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Proper holomorphic mappings between bounded complete Reinhardt domains in $\mathbf{C}^2$

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0 Introduction

This is a typical feature of several complex variables that proper holomorphic mappings between special classes of domains have strong rigidity properties. In particular, proper holomorphic self mappings of certain domains are necessarily biholomorphic. This was proved by H. Alexander in the case of the unit ball [1], by S. Pinchuk for strictly pseudoconvex domains [13] and by E. Bedford and S. Bell for pseudoconvex domains with real analytic boundaries [2]. This was also shown to be true for special kinds of Reinhardt domains by Y. Pan [12], A. Chaouech [6] and M. Landucci and G. Patrizio [10]. More generally, S. Pinchuk proved that proper holomorphic mappings between different strictly pseudoconvex domains are unbranched and this was generalized by K. Diederich and J. Fornaess for mappings from strictly pseudoconvex to smooth bounded domains [8].

Mappings between different circular domains where studied in [3] by S. Bell who proved that they must be algebraic as soon as they preserve the origin. Mappings between particular classes of Reinhardt domains were investigated by G. Dini and A. Selvaggi [9] and M. Landucci and S. Pinchuk [11].

The aim of this paper is to give a classification of proper holomorphic mappings between bounded complete Reinhardt domains in $\mathbf{C}^2$. Our main result is the following:

Theorem 1 *Let* $f: \Omega_1 \to \Omega_2$ *be a proper holomorphic mapping between bounded, complete Reinhardt domains in* $\mathbf{C}^2$. *If* f *does not have the form* $f(z, w) = (Az^m, Bw^n)$ *or* (Aw^n, Bz^m) *where* $A, B \in \mathbf{C}^*$ *and* $m, n \in \mathbf{N}$ *then,*

* The authors came independently to this problem and started to cooperate while the second was visiting University of Lille in January 1993

after normalization and a possible permutation of variables, two possibilities may occur:

a) $\Omega_1 = \Omega_2 = \Delta^2$, *where* Δ *is the unit disk in* $\mathbf{C}$.

b) $\Omega_1 = \{|z|^{2a} + |w|^{2n} < 1\}$ *and* $\Omega_2 = \{|z|^{2b} + |w|^{2r} < 1\}$ *where* $n \in \mathbf{N}$, $r \in \mathbf{Q}^+$ *and* $a, b \in \mathbf{R}^+$. *Moreover, one of the two pairs* $\left(\frac{a}{b}, \frac{n}{r}\right), \left(\frac{a}{r}, \frac{n}{b}\right)$ *must belong to* $\mathbf{N}^2$.

It is worth emphazize that, in case b), non tirvial mappings (i.e. not of the form (Az^m, Bw^n)) do exist. When $r = 1$, the automorphism group of Ω_2 is non trivial. Thus, these mappings may be obtained by composing a mapping of the form (z^k, w^l) with an appropriate automorphism of Ω_2. When the automorphism groups of Ω_1 and Ω_2 are trivial, non trivial proper holomorphic mappings still exist, as the following example (due to Dini and Selvaggi–Primicerio [9]) shows. Take $\Omega_1 = \{|z|^4 + |w|^4 < 1\}$, $\Omega_2 = \{|z| + |w|^{2/p} < 1\}$ and $f_1 = \frac{1}{2}(z^2 + w^2)^2$, $f_2 = \left(\frac{1}{\sqrt{2}}\right)^p (w^2 - z^2)^p$ where $p \in \mathbf{N}$, $p > 2$.

As a consequence of Theorem 1, we obtain the following generalization of Alexander's result for $\mathbf{C}^2$:

Theorem 2 *Among bounded, complete, Reinhardt domains in* $\mathbf{C}^2$, *the bidiscs are the only ones which admit proper holomorphic self mappings that are not automorphisms.*

The paper is organized as follows. We first consider the pseudoconvex case. In Sect. 1, we study the effect of exceptional tori in the boundaries on the structure of mapping. This is used in Sect. 2, for showing that if the mapping is not splitting then the boundaries must be real analytic and strictly pseudoconvex outside coordinate hyperplanes. The remaining of the proof is based on the study of holomorphic tangent vector fields which are generated by a non splitting mapping. In Sect. 3, we show how certain holomorphic tangent vector fields may be used to characterize ellipsoïdal hypersurfaces of the form $|z|^{2n} + |w|^{2\beta} = 1 (n \in \mathbf{N},\ \beta \in \mathbf{R}^+)$ among strictly pseudoconvex and real analytic Reinhardt hypersurfaces. In Sect. 4, after completing the proof of Theorems 1 and 2 in the pseudoconvex case, we generalize them to the non pseudoconvex case by observing that proper mappings properly extends to holomorphic hulls (see Lemma 4.1).

Notations: For any $\theta =: (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ we shall denote by $e^{i\theta} \cdot z$ the point $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$.

For any Reinhardt domain Ω in $\mathbf{C}^n$ we shall denote by $b\Omega$ the boundary of Ω and by $b\Omega^*$ the part of $b\Omega$ which does not intersect the coordinate hyperplanes: $b\Omega^* = \{\eta \in b\Omega \text{ s.t. } \eta_1 \dots \eta_n \neq 0\}$.

For any $\eta \in b\Omega^*$ there exists a n-dimensional real torus T_η which is contained in $b\Omega^*$: $T_\eta = \{e^{i\theta} \cdot \eta; \theta \in \mathbf{R}^n\}$.

For any holomorphic mapping f from $\mathbf{C}^n$ to $\mathbf{C}^n$, we shall denote f' the linear tangent map to f and its Jacobian by det f'.

1 Effect of exceptional tori

It was proved by S. Bell (see [4]) that every proper holomorphic mapping $f:\Omega_1 \to \Omega_2$ between bounded complete Reinhardt domains extends holomorphically to some neighborhood of $\overline{\Omega}_1$. In the following proposition we assume that the target domain is pseudoconvex in order to show that this extension is locally open at most points of $b\Omega_1$. We restrict ourselves to the case of $\mathbf{C}^2$.

Proposition 1.1 *Let $f:\Omega_1 \to \Omega_2$ be a proper holomorphic mapping between bounded complete Reinhardt domains in $\mathbf{C}^2$. Assume that f extends holomorphically to some neighborhood $\tilde{\Omega}_1$ of $\overline{\Omega}_1$ and that Ω_2 is pseudoconvex. Then the extension $\tilde{f}$ of f is locally open at any point $\eta_0 \in b\Omega_1$ satisfying $\tilde{f}(\eta_0) \in b\Omega_2^*$.*

Proof. As a first step, we shall prove the existence of some neighborhood V of η_0 such that $\tilde{f}(V\backslash\overline{\Omega}_1) \cap \overline{\Omega}_2 = \emptyset$. Consider the logarithmic image of Ω_2 in $\mathbf{R}^2$: $\mathscr{L}(\Omega_2) = \{(L_n|u_1|, L_n|u_2|); (u_1, u_2) \in \Omega_2\}$. Since Ω_2 is pseudoconvex, $\mathscr{L}(\Omega_2)$ is convex. Then, for any $\eta \in b\Omega_1$, such that $\tilde{f}(\eta) \in b\Omega_2^*$, we may find real numbers $\alpha_1(\eta)$ and $\alpha_2(\eta)$ such that $\alpha_1(\eta)^2 + \alpha_2(\eta)^2 = 1$ and $\alpha_1(\eta)[x_1 - Ln|\tilde{f}_1(\eta)|] + \alpha_2(\eta)[x_2 - Ln|\tilde{f}_2(\eta)|] < 0$ on $\mathscr{L}(\Omega_2)$ as a function of $x = (x_1, x_2)$. By applying the Hopf lemma to the subharmonic function $\varphi_\eta(u) = \alpha_1(\eta)[Ln|f_1(u\eta)| - Ln|\tilde{f}_1(\eta)|] + \alpha_2[Ln|f_2(u\eta)| - Ln|\tilde{f}_2(\eta)|]$ on the unit disk $\{|u| < 1\}$, we find:

$$\frac{d}{dt}[\varphi_\eta(t)]_{t=1} \geqq M > 0, \quad \forall \eta \in V \cap b\Omega_1 ,$$

where V is a small neighborhood of η_0 such that $\tilde{f}(\eta) \in b\Omega_2^*$ for any $\eta \in V \cap b\Omega_1$. On the other hand, a straightforward computation reveals that:

$$\exists R > \text{s.t.}: \forall \eta \in V \cap b\Omega_1, \quad \forall t \in [1, R]: |\frac{d^2}{dt^2}\varphi_\eta(t)| \leqq K .$$

(Here the fact that $\tilde{f}_1(\eta)\tilde{f}_2(\eta) \neq 0$ for any $\eta \in V \cap b\Omega_1$ is crucial).

By using (1), (2) and Taylor's formula we get:

$$\forall \eta \in V \cap b\Omega_1, \forall t \in]1, \min\left(1 + 2\frac{M}{K}, R\right)[: f(t_\eta) \notin \overline{\Omega_2} .$$

Shrinking V yields to the announced property.

We are now in order to show that η_0 is an isolated point in $\tilde{f}^{-1}\{\tilde{f}(\eta_0)\}$ which means that $\tilde{f}$ satisfies Osgood's condition at η_0. Indeed, the connected component A_0 of η_0 in $\{z \in \tilde{\Omega}_1 \text{ s.t. } \tilde{f}(z) = \tilde{f}(\eta_0)\}$ is an analytic set in $\tilde{\Omega}_1$ which does not intersect Ω_1 ; it therefore suffices to verify that $A_0 \subset b\Omega_1$ since in that case A_0 would be a compact (and therefore discrete) analytic set in $\tilde{\Omega}_1$. If this would not be the case, we could find a point $\eta' \in A_0 \cap b\Omega_1$ and an arbitrarily small neighborhood U of η' such that $\tilde{f}(U\backslash\overline{\Omega_1}) \cap b\Omega_2 \neq \emptyset$. This would contradict the first step of the proof.

This ends the proof of Proposition 1.1 since Osgood's condition implies local openess (see [5], p. 327, Theorem 4).

In the remaining of the paper we shall identify the mapping f with its extension $\tilde{f}$. The aim of our next proposition is to show that the mapping is splitting as soon as it exchanges "small" families of tori in boundaries.

Proposition 1.2 *Let $f: \Omega_1 \to \Omega_2$ be a proper holomorphic mapping between complete bounded Reinhardt domains in $\mathbf{C}^2$. Assume that there are subsets $\mathcal{S}_j$ of $b\Omega_j^*$ with the following properties:*

1) *$\mathcal{S}_j$ are invariant by "rotations" i.e.: $\forall \eta \in \mathcal{S}_j, \forall \theta \in \mathbf{R}^2$: $e^{i\theta} \cdot \eta \in \mathcal{S}_j$;*
2) *$\mathcal{S}_j \neq \emptyset$ and $\mathcal{S}_j^\circ = \emptyset$;*
3) *$f(\mathcal{S}_1) \cap b\Omega_2^* \subset \mathcal{S}_2$ and $f^{-1}(\mathcal{S}_2) \cap b\Omega_1^* \subset \mathcal{S}_1$.*

Then the mapping is "splitting", i.e. it has the form $(f_1(z_1), f_2(z_2))$ or $(f_1(z_2), f_2(z_1))$.

Proof. We proceed in several steps:

First step: $\forall \eta \in \mathcal{S}_2, \forall \eta' \in f^{-1}(T_\eta) \cap b\Omega_1^*: f(T_{\eta'}) \subset T_\eta$.

Consider the function $\varphi(\theta_1, \theta_2) = \sum(|f_j(e^{i\theta} \cdot \eta')|^2 - |\eta_j|^2)^2$. If φ is not constant then there exist $\theta_0 \in \mathbf{R}^2$ such that grad $\varphi(\theta_0) \neq 0$. We may also assume that det $f'(e^{i\theta_0} \cdot \eta') \neq 0$. Let us set $\theta_0 = (0, 0)$ for simplicity. The non vanishing of grad φ implies that $f(T_{\eta'})$ is not locally contained in $T_{f(\eta')}$ around $f(\eta')$. We may therefore replace η' by an arbitrarly close point η'' on $T_{\eta'}$ such that $f(\eta'') \in b\Omega_2$ and $f(T_{\eta'}) = f(T_{\eta''})$ intersects $T_{f(\eta'')}$ transversally at $f(\eta'')$. Since by properties 1) and 3) $f(T_{\eta'})$ and $T_{f(\eta'')}$ are contained in $\mathcal{S}_2$, it follows that $\mathcal{S}_2$ contains an open neighborhood of $f(\eta'')$. This contradicts 2), therefore φ is constant and $f(T_{\eta'}) \subset T_\eta$.

Second step: $\forall \eta \in \mathcal{S}_2 : f^{-1}(T_\eta) \cap b\Omega_1^* = \bigcup_{j=1}^N T_{\eta'_j}$.

It follows from the first step that $f^{-1}(T_\eta) \cap b\Omega_1^* = \bigcup_\alpha T_{\eta'_\alpha}$, we have to show that this family of tori is actually finite. Assume to the contrary that there exists an infinite family of tori $(T_{\eta'_j})_{j=1}^\infty$ such that $f(T_{\eta'_j}) \subset T_\eta$ for all j. First we may observe that only a finite number of these tori may intersect any relatively compact set in $b\Omega_1^*$. Otherwise we could find $T_{\eta'} \subset b\Omega_1^*$ such that, after taking some subsequence, $\lim T_{\eta'_j} = T_{\eta'}$. Then $f(T_{\eta'}) \subset T_\eta$ and this is impossible since there are points on $T_{\eta'}$ where f defines a local biholomorphism. Now we show that $f(T_{\eta'}) = T_\eta$ for every $T_{\eta'} \in \{T_{\eta'_\alpha}\}$. Since $f(T_{\eta'})$ is obviously closed in T_η it suffices to show that $f(T_{\eta'})$ is open in T_η. Let $a \in T_{\eta'}$ and $b = f(a)$. According to Proposition 1.1 there is an open neighborhood V of a such that $W =: f(V)$ is an open neighborhood of b. By the above observation we may assume that $(\bigcup_\alpha T_{\eta'_\alpha}) \cap V = T_{\eta'} \cap V$, thus $W \cap T_\eta = f(V \cap T_{\eta'})$ and $f(T_{\eta'})$ is open in $T_{\eta'}$ at point b.

We are now in order to show that $\bigcup_\alpha T_{\eta'_\alpha}$ is finite. Indeed, if this family is not finite then η admits an infinite number of preimages in $b\Omega_1^*$. Let $\eta'_1, \ldots, \eta'_{m+1}$ be $(m+1)$ distinct preimages of η in $b\Omega_1^*$ where m denotes the multiplicity of f in Ω_1. It follows from the proof of Proposition 1 that there are open neighborhoods V_j of η'_j such that $V_j \cap V_k = \emptyset$ for $j \neq k$ and $f(V_j)$ are open neighborhoods of η. Then any point in $(\bigcap_j f(V_j)) \cap \Omega_2$ admits $(m+1)$ preimagese in Ω_1; this is impossible.

Third step: $\forall \eta \in \mathscr{S}_2 : \bigcup_{j=1}^N D_{\eta'_j} \subset f^{-1}(D_\eta) \subset \widehat{f^{-1}(T_\eta)}$.

For any compact $K \subset \mathbf{C}^2, \hat{K}$ denote the polynomially convex hull of K. It is well known that $\hat{K}$ coincides with the hull of K with respect to the class of continuous purisubharmonic functions on $\mathbf{C}^2$: $\mathscr{P}SH_0(\mathbf{C}^2)$. For convenience we shall note $f^{-1}(T_\eta) = K$. The maximum principle applied to $\rho \circ f$, where $\rho = \max(|z_1|^2 - |\eta_1|^2, |z_2|^2 - |\eta_2|^2)$, shows that $f[\bigcup_1^N D_{\eta'_j}] \subset D_\eta$. We shall now establish the second inclusion. Let $\rho \in \mathscr{PSH}_0(\mathbf{C}^2)$ be such that $\rho \leqq 0$ on K. Since f is proper, we may define a p.s.h. function Ψ on Ω by setting : $\Psi(z) = \sup\{\rho(w), w \in \Omega \text{ and } f(w) = z\}$.

Let η_0 be a fixed point in T_η and let (z_p) be a sequence of points in Ω_2 such that $\lim z_p = \eta_0$ and $\lim_{\eta_0} \sup \Psi = \lim \Psi(z_p)$. The mapping f being finite, there are points $w_p \in \Omega_1$ such that $f(w_p) = z_p$ and $\Psi(z_p) = \rho(w_p)$. After taking some subsequence we may assume that $\lim w_p = \eta' \in K$, then $\lim_{\eta_0} \sup \Psi = \lim \Psi(z_p) = \lim \rho(w_p) = \rho(\eta') \leqq 0$ and therefore $\Psi \leqq 0$ on D_η by the maximum principle. In other words we have proved that $f^{-1}(D_\eta) \subset \hat{K}$.

Fourth step: The mapping is splitting.

It follows from the second step that $f^{-1}(T_\eta) \subset \bigcup_{j=1}^N T_{\eta'_j} \cup \{z_1 z_2 = 0\}$. On the other hand, the hull of $\bigcup_{j=1}^N T_{\eta'_j} \cup \{z_1 z_2 = 0\}$ coincides with $\tilde{K} \cup \{z_1 z_2 = 0\}$ where $\tilde{K}$ denotes the logarithmically convex hull of $\bigcup_{j=1}^N D_{\eta'_j}$ (see [15]). Thus, one sees from the third step that there are real numbers r_1, r_2, r'_1, r'_2 such that, after a possible permutation of variables, $f\{|w_1| < r_1, |w_2| = r_2\} \subset \{|z_1| < |\eta_1|, |z_2| = |\eta_2|\}$ and $f\{|w_1| = r'_1, |w_2| < r_2\} \subset \{|z_1| = |\eta_1|, |z_2| < |\eta_2|\}$. It readily follows from these inclusions that $\frac{\partial f_2}{\partial w_1}$ and $\frac{\partial f_1}{\partial w_2}$ are identically zero on Ω_1. ■

Remark 1.3. It follows from the three last steps in the above proof that, if there exists $\eta \in b\Omega_2^*$ such that $f(T_{\eta'}) \subset T_\eta$ for any $\eta' \in f^{-1}(T_\eta)$, then the mapping f is splitting. This will be useful during the proof of Proposition 2.1.

2 Regularity of boundaries outside coordinate's hyperplanes

The sets of tori in $b\Omega_j^* (j = 1, 2)$ in a neighborhood of which $b\Omega_j^*$ are not real analytic or, when $b\Omega_j^*$ are real analytic, not strictly pseudoconvex will be shown to be "small" in the sense of Proposition 1.2. This way, we shall prove that, if the mapping is not splitting, then both $b\Omega_1^*$ and $b\Omega_2^*$ are nicely regular.

Proposition 2.1 *Let Ω_1 and Ω_2 be bounded, complete pseudoconvex Reinhardt domains in $\mathbf{C}^2$. Let $f\colon \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. If f is not splitting then $b\Omega_j^*$ are real analytic and strictly pseudoconvex.*

Proof. We first establish the real analyticity of $b\Omega_j^*$. Denote by $\mathscr{R}_j$ the set of points in $b\Omega_j^*$ in a neighborhood of which $b\Omega_j$ is real analytic. Let $\mathscr{S}_j$ denote the complementary of $\mathscr{R}_j$ in $b\Omega_j^*$. The set $\mathscr{S}_1$ and $\mathscr{S}_2$ are obviously invariant by rotations. In view of Proposition 1.2, we can prove that $\mathscr{S}_j$ are empty by showing that $\mathring{\mathscr{S}}_1 = \mathring{\mathscr{S}}_2 = \emptyset$ and that $f(\mathscr{S}_1) \cap b\Omega_2^* \subset \mathscr{S}_2, f^{-1}(\mathscr{S}_2) \cap b\Omega_2^* \subset \mathscr{S}_1$. Let us show that $\mathscr{R}_2$ is dense in $b\Omega_2^*$. Pick $\eta \in b\Omega_2^*$, we may assume that $f^{-1}(T_\eta) \cap b\Omega_1^* \neq \emptyset$ since the set of points satisfying this property is clearly dense in $b\Omega_2^*$. The mapping f being not splitting, it follows from Remark 1.3 that there exists $\eta' \in b\Omega_1^*$ such that $f(\eta') = \eta$ and $f(T_{\eta'}) \not\subset T_\eta$. Thus, we may find $\eta'' \in T_{\eta'}$ such that $f(\eta'')$ is arbitrarly close to T_η and the gradient of the function $[|f_1(e^{i\theta} \cdot \eta'')|^2 - |\eta_1|^2] + [|f_2(e^{i\theta} \cdot \eta'')|^2 - |\eta_2|^2]$ does not vanish at $\theta = (0, 0)$. Assume, for instance, that $\left(\frac{d}{d\theta_1}\right)_{\theta=0} |f_1(e^{i\theta_1}\eta_1'', \eta_2'')|^2 \neq 0$. Then the mapping $\Psi\colon \mathbf{R}^3 \to b\Omega_2^*$ defined by $\Psi(t, u, v) = [e^{iu} f_1(e^{it}\eta_1'', \eta_2''), e^{iv} f_2(e^{it}\eta_1'', \eta_2'')]$ is a local parametrization of $b\Omega_2^*$ at $f(\eta'')$. (A straightforward computation shows that Ψ' has rank 3 at the origin). Thus $f(\eta'') \in \mathscr{R}_2$; we have shown the density of $\mathscr{R}_2$ in $b\Omega_2^*$ and therefore $\mathring{\mathscr{S}}_2 = \emptyset$. Let us now prove that $f(\mathscr{S}_1) \cap b\Omega_2^* \subset \mathscr{S}_2$. Let $\eta' \in \mathscr{S}_1$, since det f' cannot vanish identically on some neighborhood of η' in $T_{\eta'}$, we may find $\eta'' \in T_{\eta'}$ arbitrarly close to η' and such that det $f'(\eta'') \neq 0$. Thus $f(\eta') \in \overline{\mathscr{S}_2} \cap b\Omega_2^* = \mathscr{S}_2$. Since, by Proposition 1.1, f is locally open on $b\Omega_1^*$, $\mathring{\mathscr{S}} = \emptyset$ follows from $\mathring{\mathscr{S}}_2 = \emptyset$. It remains to show that $f^{-1}(\mathscr{S}_2) \cap b\Omega_1^* \subset \mathscr{S}_1$. Let $\eta' \in \mathscr{R}_1$, we have to prove that $\eta =: f(\eta') \in \mathscr{R}_2$. If det $f'(\eta') \neq 0$ this is clear. It not, let $Z_{\eta'}$ be the connected component of η' in $\{z \in \tilde{\Omega}_1 \text{ s.t. } \det f'(z) = 0\}$, where $\tilde{\Omega}_1$ denotes a neighborhood of $\overline{\Omega_1}$ on which f extends. Let U be a neighborhood of η' such that $U \cap b\Omega_1 \subset \mathscr{R}_1$, it suffices to show that $f(Z_{\eta'} \cap U)$ is not a neighborhood of η in T_η. If this would be the case then, after shrinking U, we would have $f(Z_{\eta'} \cap U) \subset T_\eta$ and the maximum principle applied to $\exp\left[\frac{f_1}{\eta_1} + \frac{f_2}{\eta_2} - 2\right]$ on $Z_{\eta'}$ would imply that $f \equiv \eta$ on $Z_{\eta'}$. It would follow from the first step of the proof of Proposition 1.1 that $Z_{\eta'} \subset b\Omega_1$, but this is impossible.

We now prove that $b\Omega_j^*$ are strictly pseudoconvex. Denote by $\mathscr{R}_j$ the set of strictly pseudoconvex points in $b\Omega_j^*$. Let $\mathscr{S}_j = \mathscr{R}_j^C \cap \overline{\mathscr{R}}_j \cap b\Omega_j^*$. We shall again use Proposition 1.2 in order to show that $\mathscr{S}_j$ are empty. Since $\mathscr{S}_j$ are obviously of empty interior and are invariant by rotations, we only must check that $f(\mathscr{S}_1) \cap b\Omega_2^* \subset \mathscr{S}_2$ and $f^{-1}(\mathscr{S}_2) \cap b\Omega_1^* \subset \mathscr{S}_1$. It is actually enough to prove the following inclusions: $f(\mathscr{R}_1) \cap b\Omega_2^* \subset \mathscr{R}_2$ and $f(\mathscr{R}_1^C \cap b\Omega_1^*) \subset \mathscr{R}_2^C$. Since the mapping cannot branch at strictly pseudoconvex points (see [8]. Lemma 4), the first inclusion is clear. For the second one we just use the fact that for any $\eta' \in b\Omega_1$, there are points on $T_{\eta'}$ which are arbitrarly close to η' and where det f' does not vanish. ∎

3 Holomorphic tangent vector fields on Reinhardt hypersurfaces

Let V be an open set in $\mathbf{C}^2$ and S be an hypersurface defined by $S =: \{(z, w) \in V$ s.t. $\rho(z, w) = 0\}$ where ρ is a smooth real valued function on V with non vanishing gradient on S. An holomorphic tangent vector field for S defined on V is a vector field of the form:

$$Q = a(z, w)\frac{\partial}{\partial z} + b(z, w)\frac{\partial}{\partial w}$$

where a and b are holomorphic functions on V, which satisfies the following tangency condition:

$$\rho(z, w) = 0 \Rightarrow \operatorname{Re}\left[a(z, w)\frac{\partial \rho}{\partial z}(z, w) + b(z, w)\frac{\partial \rho}{\partial w}(z, w)\right] = 0\,.$$

We shall also have to consider finite-valued holomorphic tangent vector fields. Let us describe this notion in a concrete situation. Consider a domain Ω in $\mathbf{C}^2$ and an analytic subset A of codimension 1 in Ω. An holomorphic vector field Q on Ω is said to be m-valued if:

1. it consists of m holomorphic branches near any point of $\Omega \backslash A$ and each of these branches may be holomorphically extended along any path in $\Omega \backslash A$ always staying within the values of Q.
2. the components of Q are locally bounded on Ω.

Such vector fields naturally arise by pushing forward holomorphic vector fields by proper holomorphic mappings. If $f: \Omega_1 \to \Omega_2$ is a proper holomorphic mapping and $Q = a\frac{\partial}{\partial z} + b\frac{\partial}{\partial w}$, then the vector field $f_*(Q)$ defined by $(f_* \cdot Q)(z, w) = f' \circ f^{-1}(z, w). Q$ outside $f^{-1}[f(\{\det f' = 0\})] =: A$ is an m-valued holomorphic vector field on Ω_2.

We shall say that an m-valued holomorphic vector field Q on Ω is tangent for $b\Omega$ at some point η if there exists an holomorphic vector field H, defined on some neighborhood V of η, which is tangent to for $b\Omega \cap V$ in the previous sense and agrees with a branch of Q on $V \cap \Omega$.

The main result of this section deals with the characterization of "ellipsoïdal" Reinhardt hypersurfaces in $\mathbf{C}^2$ by means of holomorphic tangent vector fields:

Theorem 3.1 *Let Ω be a complete, pseudoconvex, bounded Reinhardt domain in $\mathbf{C}^2$. Assume that $b\Omega^*$ is strictly pseudoconvex and real analytic.*

(α) *Suppose that there exists an holomorphic tangent vector field for $b\Omega$, defined on a neighborhood of some torus $T_\eta (\eta \in b\Omega^*)$ and which is not a rotation vector field. Then, after normalization and a possible permutation of variables, Ω takes the form: $\Omega = \{|z|^{2a} + |w|^{2n} < 1\}$ where $a > 0$ and $n \in \mathbf{N}$.*

(β) *Suppose that there exists an m-valued holomorphic vector field Q on Ω which is tangent for $b\Omega^*$ at any point of some torus $T_\eta (\eta \in b\Omega^*)$. If, moreover, Q has the following form: $Q = a(z^{1/r}, w^{1/\sigma})\frac{\partial}{\partial z} + b(z^{1/r}, w^{1/\sigma})\frac{\partial}{\partial w}$ where a and b are holomorphic functions on Ω and $\tau, \sigma \in \mathbf{N}^*$, then*

$\Omega = \{|z|^{2a} + |w|^{2r} < 1\}$ where $a > 0$, $r \in \mathbf{Q}^+$, after normalization and a possible permutation of variables.

We shall use the following technical lemma for proving this theorem.

Lemma 3.2 *Let $\varphi :]0, 1[\to]0, +\infty[$ be a function which satisfies the following differential equation:*

$$\text{(i)} \qquad t^{k/r}\varphi^{l/\sigma}[\varphi B_1 - t\varphi' A_1] + [\varphi B_2 - t\varphi' A_2] = 0$$

where $k, l \in \mathbf{Z}; \tau, \sigma \in \mathbf{N}$ and $A_1, A_2, B_1, B_1 \in \mathbf{C}$.

Assume, moreover, that φ satisfies the following boundary conditions:

$$\text{(ii)} \qquad \lim_{t\to 0} t\varphi'(t) = 0 \,.$$

$$\text{(iii)} \qquad \lim_{t\to 1} \frac{\varphi(t)}{\varphi'(t)} = 0 \,.$$

$$\text{(iv)} \qquad \lim_{t\to 0} \varphi(t) = 1, \quad \lim_{t\to 1} \varphi(t) = 0 \,.$$

Then, if $k^2 + l^2 \neq 0$, one has: $\varphi = (1 - t^{k/\tau})^\beta$ or $\varphi = (1 - t^\beta)^{\sigma/l}$, where $\beta \in [0, 1]$.

Let us start by the *proof of Theorem 3.1.* The boundary $b\Omega$ may be described by an equation $|w|^2 = \varphi(|z|^2)$. Since Ω is pseudoconvex, φ satisfies the conditions (ii) and (iii) of Lemma 3.2, the condition (iv) results from normalization. Let $Q =: a(z, w)\frac{\partial}{\partial z} + b(z, w)\frac{\partial}{\partial w}$ be an holomorphic vector field satisfying the assumptions of part (α) of our theorem. The holomorphic functions a and b may be expended in Laurent series in some corona neighborhood V_η of T_η. Thus, we may write:

$$Q = \sum a_{kl} z^k w^l \frac{\partial}{\partial z} + \sum b_{kl} z^k w^l \frac{\partial}{\partial w} \,.$$

After parametrizing $b\Omega^*$ by $z = re^{iu}$, $w = \rho e^{i\delta}$, $\rho^2 = \varphi(r^2)$, the tangency condition

$$\mathrm{Re}[\overline{w} b(z, w) - \overline{z}\varphi'(|z|^2) a(z, w)] \equiv 0 \quad \text{on } V_\eta \cap b\Omega$$

becomes:

$$\mathrm{Re}\left[\sum_{k,l} e^{i(ku+lv)} (b_{k,l+1}\rho^{l+2} r^k - a_{k+1,l} r^{k+2} \rho^l \varphi'(r^2))\right] \equiv 0 \,.$$

The vanishing of the above Fourier series in (u, v) yields to the following family of equations:

$$b_{k,l+1}\rho^{l+2} r^k - a_{k+1,l} r^{k+2}\rho^l \varphi'(r^2) + \overline{b}_{-k,-l+1}\rho^{-l+2} r^{-k} - \overline{a}_{-k+1,-l} r^{-k+2-l}\varphi'(r^2) = 0 \,.$$

Since φ is real analytic on $]0,1[$, these equations, which are a priori satisfied on some neighborhood of $|\eta_1| = r$, are actually satisfied on $]0,1[$. After multiplying by $r^k\rho^l$ and setting $t = r^2$ we obtain the following differential equations on $]0,1[$:

$$t^k\varphi^l[\varphi b_{k,l+1} - t\varphi' a_{k+1,l}] + [\varphi\bar{b}_{-k,1-l} - t\varphi'\bar{a}_{1-k,-l}] = 0\,.$$

Since Q is not a rotation vector field, it contains other non zero terms than $a_{1,0}z\frac{\partial}{\partial z}$ or $b_{0,1}w\frac{\partial}{\partial w}$. Thus, one of the above equations must be satisfied for $k\neq 0$ or $l\neq 0$ and the conclusion follows from Lemma 3.2 with $\sigma = \tau = 1$.

In the case of some vector field satisfying the assumptions of part (β), the same computations yield to equations of the following form:

$$t^{k/\tau}\varphi^{l/\sigma}[\varphi b_{k,l+\sigma} - t\varphi' a_{k+\tau,l}] + [\varphi\bar{b}_{-k,\sigma-l} - t\varphi'\bar{a}_{\tau-k,-l}] = 0\,.$$

and the conclusion is also obtained via Lemma 3.2. ∎

We now give a *proof of Lemma 3.2.*

Since equation (i) keeps the same form after multiplication by $t^{-k/\tau}\varphi^{-l/\sigma}$, we only have to consider the following cases:

(1) $k > 0$ and $l \neq 0$,
(2) $k > 0$ and $l = 0$,
(3) $k = 0$ and $l > 0$.

(1) Making $t \to 0$ and using (ii) and (iv) one sees that $B_2 = 0$. Dividing by φ', making $t \to 1$ and using (ii) and (iv), one gets $A_2 = 0$. Then the equation becomes $\varphi B_1 - t\varphi' A_1 = 0$, making $t \to 0$ and using (ii) and (iv) one obtains $B_1 = 0$ and $A_1 = 0$.

(2) Like in case (1) one sees that $B_2 = 0$. Equation (i) may therefore be written as $t^{k/\tau}[\varphi B_1 - t\varphi' A_1] - t\varphi' A_2 = 0$. Dividing by φ', making $t \to 1$ and using (iii) and (iv), one gets $A_2 = -A_1 =: A$. Thus the equation is: $A\varphi'[1 - t^{k/\tau}] = B_1\varphi t^{(\frac{l}{\tau}-1)}$ and, if the constants A and B_1 are not both equal to zero, φ must be of the form $(1 - t^{k/\tau})^\beta$ for some $\beta \in]0,1]$.

(3) The equation is $\varphi^{l/\sigma}[\varphi B_1 - t\varphi' A_1] + [\varphi B_2 - t\varphi' A_2] = 0$. Dividing by φ', making $t \to 1$ and using (iii) and (iv) one gets $A_2 = 0$. Then, making $t \to 0$ and using (ii) and (iv) one gets $B_2 = -B_1 =: B$. The equation may therefore be written $B(1 - \varphi^{l/\sigma}) = A_1 t\varphi'\varphi^{(\frac{l}{\sigma}-1)}$ and one has $(1 - t^\beta)\sigma/l$ for some $\beta > 0$. ∎

In order to apply Theorem 3.1 to our problem we will have to extend germs of holomorphic tangent vector fields along tori. The following proposition shows how this is possible.

Proposition 3.3 *Let $\mathscr{H}$ be a real analytic, strictly pseudoconvex Reinhardt, hypersurface in $\mathbf{C}^n$. Let $\vec{X} =: \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}$ be some holomorphic vector field which is defined in a neighborhood U of some point $\eta \in \mathscr{H}$ and tangent to $\mathscr{H}$. Then $\vec{X}$ may be extended, as an holomorphic tangent vector field for $\mathscr{H}$, along any path lying in the torus T_η.*

Proof. Let V and W be open balls in $\mathbf{C}^n$ centered at η such that $\overline{V} \subset W \subset U$. We shall show that $\vec{X}$ may be holomorphically extended on any open set of form $\bigcup_{l\in[0,\tau]} e^{il\theta} \cdot W =: W^{\theta}_{0,\tau}$ where $\tau > 0$ and $\theta \in \mathbf{R}^n$ are arbitrarily chosen. This will be sufficient since any path in T_η is contained in some open set of the form: $\bigcup_{j=1}^{N} (W_j)^{\theta_j}_{0,\tau_j}$, where $W_1 = W$ and $W_{j+1} = e^{i\tau_j\theta_j} \cdot W_j$. Let us assume that $\vec{X}$ is extendable on $W^{\theta}_{0,\tau_0}(\tau_0 > 0)$ but not on $W^{\theta}_{0,\tau}$ for any $\tau > \tau_0$. In the remaining of the proof we shall identify $\vec{X}$ with its extension on W_{0,τ_0}. Consider a sequence $(R_p)_p$ of automorphisms of $\mathscr{H}$ defined by $R_p(z) =: e^{i\frac{\varepsilon\theta}{p}} \cdot z$, $\varepsilon > 0$. The holomorphic tangent vector fields $\vec{X}_p =: (R_p)_*\vec{X}$ are defined on $R_p(W^{\theta}_{0,\tau}) = W^{\theta}_{\frac{\varepsilon}{p},\tau_0+\frac{\varepsilon}{p}}$ but are not extendable on $W^{\theta}_{\frac{\varepsilon}{p},\tau+\frac{\varepsilon}{p}}$ for any $\tau > \tau_0$. If ε is sufficiently small, we may assume that $\overline{V} \subset \bigcap_{p\geqq 1} W^{\theta}_{\frac{\varepsilon}{p},\tau_0+\frac{\varepsilon}{p}}$. We claim that $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_N, \vec{X}$ are $\mathbf{R}$-linearly independent on V for any $N \in \mathbf{N}^*$. To prove this, suppose that $\alpha_1\vec{X}_1 + \alpha_2\vec{X}_2 + \cdots + \alpha_N\vec{X}_N + \alpha\vec{X} = \vec{0}$ on V for some real numbers $\alpha_1, \ldots, \alpha_N, \alpha$. If $\alpha \neq 0$ then $\vec{X}$ coincides with some linear combination of $\vec{X}_1, \ldots, \vec{X}_N$ on V. Let us denote $\vec{Y}$ this combination. Then, $\vec{Y}$ is defined on $\bigcap_{p=1}^{N} W^{\theta}_{\frac{\varepsilon}{p},\tau_0+\frac{\varepsilon}{p}} = W^{\theta}_{\varepsilon,\tau_0+\frac{\varepsilon}{N}}$, but $\vec{X} = \vec{Y}$ on $V \subset W^{\theta}_{0,\tau_0} \cap W^{\theta}_{\varepsilon,\tau_0+\frac{\varepsilon}{N}}$, this shows that $\vec{X}$ may be holomorphically extended on $W^{\theta}_{0,\tau_0+\frac{\varepsilon}{N}}$. Since this is not, we have $\alpha = 0$ and $\alpha_1\vec{X}_1 + \cdots + \alpha_N\vec{X}_N = \vec{0}$. By iterating this process, we find that $\alpha_N = 0, \ldots, \alpha_1 = 0$. This leads to some contradiction since the Lie algebra of holomorphic tangent vector field to $\mathscr{H} \cap V$ is finite dimensional (see [7] or [16]). ∎

4 Proofs of the main results

Proof of Theorem 1. We first consider the case of pseudoconvex domains. Then we shall extend our result to non pseudoconvex domains by using the following lemma whose proof is defered until the end of the section.

Lemma 4.1 *Let $f: \Omega_1 \to \Omega_2$ be a proper holomorphic mapping between domains in $\mathbf{C}^n$. Assume that the hulls of holomorphy $\hat{\Omega}_1, \hat{\Omega}_2$ of Ω_1, Ω_2 are domains in $\mathbf{C}^n$. Then f extends to some proper holomorphic mapping $\hat{f}: \hat{\Omega}_1 \to \hat{\Omega}_2$.*

Let Ω_1 and Ω_2 be bounded, pseudoconvex Reinhardt domains in $\mathbf{C}^2$. After normalization we may assume that

$$\{(0,\omega); w \in \Delta\} \cup \{(z,0), z \in \Delta\} \subset \Omega_j \subset \Delta^2 \quad \text{for } j = 1,2$$

where Δ denotes the unit disc in $\mathbf{C}$. Let $f: \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Let us recall that, according to Bell [4], f holomorphically extends to some neighborhood of $\overline{\Omega_1}$. When f is splitting and $\Omega_1 = \Delta^2$, one easily checks that $\Omega_2 = \Delta^2$ too. In that case it is well known that the components of f are finite Blashke products. Thus, all we have to show is that, if f is splitting and $\Omega_1 \neq \Delta^2$, then f has the form $(e^{i\nu}z^m, e^{i\tau}w^n)$ and, that if f is not splitting, then Ω_1, Ω_2 are ellipsoids.

Let us first assume that $f = (f_1(z), f_2(w))$. Since $\{(f_1(0), f_2(e^{i\theta})); \theta \in [0, 2\pi]\} \subset b\Omega_2$, one has $|h_2(e^{i\theta})| \equiv R_1$ on $[0, 2\pi]$ for some $R_1 \in]0, 1]$. If $\Omega_1 \neq \Delta^2$ we may find $z_0 \in \Delta$ and $r \in]0, 1[$ such that $S =: \{(z_0, re^{i\theta}); \theta \in [0, 2\pi]\} \subset b\Omega_1$. Then $f(S) \subset b\Omega_2$ and therefore $|f_2(re^{i\theta})| \equiv R_2$ on $[0, 2\pi]$ for some $R_2 \in]0, 1]$. Using the maximum modulus principle one sees that $R_2 < R_1$ and that f_2 properly maps the corona $\{r \leqq |w| \leqq 1\}$ to the corona $\{R_2 \leqq |w| \leqq R_1\}$. It follows that f_2 has the form $f_2(w) = B\omega^n$ where $B \in \mathbf{C}^*$ and $n \in \mathbf{N}$ (see, for instance, the proof of Theorem 14.22 in [14]). The same arguments show that $f_1(z) = Az^m$ and, by our normalization, $|A| = |B| = 1$.

Let us now assume that f is not splitting. By Proposition 2.1 $b\Omega_1^*$ and $b\Omega_2^*$ are real analytic and strictly pseudoconvex. Let $\eta \in b\Omega_2^* \cap f(b\Omega_1^*)$, according to the Remark 1.3 we may find $\eta' \in b\Omega_1^*$ such that $f(T_{\eta'}) \not\subset T_\eta$. Since $f^{-1}(T_\eta) \cap T_{\eta'}$ is closed in $T_{\eta'}$ and $T_{\eta'}$ is connected, $f^{-1}(T_\eta) \cap T_{\eta'}$ cannot be open. Thus we may find $\eta'' \in T_{\eta'}$ such that $f(\eta'') \in T_\eta$ but $f(V \cap T_{\eta''}) \not\subset T_\eta$ for any neighborhood V of η''. Without any loss of generality we may assume that $\eta'' = \eta'$. Since $b\Omega_1^*$ and $b\Omega_2^*$ are strictly pseudoconvex, f induces a local biholomorphism from some neighborhood of η' onto some neighborhood of η, let us call g its inverse. Then, for every (sufficiently small) neighborhood U of η, one has $g(U \cap T_\eta) \not\subset T_{\eta'}$. Otherwise g would induce a diffeomorphism between some neighborhood of η in T_η to some neighborhood of η' in $T_{\eta'}$ and we would have $f(V \cap T_\eta) \subset T_\eta$ for V small enough. As a consequence we may find $\alpha, \beta \in \mathbf{R}$ and $\varepsilon > 0$ such that the curve $\{g(e^{i\alpha t}\eta_1, e^{i\beta t}\eta_2); |t| < \varepsilon\}$ is not contained in $T_{\eta'}$ and, therefore, the vector field $g_*[i\alpha z\frac{\partial}{\partial z} + i\beta w\frac{\partial}{\partial w}]$ cannot coincide with a rotation vector field near η'.

Let $\tilde{\Omega}_1$ be an open neighborhood of $\overline{\Omega}_1$ on which f holomorphically extends. Consider the analytic sets Z_1 and Z_2 in $\tilde{\Omega}_1$ which are respectively defined by $\{\det f' = 0\}$ and $\{f_1 f_2 = 0\}$. Since f is locally biholomorphic on $\tilde{\Omega}_1 \backslash Z_1$, we may define an holomorphic vector field Q on $\tilde{\Omega}_1 \backslash Z_1$ by pulling back the vector field $i\alpha z\frac{\partial}{\partial z} + i\beta w\frac{\partial}{\partial w}$ by f:

$$Q = f^*\left[i\alpha z\frac{\partial}{\partial z} + i\beta w\frac{\partial}{\partial w}\right] .$$

By construction, Q is an holomorphic tangent vector field for $b\Omega_1^*$ on $\tilde{\Omega}_1 \backslash (Z_1 \cup Z_2)$. According to Proposition 3.3, every germ of Q at some point $\eta'' \in T_{\eta'} \backslash (Z_1 \cup Z_2)$ may be holomorphically extended along any path in $T_{\eta'}$. It follows that the components of Q are locally bounded holomorphic functions on $V \backslash (Z_1 \cup Z_2)$ for some neighborhood V of $T_{\eta'}$. By the Riemann Removable Singularity Theorem, Q extends across $Z_1 \cup Z_2$ and defines an holomorphic tangent vector field for $b\Omega_1^*$ on V. This vector field is not a rotation vector field since, by our choice of α and β, it does not coincide with such a vector field near η'. Thus, part (α) of Theorem 3.1 shows that Ω_1 is an ellipsoïd of the form $\{|z|^{2a} + |w|^{2n} < 1\}$, $a > 0, n \in \mathbf{N}$.

We shall now end the proof in the pseudoconvex case by showing that Ω_2 is an ellipsoïd too. We may find points $\eta'_0 \in b\Omega_1^*$ and $\eta_0 \in b\Omega_2^*$ such that $\eta_0 = f(\eta'_0)$ and a vector field $i\alpha z\frac{\partial}{\partial z} + i\beta w\frac{\partial}{\partial w} =: X$ such that $f_*(X)$ does not coincide with a rotation vector field near η'_0. This follows from the fact that f is not splitting by an argument similar to the argument used in the study of Ω_1. Now, the vector field $Q =: f_*(X)$ is a well defined m-valued vector field on Ω_2. Let us show that Q is tangent for $b\Omega_2$ near any point $\eta \in b\Omega_2^*$. If $\eta \in b\Omega_2^*$, there exists $\eta' \in b\Omega_1^*$ such that $f(\eta') \in T_\eta$ (since $f(b\Omega_1 \cap \{zw = 0\})$ cannot contain T_η), then f induces a local biholomorphism from some neighborhood of η' to some neighborhood of $f(\eta') =: \tilde{\eta}$. Let us still denote by f this local biholomorphism, the vector field $f_*(X)$ is tangent to $b\Omega_2$ near $\tilde{\eta}$ and extends Q. This shows that Q is tangent to $b\Omega_2$ near $\tilde{\eta}$; using Proposition 3.3 we see that Q may be holomorphically extended to some tangent vector field to $b\Omega_2$ near any point of $T_{\tilde{\eta}} = T_\eta$. It also follows from the above arguments that the branch set A of Q cannot intersect $b\Omega_2^*$; the maximum modulus principle applied to the function zw on the analytic set A shows that $A \subset \{zw = 0\}$. This implies that Q satisfies the assumptions of part(β) of Theorem 3.1, thus Ω_2 is an ellipsoïd of the form: $\{|w|^{2r} + |z|^{2b} < 1\}$ where $b > 0$ and $r \in \mathbf{Q}^+$. The relation between the exponents (a, n) and (r, b) follows directly from the result of Dini and Primicerio ([9]).

Finally, it remains to remove the assumption of pseudoconvexity. Let $f: \Omega_1 \to \Omega_2$ be a proper holomorphic mapping between bounded, complete Reinhardt domains in $\mathbf{C}^2$. By Lemma 4.1 f extends to some proper holomorphic mapping $\hat{f}: \hat{\Omega}_1 \to \hat{\Omega}_2$ between bounded, complete pseudoconvex Reinhardt domains. We must prove that, if one of the domains Ω_1, Ω_2 is not pseudoconvex, then f has the form (Az^m, Bw^n). If Ω_1 or Ω_2 is not pseudoconvex then $b\hat{\Omega}_1$, or $b\hat{\Omega}_2$ contains analytic sets. In that case, one of the domains $\hat{\Omega}_1$, $\hat{\Omega}_2$ is not an ellipsoïd and therefore, by the first part of the proof, the mapping $\hat{f}: \hat{\Omega}_1 \to \hat{\Omega}_2$ is splitting. We now proceed by contradiction. Suppose that the mapping f does not have the form (Az^m, Bw^n). Then $\hat{f}$ does not have this form too and, always by the first part of the proof, one has $\hat{\Omega}_1 = \hat{\Omega}_2 = \Delta^2$ after some normalization. Moreover, Ω_1 cannot coincide with Δ^2 otherwise we would also have $\Omega_2 = \Delta^2$ since f is splitting. Thus $b\Omega_1$ contains a circle $\{(z_0, re^{i\theta}); \theta \in [0, 2\pi]\} =: S$ where $0 < r < 1$. Since $f(S) \subset b\Omega_2$, one easily sees that $w \mapsto (\hat{f}_1(z_0), \hat{f}_2(w))$ properly maps the corona $\{r \leqq |w| \leqq 1\}$ to some corona $\{R \leqq |w| \leqq 1\}$. It follows that $\hat{f}_2$ has the form Bw^n. Similar arguments show that $\hat{f}_1(z) = Az^m$; this contradicts the above assumption. ∎

Proof of Theorem 2. According to Lemma 4.1, we may restrict ourselves to the pseudoconvex case. Let $f: \Omega \to \Omega$ be a self proper holomorphic mapping of some bounded, complete pseudoconvex Reinhardt domain in $\mathbf{C}^2$. Assume that Ω is not a bidisc. By Theorem 1, two cases are in order : Ω is an ellipsoïd of the form $\{|z|^{2b} + |w|^2 < 1\}$ or f has the form (Az^m, Bw^n). In the first case, one knows that f must be an automorphism (see [9]). In the second, after some normalization, we may assume that $\{(0, w); w \in \Delta\} \cup \{(z, 0), z \in \Delta\} \subset \Omega \subset \Delta^2$. Since the circles $\{(0, e^{i\theta}), \theta \in [0, 2\pi]\}$ and $\{(e^{i\theta}, 0), \theta \in [0, 2\pi]\}$ are mapped to

$b\Omega$ one has $|A| = |B| = 1$. As $\Omega \neq \Delta^2$, we may find a point $(z_0, w_0) \in b\Omega$ such that $|w_0| < 1$ and $|z_0| < 1$. Considering the sequence of iterates $f^k(z_0, w_0)$ one sees that, if $(m, n) \neq (1, 1)$, then one of the points $(0, 0)$, $(0, w_0)$, $(z_0, 0)$ must belong to $b\Omega$. Since this is not, one has $m = n = 1$ and f is a rotation.

We now close this section with the *proof of Lemma 4.1.*

The mapping f extends to a mapping $\hat{f}$ which is defined on $\hat{\Omega}_1$. The inverse of f is an algebroid mapping which we denote $g : \Omega_2 \to \Omega_1$. Each component g_j of g is an algebroid function on Ω_2 whose values over a point $w \in \Omega_2$ are solutions of an equation $Q_j(w, z) = 0$ where $Q_j(w, z) = w^m + a_{j,m-1}(z)w^{m-1} + \cdots + a_{j,0}(z)$ is a pseudopolynomial with coefficients in $\mathcal{O}(\Omega_2)$. Since the functions $a_{j,l}$ extend holomorphically to $\hat{\Omega}_2$, we see that g extend to some algebroid mapping $\hat{g}$ on $\hat{\Omega}_2$.

The properness of $\hat{f}$ immediately follows from the two following inclusions: $\hat{g}(\hat{\Omega}_2) \subset \hat{\Omega}_1$ and $\hat{f}(\hat{\Omega}_2) \subset \hat{\Omega}_2$. Let us justify the first one. Assume that there exists $z'_0 \notin \hat{\Omega}_1$, such that $z'_0 \in \{\hat{g}(w'_0)\}$ for some $w'_0 \in \hat{\Omega}_2$. If $z'_0 \notin \overline{\hat{\Omega}_1}$ and since $\{\hat{g}(w)\}$ depends continuously on w, we may replace w'_0 by an arbitrarily close point which lies outside of the branch set of $\hat{g}$. Let $a \in \Omega_1$ be some point in a neighborhood of which f defines a local biholomorphism. Since g extends to some algebroid mapping on $\hat{\Omega}_2$, there exists a path $\gamma : [0, 1] \to \hat{\Omega}_2$ from $f(a)$ to w'_0, which does not intersect the branch set of $\hat{g}$, such that the branch of g which maps $f(a)$ to a extends holomorphically on some neighborhood of $\gamma([0, 1])$. Let us not $\hat{g}_\gamma$ this extension. Let $t_0 \in]0, 1[$ be such that $\hat{g}_\gamma(t_0) \in b\hat{\Omega}_1$ and $\hat{g}_\gamma([0, t_0]) \subset \hat{\Omega}_1$. Since $\hat{\Omega}_1$ is a domain of holomorphy there exists an holomorphic function φ on $\hat{\Omega}_1$, which is unbounded on $\hat{g}_\gamma([0, t_0[)$. The function $\tilde{\varphi} =: \varphi \circ g$ is algebroid on Ω_2 and therefore extends as an algebroid function on $\hat{\Omega}_2$. The branch of $\tilde{\varphi}$ which takes the value $\varphi(a)$ at $f(a)$ extends along $\gamma([0, t_0[)$ where it coincides with $\varphi \circ \hat{g}_\gamma$. This is absurd since $\varphi \circ \hat{g}_\gamma$ is not bounded on $\gamma[(0, t_0[)$.

So far we have shown that $\hat{g}(\hat{\Omega}_2) \subset \overline{\hat{\Omega}_1}$. If there exists $z'_0 \in b\hat{\Omega}_1$ such that $z'_0 \in \{\hat{g}(w'_0)\}$ for some $w'_0 \in \hat{\Omega}_2$, we may find a path $\gamma : [0, 1] \to \hat{\Omega}_2$ from z_0 to w'_0 such that $\gamma([0, 1[)$ does not intersect the branch set of $\hat{g}(w'_0$ may belong to the branch set of $\hat{g})$. As above we can extend a branch of $\hat{g}$ along $\gamma([0, 1[)$ and obtain a contradiction by the same method.

The second inclusion may be proved by following the same line, the argument is even simpler since, in that case, we are dealing with mono-valued mappings and functions. ■

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