Math. Z. 219, 539 – 565 (1995)

Mathematische Zeitschrift

The representation theory of the Temperley–Lieb algebras

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Received 28 April 1993; in final form 4 March 1994

Contents

0 Introduction

The Temperley–Lieb algebras were first introduced in (Temperley and Lieb [1971]) where it was noted the single bond transfer matrices for the ice model and for the Potts model both satisfied the relations

$$\begin{aligned} u_i^2 &= \delta\, u_i \\ u_i u_{i\pm 1} u_i &= u_i \\ u_i u_j &= u_j u_i \ \text{ if } |i-j| > 1 \end{aligned}$$

where δ is a parameter that depends on the model. Subsequently it was realised that the single bond transfer matrices in other square lattice exactly solvable lattice models also satisfied these relations. Furthermore the expression for the transfer matrix in any of these models does not depend on the model.

It is then natural to consider the abstract algebras over the complex field defined by these relations and to study their representation theory. In particular these algebras act on the state space of each of these models and if the representation for some model can be decomposed then this gives a block diagonalisation of the transfer matrices for all values of the spectral parameter.

These relations were then found independently and studied in the fundamental paper (Jones 1983). In this paper the values $\delta = 2\cos\pi/e$ for e an integer are special. These are also the values of δ for lattice models which have a phase transition. In this paper Jones also defined a trace on these algebras which he subsequently used to define the Jones polynomial of a link.

The Jones polynomial of a link is a Laurent polynomial with integer co-efficients. In fact, to define the trace Jones constructed a basis of these algebras with the remarkable property that the product of any two basis elements is a power of δ times another basis element. In particular this shows that the algebras over the complex field are obtained from algebras over the polynomial ring $\mathbb{Z}[\delta]$ by tensoring with the complex numbers. This construction can be generalised to define a sequence of algebras over any commutative ring with a distinguished element. An alternative description of these algebras over $\mathbb{Z}[\delta]$ with this basis is implicit in the states model for the Jones polynomial given in (Kauffman 1987). This is the description of these algebras that we will be working with in this paper.

These algebras are also quotients of the Hecke algebras of the general linear groups. These are another sequence of finite dimensional algebras defined over $\mathbb{Z}[\delta]$. The representation theory of these algebras over arbitrary fields is a generalisation of the modular representation theory of the symmetric groups. This is a well developed branch of algebra. One of the purposes of this paper is to provide an introduction to this subject by using the Temperley–Lieb algebras as a worked example. With this in mind we have given outlines of proofs of all results used in this paper even where the result and the proof is the same as in published work by other authors. In several places we state theorems which are special cases of theorems which are known for the Hecke algebras. In these cases our proofs follow the proof of the more general result but may be more explicit.

The main objective of this paper is to determine all solutions of the Temperley–Lieb relations for matrices with complex entries. It is clear that, given a solution to the Temperley–Lieb relations, simultaneous conjugation by an invertible matrix gives another solution. This just corresponds to a change of basis and two solutions related in this way are regarded as equivalent. It is also clear that given two solutions taking the direct sums of the matrices gives another solution. A solution which can be written in this form after a change of basis is called decomposable. The precise sense in which we determine all solutions of the Temperley–Lieb relations is that we give a finite list of solutions with the property that any solution can be obtained from this list using these two operations of direct sum and change of basis. The result that such a list exists is a special feature of these algebras.