partial_\alpha\phi(t,x,\xi)\partial_\beta\phi(t,x,\eta)a_{-1}(t,x,\xi)b_{-1}(t,x,\eta) \quad (3.19)$$

in the case of the form (2.20), or:

$$\begin{aligned} q_0(t,x,\xi,\eta) &= (\partial_\alpha\phi(t,x,\xi)\partial_\beta\phi(t,x,\eta)-\partial_\beta\phi(t,x,\xi)\partial_\alpha\phi(t,x,\eta)) \\ &\cdot a_{-1}(t,x,\xi)b_{-1}(t,x,\xi) \end{aligned} \quad (3.20)$$

in case of the forms (2.21). In both cases we notice that the main symbol vanishes over the set of vectors such that $\frac{\xi}{|\xi|}=\frac{\eta}{|\eta|}\in S^2$:

$$q_0(t,x,\frac{\xi}{|\xi|},\frac{\xi}{|\xi|})=0. \quad (3.21)$$

More precisely,one can prove that for every pair of vectors $\xi,\eta \neq 0$,we have

$$|q_0(t,x,\xi,\eta)| \le c|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}| \quad (3.22)$$

and for every multiindex α, $|\alpha|\ge 1$:

$$|\partial^\alpha_{t,x}q_0(t,x,\xi,\eta)| \le c_\alpha|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|. \quad (3.23)$$

This follows from the smoothness of the symbols and relation (3.21).

Let now $Q_0(f,h)$ be the oscillatory integral (3.15) with amplitude $q(t,x,\xi,\eta)$ replaced by $q_0(t,x,\xi,\eta)$. We shall prove that:

$$||Q(f,h)-Q_0(f,h)||_{L^2(\mathbb{R}^3\times\mathbb{R})} \le c||f||_{L^2}\cdot||h||_{H^1}. \quad (3.24)$$

Indeed, by noting that $ab-a_0b_0=(a-a_0)b_0+a(b-b_0)$ the remainder term will be a linear combination of product terms of type:

$$\mathcal{T}_0(f).\mathcal{T}_{-1}(h) \text{ and } \mathcal{T}_{-1}(f).\mathcal{T}_0(h), \quad (3.25)$$

where $\mathcal{T}_k(f)(t,x)$ denotes a local Fourier integral operator of the kind:

$$\mathcal{T}_k(f)(t,x) = \int_{\mathbf{R}^3} \exp i\phi(t,x,\xi) a_k(t,x,\xi)\hat{f}(\xi)d\xi \tag{3.26}$$

with $\phi(t,x,\xi)$ solving equation (3.8) and $a_k \in S^k(\mathbb{R}^{3+1} \times \mathbb{R}^3)$ a classical symbol homogeneous of degree k.

The first term can be estimated by applying the L^2-continuity (for fixed time t) of the Fourier integral operator $\mathcal{T}_0$ (cf. [8]):

$$\begin{aligned} ||\mathcal{T}_0(f)\mathcal{T}_{-1}(h)||_{L^2(\mathbf{R}^3_x)} &\leq c||\mathcal{T}_0(f)||_{L^2} \cdot |\mathcal{T}_{-1}(h)|_{L^\infty} \\ &\leq c||\hat{f}||_{L^2} \cdot ||\frac{\hat{h}}{(1+|\eta|^2)^{1/2}}||_{L^1} \\ &\leq c||f||_{L^2} \cdot ||\frac{1}{1+|\eta|^2}||_{L^2} \cdot ||h||_{H^1} \end{aligned} \tag{3.27}$$

and then, integrating over $t \in [0,\delta]$:

$$||\mathcal{T}_0(f)\mathcal{T}_{-1}(h)||_{L^2(\mathbf{R}^{3+1})} \leq c(\delta)||f||_{L^2} \cdot ||h||_{H^1}. \tag{3.28}$$

To estimate the term

$$\mathcal{T}_{-1}(f)\mathcal{T}_0(h) \tag{3.29}$$

we need to resort to a generalization of the usual Strichartz estimate ([14]):

Proposition 3.1 (cf. [1]) *If $\mathcal{T}_k(f)$ denotes the Fourier integral operator (3.26), then:*

$$||\mathcal{T}_0(f)||_{L^4(\mathbf{R}^{3+1})} \leq c||f||_{H^{1/2}(\mathbf{R}^3)}, \tag{3.30}$$
$$||\mathcal{T}_{-1}(f)||_{L^4(\mathbf{R}^{3+1})} \leq c||f||_{H^{-1/2}(\mathbf{R}^3)}. \tag{3.31}$$

From these two estimates it follows directly that:

$$\begin{aligned} ||\mathcal{T}_{-1}(f)\mathcal{T}_0(h)||_{L^2(\mathbf{R}^{3+1})} &\leq ||\mathcal{T}_{-1}(f)||_{L^4(\mathbf{R}^{3+1})} \cdot ||\mathcal{T}_0(h)||_{L^4(\mathbf{R}^{3+1})} \\ &\leq c||f||_{H^{-1/2}} \cdot ||h||_{H^{1/2}} \\ &\leq c||f||_{L^2} \cdot ||h||_{H^1} \end{aligned} \tag{3.32}$$

and this concludes the proof of estimate (3.24).

Finally, in order to estimate the leading term $Q_0(f,h)$ we need to use the specific form of the null forms. We shall carry out this in detail in the next section.

4 L^2- estimate of the leading oscillatory integrals

To complete the proof of Proposition 2.1 we introduce a Paley-Littlewood partition of unity for the initial data:

$$f = \sum_{j\geq 0} f_j \quad , \quad h = \sum_{k\geq 0} h_k \tag{4.1}$$

with

$$\hat{f_j}(\xi) = \varphi(\frac{\xi}{2^j})\hat{f}(\xi) \quad , \quad \hat{h_k}(\xi) = \varphi(\frac{\xi}{2^k})\hat{h}(\xi) \ , \ j,k \geq 1 \tag{4.2}$$

$$\hat{f_0}(\xi) = \sum_{j\leq 0} \varphi(\frac{\xi}{2^j})\hat{f}(\xi) \quad , \quad \hat{h_0}(\xi) = \sum_{j\leq 0} \varphi(\frac{\xi}{2^j})\hat{h}(\xi) \tag{4.3}$$

and $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying:

$$\text{supp}\varphi \subset \{1/2 \leq |\xi| \leq 2\} \tag{4.4}$$

and:

$$\sum_j \varphi(\frac{\xi}{2^j}) = 1. \tag{4.5}$$

The important thing to notice is that at most three of the terms in the sum:

$$f = \sum_{j\geq 0} f_j \tag{4.6}$$

(similarly for h) have supports intersecting non-trivially, so that for any Sobolev norm H^s, $s \in \mathbb{R}$ we have:

$$\frac{1}{C}\sum_{j\geq 0} ||f_j||^2_{H^s} \leq ||f||^2_{H^s} \leq C\sum_{j\geq 0} ||f_j||^2_{H^s}. \tag{4.7}$$

In order to estimate the space-time L^2-norm of the oscillatory integral :

$$Q_0(f,h) = \int\int_{\mathbf{R}^3\times\mathbf{R}^3} e^{i\Phi(t,x,\xi,\eta)} Q_0(t,x,\xi,\eta)\hat{f}(\xi)\hat{h}(\eta)d\xi d\eta \tag{4.8}$$

with q_0 denoting the main amplitude (3.18) we introduce the partition of unity (4.1), property (4.7) and the bilinearity to reduce proving the estimate:

$$||Q_0(f,h)||_{L^2(\mathbf{R}^{3+1})} \leq C||f||_{L^2}.||h||_{H^1} \tag{4.9}$$

to the estimates:

$$||Q_0(f_j,h_k)||_{L^2(\mathbf{R}^{3+1})} \leq C2^{-|j-k|/2}||f_j||_{L^2}.||h_k||_{H^1} \tag{4.10}$$

with a numerical constant C independent of j and k. It is easy to check estimate (4.10) in case the integer k dominates, that is to say, when

$$j \leq k + N_0 \tag{4.11}$$

for some sufficiently large integer N_0 to be specified later. Indeed, in this case we can represent $Q_0(f_j, h_k)$ as a product $\mathcal{T}_0(f_j)\mathcal{T}_0(h_k)$ of fourier integral operators of type (3.6) and apply directly the Strichartz estimate (3.30) to obtain:

$$\begin{aligned} ||Q_0(f_j, h_k)||_{L^2(\mathbf{R}^{3+1})} &\leq ||\mathcal{T}_0(f_j)||_{L^4(\mathbf{R}^{3+1})}.||\mathcal{T}_0(h_k)||_{L^4(\mathbf{R}^{3+1})} \\ &\leq C||f_j||_{H^{1/2}}.||h_k||_{H^{1/2}}. \end{aligned} \tag{4.12}$$

Using the fact that ξ is equivalent to 2^j on the support supp$\hat{f}_j(\xi)$ and the relation (4.11) we get:

$$\begin{aligned} ||f_j||_{H^{1/2}}.||h_k||_{H^{1/2}} &\leq C(2^j)^{1/2}(2^k)^{1/2}||f_j||_{L^2}||h_k||_{L^2} \\ &\leq C2^{j-k/2}.||f_j||_{L^2}2^k||h_k||_{L^2} \\ &\leq C2^{-|j-k|/2}||f_j||_{L^2}.||h_k||_{H^1} \end{aligned} \tag{4.13}$$

and this proves estimate (4.10) in case the condition (4.11) between j and k is verified. The remaining case

$$j \geq k + N_0 \tag{4.14}$$

can be treated by rewriting the oscillatory integral $Q_0(f_j, h_k)$ (defined through equation (4.8)) as a Fourier integral operator $\mathcal{Q}_{(jk)}(f \otimes (1-\Delta)^{1/2}h)$ acting on the pair $f \otimes (1-\Delta)^{1/2}h \in L^2 \times L^2$:

$$\begin{aligned} \mathcal{Q}_{(jk)}(t,x) &= Q_0(f_j, h_k)(t,x) \\ &= \int\int_{\mathbf{R}^3\times\mathbf{R}^3} e^{i\Phi(t,x,\xi,\eta)} \frac{q_0(t,x,\xi,\eta)}{(1+|\eta|^2)^{1/2}} \varphi(\frac{\xi}{2^j})\varphi(\frac{\eta}{2^k}) \\ &\quad \hat{f}(\xi)[(1-\Delta)^{1/2}h]\hat{}(\eta)d\xi d\eta. \end{aligned} \tag{4.15}$$

By a duality argument we see that estimate (4.10) will follow from:

$$||\mathcal{Q}^*_{(jk)}(\chi)||_{L^2(\mathbf{R}^3_\xi\times\mathbf{R}^3_\eta)} \leq c||\chi||_{L^2(\mathbf{R}^{3+1})}, \tag{4.16}$$

where

$$\mathcal{Q}^*_{(jk)}(\chi)(\xi,\eta) = \int\int_{\mathbf{R}^{3+1}} e^{-i\Phi(t,x,\xi,\eta)} \frac{\overline{q_0(t,x,\xi,\eta)}}{(1+|\eta|^2)^{1/2}} \varphi(\frac{\xi}{2^j})\varphi(\frac{\eta}{2^k})\chi(t,x)dtdx \tag{4.17}$$

is the adjoint operator to (4.15).

Now we rescale the variables $\xi \mapsto 2^j\xi$, $\eta \mapsto 2^j\eta$ so that the left hand side of (4.16) becomes:

$$(2^j)^3(\int\int_{\mathbf{R}^3\times\mathbf{R}^3} |\mathcal{Q}^*_{(jk)}(\chi)(2^j\xi, 2^j\eta)|^2 d\xi d\eta)^{1/2}. \tag{4.18}$$

Now we follow the references [2] and [10] to introduce a change of variables:

$$\zeta = \xi + \eta \ , \ \tau = |\eta| \ , \ \sigma = \frac{\eta}{|\eta|} \in S^2 \tag{4.19}$$

so that

$$\mathcal{Q}^*_{(jk)}(\chi)(2^j\xi, 2^j\eta) = \frac{\mathcal{T}_{(jk)}(\chi)(\zeta,\tau;\sigma)}{(1+(2^j\eta)^2)^{1/2}}, \tag{4.20}$$

where

$$\mathcal{T}_{(jk)}(\chi)(\zeta,\tau;\sigma) = \int\int_{\mathbb{R}^{3+1}} e^{-i\Phi(t,x,\zeta-\tau\sigma,\tau\sigma)2^j} q_{(jk)}(t,x,\zeta-\tau\sigma,\tau;\sigma)\chi(t,x)dtdx \tag{4.21}$$

is a Fourier integral operator with symbol

$$q_{(jk)}(t,x,\zeta,\tau;\sigma) = q_0(t,x,\zeta-\tau\sigma,\tau\sigma)\varphi(\zeta-\tau\sigma)\varphi(\frac{\tau\sigma 2^j}{2^k}) \tag{4.22}$$

depending on a parameter $\sigma \in S^2$ and such that:

$$\operatorname{supp} q_{(jk)}(t,x,\cdot,\cdot;\sigma) \subset \{\frac{1}{2} \leq |\zeta-\tau\sigma| \leq 2\ ,\ \frac{2^k}{2^{j+1}} \leq \tau \leq \frac{2^{k+1}}{2^j}\}. \tag{4.23}$$

The L^2-norm in the left-hand side of (4.16) can be rewritten as:

$$(2^j)^2(\int_{S^2_\sigma}[\int\int_{\mathbb{R}^3\times\mathbb{R}} |\mathcal{T}_{(jk)}(\chi)(\zeta,\tau;\sigma)|^2 d\zeta d\tau]d\sigma)^{1/2} \tag{4.24}$$

since $\frac{d\xi d\eta}{|\eta|^2} = d\zeta d\tau d\sigma$. Therefore, the desired estimate (4.16) will be a consequence of the following family of inequalities with $\sigma \in S^2$ considered as a parameter:

$$||\mathcal{T}_{(jk)}(\chi)(\cdot,\cdot,\sigma)||_{L^2(\mathbb{R}^3_\zeta\times\mathbb{R}^3_\tau)} \leq \frac{c}{(2^j)^2}||\chi||_{L^2(\mathbb{R}^{3+1})}.. \tag{4.25}$$

The oscillatory integral in (4.21) can be estimated with the aid of an inequality derived in [13] for some general oscillatory integral operators . We shall employ the following technical tool:

Proposition 4.1 *Let $\lambda \geq 1$ be a large real parameter . For $\zeta \in \mathbb{R}^N$ consider the following oscillatory integral operator:*

$$\mathcal{T}_\lambda(\chi)(\zeta) = \int_{\mathbb{R}^N} e^{i\Phi(x,\zeta)\lambda} q(x,\zeta)\chi(x)dx, \tag{4.26}$$

where $q(x,\zeta) \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. Assume that the phase function $\Phi(x,\zeta)$ in (4.26) has a non-vanishing Hessian:

$$det[\frac{\partial^2\Phi(x,\zeta)}{\partial x\partial\zeta}] \neq 0. \tag{4.27}$$

Then for any λ, the operator $\mathcal{T}_\lambda$

$$\mathcal{T}_\lambda : L^2(\mathbb{R}^N_x) \longrightarrow L^2(\mathbb{R}^N_\zeta) \tag{4.28}$$

is bounded and the following estimate is verified:

$$||\mathcal{T}_\lambda(\chi)||_{L^2(\mathbb{R}^N)} \leq c\lambda^{-N/2}||\chi||_{L^2(\mathbb{R}^N)} \tag{4.29}$$

In the proof of (4.29) only an integration by parts in the x-coordinates is used and we need that the symbol $q(x,\zeta)$ is uniformly bounded in the spatial variables: $|\partial_x^\alpha q(x,\zeta)| \le C_\alpha$, for $|\alpha| \le k_0$, k_0 some large enough natural number, uniformly in $\zeta \in \operatorname{supp} q(x,\cdot)$. When applied to our oscillatory integral (4.21), this condition is guaranteed exactly by the vanishing properties (3.22)-(3.23) of the symbols of the null forms. As a matter of fact, it turns out that Proposition 4.1 is not directly applicable, as the second Hessian:

$$\det \begin{bmatrix} \frac{\partial^2 \Phi}{\partial t \partial \tau} & \frac{\partial^2 \Phi}{\partial t \partial \zeta} \\ \frac{\partial^2 \Phi}{\partial x \partial \tau} & \frac{\partial^2 \Phi}{\partial x \partial \zeta} \end{bmatrix} \tag{4.30}$$

can vanish at $(t,x) = 0$. A simple computation based on the eikonal equation (3.8) shows that this Hessian vanishes (at $(t,x) = 0$) exactly at $\frac{\zeta}{|\zeta|} = \sigma$. To overcome this difficulty we change the variables exactly in the same manner of the flat space case, following Klainerman and Machedon ([10]). Consider the (singular) change of variables :

$$(\zeta,\tau) \to (\zeta,\tilde{\tau}) \; , \; \tilde{\tau} = \tau + |\zeta - \tau\sigma|. \tag{4.31}$$

Note that the Jacobian of this change of variables is proportional to

$$\frac{d\tilde{\tau}}{d\tau} = 1 + \frac{\tau - \zeta\cdot\sigma}{|\zeta - \tau\sigma|} = \frac{|\zeta - \tau\sigma| + \tau - \zeta\cdot\sigma}{|\zeta - \tau\sigma|}. \tag{4.32}$$

The denominator $|\zeta - \tau\sigma|$ never vanishes on the support of $q_{(jk)}(t,x,\cdot,\cdot;\sigma)$ in view of (4.23) and the assumption (4.14). This enables one to conclude that $\frac{d\tilde{\tau}}{d\tau} = 0$ on the support of $q_{(jk)}(t,x,\cdot,\cdot;\sigma)$ if and only if $\frac{\zeta}{|\zeta|} = \sigma$. In order to compute the Hessian (4.27) of the phase function $\Phi(t,x,\xi,\eta) = \phi(t,x,\xi) + \phi(t,x,\eta)$ in the new coordinates we use the fact that ϕ solves the eikonal equation (3.8) and that $g^{jk}(0) = \delta^{jk}$ to write:

$$\phi(t,x,\xi) = t|\xi| + x\cdot\xi + \psi(t,x,\xi) \tag{4.33}$$

with remainder ψ satisfying for every multiindex α:

$$|\partial^\alpha \psi(t,x,\xi)| \le C|\xi|(t^2 + |x|^2). \tag{4.34}$$

The phase function $\Phi(t,x,\xi,\eta)$ in the new coordinates $(\zeta,\tilde{\tau})$ becomes:

$$\begin{aligned} \Phi(t,x,\zeta,\tilde{\tau}) &= t\tilde{\tau} + x\cdot\zeta + \psi(t,x,\zeta - \tau\sigma) + \psi(t,x,\tau\sigma) \\ &= t\tilde{\tau} + x\cdot\zeta + \psi(t,x,\frac{\zeta - \tau\sigma}{|\zeta - \tau\sigma|})|\zeta - \tau\sigma| + \psi(t,x,\sigma)\tau \end{aligned} \tag{4.35}$$

(recall (4.31) $\tilde{\tau} = \tau + |\zeta - \tau\sigma|$). Now we Taylor expand the function $\psi(t,x,\frac{\zeta-\tau\sigma}{|\zeta-\tau\sigma|})$ near $\sigma \in S^2$:

$$\begin{aligned} \psi(t,x,\frac{\xi}{|\xi|}) &= \psi(t,x,\sigma) + \psi'_\xi(t,x,\sigma)\cdot(\frac{\xi}{|\xi|} - \sigma) \\ &+ \frac{1}{2!}(\frac{\xi}{|\xi|} - \sigma)^T R(t,x,\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|} - \sigma) \end{aligned} \tag{4.36}$$

with Taylor remainder:

$$R(t,x,\frac{\xi}{|\xi|}) = \int_0^1 \psi''_\xi(t,x,\mu\sigma + (1-\mu)\frac{\xi}{|\xi|})d\mu. \tag{4.37}$$

Using the relation

$$\zeta - \tau\sigma - |\zeta - \tau\sigma|\sigma = \zeta - \tilde{\tau}\sigma, \tag{4.38}$$

obtain finally:

$$\begin{aligned} \Phi(t,x,\zeta,\tilde{\tau}) &= t\tilde{\tau} + x\cdot\zeta + \psi(t,x,\sigma)\tilde{\tau} + \psi'_\xi(t,x,\sigma)\cdot(\zeta - \tilde{\tau}\sigma) \\ &+ \frac{1}{2!}(\frac{\zeta-\tau\sigma}{|\zeta-\tau\sigma|} - \sigma)^T R(\frac{\zeta-\tau\sigma}{|\zeta-\tau\sigma|} - \sigma)\cdot|\zeta - \tau\sigma|. \end{aligned} \tag{4.39}$$

It is our task now to compute:

$$\det \begin{bmatrix} \frac{\partial^2\Phi}{\partial t\partial\tilde{\tau}} & \frac{\partial^2\Phi}{\partial t\partial\zeta} \\ \frac{\partial^2\Phi}{\partial x\partial\tilde{\tau}} & \frac{\partial^2\Phi}{\partial x\partial\zeta} \end{bmatrix}. \tag{4.40}$$

For this purpose we appeal to the following technical lemma:

Lemma 4.1 *Set $\xi = \zeta - \tau\sigma$ and $\tilde{\tau} = \tau + |\zeta - \tau\sigma|$. Then for any ζ with $|\zeta| \geq M\tau$, with $M > 0$ a sufficiently large constant, the following estimates are verified:*

$$\begin{aligned} |\partial_{\zeta,\tilde{\tau}}\xi| &\leq c|\frac{\xi}{|\xi|} - \sigma|^{-2}, & (4.41)\\ |\partial_{\zeta,\tilde{\tau}}(\xi - |\xi|\sigma)| &\leq c, & (4.42)\\ |\partial_{\zeta,\tilde{\tau}}(\frac{\xi}{|\xi|} - \sigma)| &\leq c|\frac{\xi}{|\xi|} - \sigma|^{-1}|\xi|^{-1}. & (4.43) \end{aligned}$$

Proof of Lemma 4.1. Given any function $f(\zeta,\tau)$, set:

$$f(\tilde{\zeta},\tilde{\tau}) = f(\zeta,\tau(\zeta,\tilde{\tau})) \tag{4.44}$$

where $\tau(\zeta,\tilde{\tau})$ is defined by the implicit function theorem from the equation (4.31) $\tilde{\tau} = \tau + |\zeta - \tau\sigma|$. The chain rule gives:

$$\partial_{\tilde{\tau}}\tilde{f} = \frac{\partial_\tau}{\partial_{\tilde{\tau}}}\partial_\tau f\ ,\ \partial_\zeta\tilde{f} = \partial_\zeta f + \frac{\partial_\tau}{\partial_\zeta}\partial_\tau f. \tag{4.45}$$

A straightforward differentiation with respect to $\tilde{\tau}$ and ζ leads to:

$$\begin{aligned} \frac{\partial\tau}{\partial\tilde{\tau}} &= 2|\frac{\xi}{|\xi|} - \sigma|^{-2}, & (4.46)\\ \frac{\partial\tau}{\partial\zeta} &= -2\frac{\xi}{|\xi|}|\frac{\xi}{|\xi|} - \sigma|^{-2}. & (4.47) \end{aligned}$$

From the chain rule (4.45) and the above relations we get:

$$\partial_{\tilde{\tau}}\xi = -2\left|\frac{\xi}{|\xi|} - \sigma\right|^{-2}\sigma \tag{4.48}$$

$$\partial_{\zeta}\xi = Id + 2\left(\frac{\xi}{|\xi|} \otimes \sigma\right)\left|\frac{\xi}{|\xi|} - \sigma\right|^{-2} \tag{4.49}$$

and then property (4.41) follows. Property (4.42) follows directly then by the formula:

$$\xi - |\xi|\sigma = \zeta - \tilde{\tau}\sigma. \tag{4.50}$$

Finally, property (4.43) follows from (4.41) and (4.42). This completes the proof of the lemma.

Using this lemma we can estimate the Hessian determinant of the remainder term:

$$\mathscr{R}(t, x, \zeta, \tilde{\tau}; \sigma) = \frac{1}{2!}\left(\frac{\zeta - \tau\sigma}{|\zeta - \tau\sigma|} - \sigma\right)^T R\left(\frac{\zeta - \tau\sigma}{|\zeta - \tau\sigma|} - s\right) \cdot |\zeta - \tau\sigma|. \tag{4.51}$$

Using the asymptotic relations (4.34) and (4.41)-(4.43) obtain finally:

$$\left|\det \begin{bmatrix} \frac{\partial^2 \mathscr{R}}{\partial t \partial \tilde{\tau}} & \frac{\partial^2 \mathscr{R}}{\partial t \partial \zeta} \\ \frac{\partial^2 \mathscr{R}}{\partial x \partial \tilde{\tau}} & \frac{\partial^2 \mathscr{R}}{\partial x \partial \zeta} \end{bmatrix}\right| \leq c(t + |x|) \tag{4.52}$$

for some constant C independent of $t, x, \sigma, \zeta, \tau$. It follows then that:

$$\begin{bmatrix} \frac{\partial^2 \Phi}{\partial t \partial \tilde{\tau}} & \frac{\partial^2 \Phi}{\partial t \partial \zeta} \\ \frac{\partial^2 \Phi}{\partial x \partial \tilde{\tau}} & \frac{\partial^2 \Phi}{\partial x \partial \zeta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Id \end{bmatrix} + O(t + |x|). \tag{4.53}$$

And then, in view of the fact that the support suppχ can be chosen with arbitrarily small diameter on the sphere S^3, then we can arrange the invertibility of the Hessian and be in position to apply Proposition 4.1, provided we have the boundedness (together with derivatives) of the amplitude function in the new coordinates $\zeta, \tilde{\tau}$. Now, changing variables in (4.25) we have:

$$||\mathscr{T}_{(jk)}(\chi)(\cdot,\cdot,\sigma)||_{L^2(\mathbf{R}^3_\zeta \times \mathbf{R}^1_\tau)} = ||\mathscr{T}_{(jk)}(\chi)(\cdot,\cdot,\sigma)\sqrt{\frac{\partial \tau}{\partial \tilde{\tau}}}||_{L^2(\mathbf{R}^3_\zeta \times \mathbf{R}^1_{\tilde{\tau}})} \tag{4.54}$$

where now

$$\mathscr{T}_{(jk)}(\chi)(\cdot,\cdot,\sigma)\sqrt{\frac{\partial \tau}{\partial \tilde{\tau}}} \tag{4.55}$$

is a Fourier integral operator with symbol (in the new coordinates):

$$q_{(jk)}\sqrt{\frac{\partial \tau}{\partial \tilde{\tau}}}. \tag{4.56}$$

Recalling now the relation:

$$\left(\frac{\partial \tau}{\partial \tilde{\tau}}\right)^{1/2} = \sqrt{2}\left|\frac{\xi}{|\xi|} - \sigma\right|^{-1} \tag{4.57}$$

and the property (3.22) satisfied by the symbols of the null condition nonlinearities, we obtain directly the desired boundedness of the symbol and we can apply Proposition 4.1 with $N = 4$ and $\lambda = 2^j$. This concludes the proof of the Proposition 2.1.

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