Math. Z. 219, 203 – 213 (1995)

Mathematische Zeitschrift

Higher homotopy commutativity in localized groups

Laia Saümell

Department de Matemàtiqũes, Universitat Aũtònoma de Barcelona, E-08103 Bellaterra, Barcelona, Spain

Received: 8 September 1993; in final form: 31 Jaunary 1994

1 Introduction and statement of results

Let G be a simply connected Lie group of rank $\geqq 1$. It is well known that G is not homotopy commutative. Instead, if we consider G localized at a prime p, $G_{(p)}$, the localized Lie multiplication whether may be homotopy commutative or not.

In [6] McGibbon give a complete description of the homotopy commutativity or not of $G_{(p)}$, for all primes p. Since the localization commutes with the products and a product of topological groups is homotopy commutative if and only if each factor is also homotopy commutative, it suffices to study the case G simple.

1.1 Theorem (McGibbon [6]). *Let G be a 1-connected, simple Lie group of type* $(2n_1, \ldots, 2n_l)$ *where* $n_1 \leqq \cdots \leqq n_l$. *Then*

i) *If* $p > 2n_l$ *then* $G_{(p)}$ *is homotopy commutative.*
ii) *If* $p < 2n_l$ *then* $G_{(p)}$ *is not homotopy commutative, except in two cases:* $Sp(2)$, *or equivalently* $Spin(5)$ *at* $p = 3$ *and* G_2 *at* $p = 5$.

where the type means that $G_{(0)} \simeq (S^{2n_1-1} \times \cdots \times S^{2n_l-1})_{(0)}$. ∎

In the same paper McGibbon gives also a partial result for finite loop spaces and loop multiplications:

1.2 Theorem (McGibbon [6]). *Let* (X, μ) *be a loop space where X has the homotopy type of 1-connected, p-local CW-complex, with type* $(2n_1, \ldots, 2n_l)$ *where* $n_1 \leqq \cdots \leqq n_l$. *Then*

i) *If* $p > 2n_l$ *then* μ *is homotopy commutative.*
ii) *If* $n_l < p < 2n_l$ *then* μ *is not homotopy commutative.*

where the type means $X_{(0)} \simeq (S^{2n_1-1} \times \cdots \times S^{2n_l-1})_{(0)}$. ∎

Recall that there are some generalizations of the concept of homotopy commutativity to higher homotopy commutativity, as are the C^n-spaces of Sugawara and the C_n-spaces of Williams, both for $n = 2$ correspond to the usual concept of homotopy commutativity.

In this paper we generalize the results of McGibbon on homotopy commutativity to higher homotopy commutativity in the sense of Williams. The reason why we work with the C_n-spaces of Williams, and not with the C^n-spaces of Sugawara (which is a stronger concept of higher homotopy commutativity, for example, for loop spaces "C^n-spaces of Sugawara" implies "C_n-spaces of Williams" [7]) is owing to the fact that we can relate it with the generalized higher order Whitehead product, and it is also more operative to use.

The results obtained are the followings, where in both cases the type also means that the rationalization is homotopy equivalent to $(S^{2n_1-1} \times \cdots \times S^{2n_l-1})_{(0)}$.

Theorem A. *Let G be a 1-connected, compact, simple Lie group, different from G_2 at $p = 5$, of type $(2n_1, \ldots, 2n_l)$ where $n_1 \leqq \cdots \leqq n_l$, and let $k \geqq 2$ be an integer. Then*

i) *If $p > kn_l$ then $G_{(p)}$ is a C_k-space.*

ii) *If $p < kn_l$ then $G_{(p)}$ is not a C_k-space, except in the case* Sp(2), *or equivalently Spin(5) at $p = 3$ and $k = 2$.* ∎

For the case G_2 at $p = 5$ McGibbon [6] proved that is homotopy commutative, and Hemmi [2] conjectures that is a C_4-space but not a C_5-space.

And we also prove:

Theorem B. *Let (X, μ) be a loop space where X has the homotopy type of 1-connected, p-local CW-complex, with type $(2n_1 \ldots, 2n_l)$ where $n_1 \leqq \cdots \leqq n_l$, and let $k \geqq 2$ be an integer. Then*

i) *If $p > kn_l$ then (X, μ) is a C_k-space.*

ii) *If $n_l < p < kn_l$ then (X, μ) is not a C_k-space.* ∎

The proves are essentially founded in the proofs of McGibbon for the usual homotopy commutativity. We generalize his results.

To do it, in Sect. 2 we introduce the generalized higher order Whitehead product. Section 3 contains the definition and the background what we need of the higher homotopy commutativity in the sense of Williams and their connection with the higher order Whitehead products. Section 4 concerns the use of Steenrod operations in order to prove the non-higher homotopy commutativity. And finally, Sect. 5 is devoted to the proof of Theorem B and Theorem A.

2 Generalized higher order Whitehead products

The notation and definitions which we will introduce in this section is following the one of Porter in [8], where he introduced the concept of higher order Whitehead products.

We shall assume that all spaces are countables, connected CW-complexes with base point, and all maps are continuous and base point preserving.

First we will stablish some notation. Let be $A_1, \dots, A_n$ topological spaces, we will denote by $T_i(A_1, \dots, A_n)$ the subspace of $A_1 \times \cdots \times A_n$ consisting of those points with at least i co-ordinates at base point. Note that T_0 is the cartesian product, T_1 is so called the fat wedge, T_{n-1} is the one point union, and T_0/T_1 is the smash product. And, of course, there is a natural transformation $T_{i-1} \to T_i$.

We say that a map $\varphi : T_1(\Sigma A_1, \dots, \Sigma A_n) \to X, i < n$, *is of type* $(f_1, \dots, f_n)$ or more briefly $\varphi \in (f_1, \dots, f_n)$, if $\varphi k_j \sim f_j$ for $j = 1, \dots, n$, where $k_j : \Sigma A_j \to T_i(\Sigma A_1, \dots, \Sigma A_n)$ is the canonical inclusion.

Porter defined the generalized higher order Whitehead products by using natural transformations of functors and categories, but we can look at it as:

2.1 Definition [9]. *Given a map* $\varphi : T_1(\Sigma A_1, \dots, \Sigma A_n) \to X, n \geqq 2$, **the generalized n^{th}-Whitehead product** $\omega(\varphi)$ *is an element of* $[\Sigma^{n-1} A_1 \wedge \ldots \wedge A_n, X]$ *and it is the obstruction to extend* f *to* $\Sigma A_1 \times \cdots \times \Sigma A_n$.

2.2 Definition [9]. The set of n^{th} order Whitehead products of type $(f_1, \dots, f_n)$ *is denoted* $[f_1, \dots, f_n]$ *and defined by*

$$[f_1, \dots, f_n] = \{\omega(\varphi) | \varphi : T_1(\Sigma A_1, \dots, \Sigma A_n) \to X, \varphi \in (f_1, \dots, f_n)\} .$$

Note that $\omega(\varphi)$ is well defined element while $[f_1, \dots, f_n]$ is a subset (perhaps empty) of $[\Sigma^{n-1} A_1 \wedge \cdots \wedge A_n, X]$. And, of course, if $n = 2$ it is agree to the usual generalized Whitehead product.

Let $(\Sigma A)_n$ be the n^{th} reduced product space of ΣA obtained by identifying points of $(\Sigma A)^n$ with each other if and only if they are the same when occurrences of the base points are disregarded. Then there is a natural map.

$$T_i(\Sigma A, \overset{n)}{\dots}, \Sigma A) \xrightarrow{p_i} (\Sigma A)_{n-i} .$$

And hence we can define

2.3 Definition [9]. *A map* $\varphi : T_i(\Sigma A, \overset{n)}{\dots}, \Sigma A) \to X$ *is said to be* **reducible** *if there exists a map* $\psi : (\Sigma A)_{n-i} \to X$ *such that* $\phi \sim \psi \circ p_i$.

Porter in [8] proved as a corollary of a Theorem:

2.4 Proposition (Porter [9]). *If a reducible map* $\varphi : T_i(\Sigma A, \overset{n)}{\dots}, \Sigma A) \to X$ *may be extended to* $T_{i-1}(\Sigma A, \overset{n)}{\dots}, \Sigma A)$ *then the extension may be chosen to be reducible.* ∎

Then we can prove the following result, which we will need in order to prove Theorem B:

2.5 Proposition. *Let* $f : \Sigma A \to X$ *be a map.*

If $[\Sigma^{k-1} A \wedge \overset{k)}{\dots} \wedge, X] = 0$, *for all* k, $2 \leqq k \leqq r$, *then* $0 = [f, \overset{k)}{\dots}, f]$ *for all* k, $2 \leqq k \leqq r$.

Proof. Since $[f,\overset{k)}{\ldots},f] \in [\Sigma^{k_1}A \wedge \overset{k)}{\ldots} \wedge, X] = 0$, it is plain that if $[f,\overset{k)}{\ldots},f]$ is not empty, is zero. Hence, it suffices to prove that for all k, $2 \leqq k \leqq r$ there exists a map

$$\varphi_k : T_1(\Sigma A,\overset{k)}{\ldots},\Sigma A) \to X$$

of type $(f,\overset{k)}{\ldots},f)$.

For $k = 2$ is trivial, because $[f,f] \in [\Sigma A \wedge A, X] = 0$ is the usual Whitehead product, and of course it can not be empty. Hence there exists an extension $\tilde{\varphi}_2 : \Sigma A \times \Sigma A \to X$ which it is clearly reducible, and consequently we can construct a map

$$\begin{array}{ccccc} \varphi_3 : T_1(\Sigma A,\Sigma A,\Sigma A) & \to & (\Sigma A)_2 & \to & X \\ & & \uparrow & \nearrow_{\tilde{\varphi}_2} & \\ & & \Sigma A \times \Sigma A & & \end{array}$$

which is a reducible map of type (f,f,f), is reducible, and $\omega(\varphi_3) = 0$.

Assume now by induction that there exists a reducible extension

$$\varphi_{k-1} : T_1(\Sigma A \overset{k-1)}{\ldots}, \Sigma X) \to X$$

of type $(f,\overset{k-1)}{\ldots},f)$. Since $\omega(\varphi_{k-1}) \in [\Sigma^{k-2}A \wedge \overset{k-1)}{\ldots} \wedge A, X] = 0$, then $\omega(\varphi_{k-1}) = 0$. And by Proposition 2.5. we obtain that there exists a reducible map $\tilde{\varphi}_{k-1}$

$$\tilde{\varphi}_{k-1} : \Sigma A \times \overset{k-1)}{\ldots} \times \Sigma A \to (\Sigma A)_{k-1} \overset{\phi_{k-1}}{\to} X .$$

Consider now the map

$$\tilde{\varphi}_k : T_1(\Sigma A,\overset{k)}{\ldots},\Sigma A \overset{p_1}{\to} (\Sigma A)_{k-1} \overset{\phi_{k-1}}{\to} X$$

which is of type $(f,\overset{k)}{\ldots},f)$, is reducible and $\omega(\varphi_k) \in [\Sigma^{k-1}A \overset{k)}{\ldots} \wedge A, X] = 0$, then φ_k can be extended to $\Sigma A \times \overset{k)}{\ldots} \times \Sigma A, 0 = \omega(\varphi_k) \in [f,\overset{k)}{\ldots},f]$, and furthermore the extension may be chosen reducible. ∎

3 Higher homotopy commutativity

Williams in [12] introduced a concept of higher homotopy commutativity in the category of CW-monoids. He has given ([11]), [12]) many different but equivalent descriptions. The following theorem recalls some of his results and we can use it as definition.

3.1 Theorem-Definition [7]. *Given a 1-connected countable CW-complex X, and an integer $n \geqq 2$, the following statements are equivalent:*

a) *The loop space ΩX admits certain forms*

$$Q_i : K_i \times (\Omega X)^i \to \Omega X \quad i = 2,\ldots,n$$

defined by Williams [11], *using Milgram's permutahedrons K_i.*

b) *There is a sequence of maps*

$$(\Sigma\Omega X)_k \to P_k(\Omega X) \quad k = 1, \dots, n$$

where $(\ )_n$ *denotes the James n-fold reduced product, and* P_k *the k-projective space, that starts with the identity map when* $k = 1$, *and commutes with the usual inclusions.*

c) *The composition*

$$\Omega X \to \Omega\Sigma\Omega X = \Omega[(\Sigma\Omega X)_1] \stackrel{\Omega_{\iota_n}}{\to} \Omega[(\Sigma\Omega X)_n]$$

has a left inverse that is an A_n*-map*

d) *Every generalized Whitehead product on X of order* $\leqq n$, *contains zero.*

e) *Every generalized Whitehead product on X of order* $\leqq n$, *equals zero.*

We say that a loop space ΩX *with the above properties is a* C_n*-space of Williams, or to simplify, a* C_n*-space.* ▮

As in the case $n = 2$, we prove a connection between higher homotopy commutativity in a loop space X and higher Whitehead products on its classifying space BX, given by:

3.2 Theorem. *Let X be a loop space with the homotopy type of a countable connected CW-complex, the following statements are equivalent:*

1) *X is a* C_n*-space.*

2) *The Whitehead product* $0 \in [\iota, \overset{n)}{\dots}, \iota]$, *where* $\iota : \Sigma X \to BX$ *is the adjoint of* $X \stackrel{id}{=} X \simeq \Omega BX$.

3) *There is a map* $\phi_n : \Sigma X \times \overset{n)}{\dots} \times \Sigma X \times \to BX$ *whose restriction to each factor is homotopic to* ι.

Proof. 1) ⇒ 2) It is an obvious consequence of Theorem-Definition 3.1.

2) ⇒ 3) Since we assume that $0 \in [\iota, \overset{n)}{\dots}, \iota]$, there is at least one map

$$\varphi : T_1(\Sigma X, \overset{n)}{\dots}, \Sigma X) \to BX$$

such that $\omega(\varphi) = 0$, and $\varphi \in (\iota, \overset{n)}{\dots}, \iota)$.

Now, because of $\omega(\varphi) = 0$, there is a map

$$\phi : \Sigma X \times \overset{n)}{\dots} \times \Sigma X \to BX$$

which extends φ and of course, whose restriction to each factor is homotopic to ι.

3) ⇒ 1) Let be $f_i : \Sigma A_i \to BX$ a map, $i = 1, \dots, k$, where $k \leqq n$.

Consider $\tilde{f}_i : A_i \to \Omega BX \simeq X$ the adjoint of f_i, then

$$\Sigma A_i \stackrel{\Sigma\tilde{f}_i}{\to} \Sigma X \stackrel{\iota}{\to} BX$$

is homotopic to f_i.

By the assumptions, there exists a map $\phi_n : \Sigma X \times \overset{n)}{\ldots} \times \Sigma X \to BX$ whose restriction to each factor is homotopic to ι. And obviously there exists also

$$\phi_k : \Sigma X \times \overset{k)}{\ldots} \times \Sigma X \hookrightarrow \Sigma X \times \overset{n)}{\ldots} \times \Sigma X \overset{\phi_n}{\to} BX \quad k \leqq n$$

whose restriction to each factor is homotopic to ι.

Consider now the map

$$\tilde{\phi} : \Sigma A_1 \times \ldots \times \Sigma A_k \overset{\Sigma \tilde{f}_1 \times \cdots \Sigma \tilde{f}_k}{\longrightarrow} \Sigma X \times \overset{k)}{\ldots} \times \Sigma X \overset{\phi_k}{\to} BX$$

whose restriction to each ΣA_i is homotopic to f_i. Let $\tilde{\varphi}$ be the composition

$$\tilde{\varphi} : T_1(\Sigma A_1, \ldots, A_k) \hookrightarrow \Sigma A_1 \times \cdots \times \Sigma A_k \overset{\tilde{\phi}}{\to} BX \quad k \leqq n$$

hence $\tilde{\varphi} \in (f_1, \ldots, f_k)$ and since $\tilde{\phi}$ is an extension of $\tilde{\varphi}, \omega(\tilde{\varphi}) = 0$. Consequently we get that $0 \in [f_1, \ldots, f_k]$ for $k \leqq n$. Moreover by the Theorem-Definition 3.1. (d) we obtain that X is a C_n-space in the sense of Williams. ∎

4 Whitehead products and cohomology operations

In general, it is very difficult to compute generalized Whitehead products, but in certain cases is easy to prove that $[\iota, \ldots, \iota]$ does not contain the zero. To be exact, the non existence of the map ϕ of Theorem 3.2 (3) can be established with the aid of primary cohomology operations. The next result is a generalization of Lemma 3.2. [6] of McGibbon, and the proof is also a generalization of the proof of his Lemma.

4.1 Proposition. *Let X be a loop space with the homotopy type of finite CW-complex (or the localization of such a complex). Let $k \geqq 2$ be an integer and $p > k$ a prime, and assume that $H_*(X, \boldsymbol{Z})$ has no p-torsion. If there exists a class $x \in H^*(BX, \boldsymbol{F}_p)$ and an integer $t > 0$ such that:*

1) $P^t(x) = 0$ *mod* F^k.

2) $P^t(x) \neq$ *mod* $\langle F^{k+1} + P^t F^k \rangle$

where $F^n = F^1 \overset{n)}{\ldots} F^1$ is the n-fold product ideal of $F^1 = \tilde{H}^*(\;, \boldsymbol{F}_p)$ and $\langle F^{k+1} + P^t F^k \rangle$ denotes the graded vector space in $H^*(BX, \boldsymbol{F}_p)$ spanned by the homogeneous elements of the subspace F^{k+1} and $P^t F^k$.

Then $[\iota, \overset{k)}{\ldots}, \iota]$ does not contain the zero at p.

Proof. All cohomology groups considered here will have Z/pZ coefficients. and to simplify we will denote $H^*(\;) = H^*(\;, \mathbf{F}_p)$. Since we suppose that $H^*(X, \mathbf{Z})$ has no p-torsion, then $H^*(BX)$ is a polynomial algebra with even dimensional generators.

Assume that the result what we want to prove is false, then by Theorem 3.1. there is a map

$$\Sigma X \times \overset{k)}{\dots} \times \Sigma X \xrightarrow{\varphi} BX$$

whose restriction to each factor is homotopic to ι.

We will denote by $p_i : \Sigma X, \times \cdots \times \Sigma X \to \Sigma X$ the projection on the i factor, and by $\varphi_i(x) = p_i^* \iota^*(x)$ for a class $x \in H^*(BX)$.

Since φ restricted to each factor is homotopic to ι, we may write

$$\varphi^*(x) = \sum_{i=1}^{k} \varphi_i(x) + \sum_i y_{i_1} \dots y_{i_k}$$

where each y_{i_j} belongs to the image of p_j^*.

Suppose now that the class x satisfies the hypothesis of the proposition, since the cup products are trivials in $H^*(\Sigma X)$, it follows from the condition (1) that $P^t \varphi_i(x) = 0$ for all i, and hence

$$P^t \varphi^*(x) = P^t \left(\sum_i y_{i_1} \dots y_{i_k} \right).$$

From now on, we want to prove that $P^t(\sum_i y_{i_1} \dots y_{i_k}) \subset P^t \varphi^* F^k$, and since the kernel of $\varphi^* \subset F^{k+1}$ this will be a contradiction to condition (2).

For this is necessary to recall that there is a splitting [3]

$$\widetilde{H}^*(\Sigma X) = V' \oplus V''$$

where $V' = \iota^* \widetilde{H}^*(BX)$ and $V'' = \sigma F^2 H^*(X)$, where σ is the suspension isomorphism, and furthermore, remark that this splitting is a splitting as modules over the Steenrod algebra $\mathcal{A}^*_{(p)}$.

If we apply p_i^* to the splitting and we express $y_{i_j} = y'_{i_j} + y''_{i_j}$ where $y'_{i_j} \in V'$ and $y''_{i_j} \in V''$, then we can rewrite

$$P^t \varphi^*(x) = P^t(\omega) + \sum_i P^t(y'_{i_1} \dots y'_{i_k})$$

where all the monomials in ω involve some y''_{i_j}. And we claim that $P^t(\omega) = 0$.

To show this, and with the same notation as before, consider the decomposition

$$F^k(H^*(\Sigma X \times \cdots \times \Sigma X)) = \bigoplus_{\varepsilon_i, \dots \varepsilon_k \in \{\prime, \prime\prime\}} V^{\varepsilon_1} \dots V_k^{\varepsilon_k}$$

where $V^{\varepsilon_1} \dots V_k^{\varepsilon_k}$ denotes the subspace spanned by $(p_1^* y_1) \dots (p_k^* y_k)$, where $y_i \in V_i^{\varepsilon_i}$ for all i.

Moreover by naturality and the Cartan formula it follows that this splitting is also a splitting as modules over the Steenrod algebra $\mathcal{A}^*_{(p)}$.

Note that $P^t\varphi^*(x)V_1'\dots V_k'$, because suppose, to simplify, that $P^t(x) = a_1\dots a_k$ is only a monomial (the general case differs only in notational complexity), then

$$\varphi^*P^t(x) = P^t\varphi^*(x) = \left(\sum_j \varphi_j(a_1) + \sum_j a_{1,j_1}\dots a_{1,j_k}\right)\cdots\left(\sum_j \varphi_j(a_k) + \sum_j a_{k,j_1}\dots a_{k,j_k}\right)$$

and since the cup products are zero in $H^*(\Sigma X)$, we get

$$P^t\varphi^*(x) = \sum_{\substack{j_1,\dots,j_k\in\{1,\dots,k\}\\ j_i\neq j_l \text{ if } i\neq l}} \varphi_{j_1}(a_1)\dots\varphi_{j_k}(a_k)$$

but since $\varphi_{j_i}(a_i) \in V_i'$ then all the sumands belong to $V_1'\dots V_k'$ while by definition, $P^t(\omega)$ belongs to its complement, and hence $P^t(\omega) = 0$

Now, if we choose classes $\bar{y}_{i_j} \in H^*(BX)$ such that $y_{i_j}' = \varphi_j(\bar{y}_{i_j})$ (and this choice is possible because $y_{i_j}' \in V'$), we obtain

$$P^t\varphi^*(x) = \sum_i P^t(y_{i_1}'\dots y_{i_k}') = \sum_i P^t(\varphi_1(\bar{y}_{i_1})\dots\varphi_k(\bar{y}_{i_k}))\,.$$

Let θ be a permutation of k letters, and

$$T_\theta : \Sigma X\times\cdots\times\Sigma X \longrightarrow \Sigma X\times\cdots\times\Sigma X$$

the map which change the factors according to θ. Consequently T_θ^* interchange the components $\varphi_j(a)$ of a class $\varphi^*(a)$, and owing to the fact that all the generators of $H^*(BX)$ have even degree, that in $H^*(\Sigma X)$ all the cup products are zero and that $P^t(x) \equiv 0 \bmod F^k$ by hypothesis, we obtain that T_θ^* fixes $P^t\varphi^*(x)$, thus

$$P^t\varphi^*(x) = T_\theta^*P^t\varphi^*(x) \quad \forall\theta\in\mathcal{S}_k$$

and consequently

$$k!P^t\varphi^*(x) = \sum_{\theta\in S_k} T_\theta^*P^t\varphi^*(x)\,.$$

Since $p > k$ we can divide by $k!$ and we obtain

$$\begin{aligned}P^t\varphi^*(x) &= \frac{1}{k!}\sum_{\theta\in S_k} T_\theta^*P^t\varphi^*(x) = \frac{1}{k!}\sum_{\theta\in S_k} T_\theta^*P^t\left(\sum_i \varphi_1(\bar{y}_{i_1})\dots\varphi_k(\bar{y}_{i_k})\right)\\ &= \frac{1}{k!}P^t\left(\sum_i\sum_{\theta\in S_k}\varphi_{\theta(1)}(\bar{y}_{i_1})\dots\varphi_{\theta(k)}(\bar{y}_{i_k})\right) = \frac{1}{k!}P^t\left(\sum_i \varphi^*(\bar{y}_{i_1}\dots\bar{y}_{i_k})\right)\\ &= \frac{1}{k!}\varphi^*\sum_i P^t(\bar{y}_{i_1}\dots\bar{y}_{i_k})\end{aligned}$$

and since the kernel of $\varphi^* \in F^{k+1}$ hence $P^t(x) \in \langle F^{k+1} + P^tF^k\rangle$, which contradicts the condition (2). ∎

5 Proof of main Theorems

Proof of Theorem B, part (1). Since we suppose that $p > kn_l \geqq 2n_l$ we can apply the results of regularity of McCleary [4] and we obtain that $X \simeq \prod_{i=1}^{l} S^{2n_i-1}$.

If we consider $W = \bigvee_{i=1}^{l} S^{2n_i-1}$ and $i : W \longrightarrow X$ the canonical inclusion, since ΣW is a retract of ΣX, it has a homotopic left inverse r.

Let $j = \imath \circ \Sigma_i$. It is no difficult [6] to see that, due to the fact that j^* takes the module of indecomposables $QH^*(BX)$ isomorphically onto $H^*(\Sigma X)$, r may be chosen so that the following diagram commutes homotopically

$$\begin{array}{ccc} \Sigma W & & \\ r \uparrow & \overset{j}{\searrow} & \\ \Sigma X & \underset{\imath}{\rightarrow} & BX \end{array}$$

And consequently

$$0 \in [j, \overset{n)}{\ldots}, j] \Longleftrightarrow 0 \in [\imath, \overset{n)}{\ldots}, \imath] \quad \forall 2 \leqq n \leqq k\,.$$

By the proposition 2.5 we need to compute the p-component of

$$[\Sigma^{r-1} W \wedge \overset{r)}{\ldots} \wedge W, BX] \cong [\Sigma^{r-2} \wedge W \wedge \overset{r)}{\ldots} \wedge W, X] \quad \text{for } 2 \leqq r \leqq k$$

but since $W = \bigvee_{i=l}^{l} S^{2n_i-1}$, then

$$\begin{aligned} [\Sigma^{r-2} W \wedge \overset{r)}{\ldots} \wedge W, X] &\cong \prod [S^{r-2} \wedge S^{2n_{i_1}-1} \wedge \ldots \wedge S^{2n_{i_r}-1}, X] \cong \\ &\cong \prod \pi_{2n_{i_1}+\ldots+2n_{i_r}-2}(X) \end{aligned}$$

where $n_{i_1}, \ldots, n_{i_r} \in \{n_1, \ldots, n_l\}$.

And since $2n_{i_1} + \ldots + 2n_{i_r} - 2$ is even, the only possibility is that this groups was of p-torsion. But the bigger group to consider is π_{2rn_l-2}, and the first group of p-torsion is in our case [10] is $\pi_{2n_1+2(p-1)-1}$, and since we have

$$2n_1 - 1 + 2(p-1) > 2n_1 + 2kn_l - 4 \geqq 2kn_l > 2rn_l - 2$$

all groups are zero, and consequently $0 \in [j, \ldots, j]$.

Proof of Theorem B, part (2). Let $n_l < p < kn_l$. By the assumptions [4] $H^*(X, \mathbf{Z})$ has no p-torsion and $H^*(BX, \mathbf{F}_p) = \mathbf{F}_p[x_1, \ldots, x_l]$ where $|x_i| = 2n_i$ and $n_1 \leqq \cdots \leqq n_l$.

We will prove the result by induction. For $k = 2$ is the Theorem 1.2. of McGibbon.

Assume now that it is true for $2 \leqq r \leqq k-1$. As a consequence of that for $n_l < p < (k-1)n_l$ X is not C_{k-1}-space, and consequently it is not C_k-space too.

Only remains the case $(k-1)n_l < p < kn_l$, but in this case we can generalize the proof of the Theorem 1.2. of McGibbon.

Let $\mathcal{J}$ be the ideal generated by the classes x_i where $|x_i| < 2n_l$. There must exist some x_i where $1 \leqq i \leqq l$ such that $P^1(x_i) \notin \mathcal{J}$, because suppose not, then $P^1(\mathcal{J}) \subseteq \mathcal{J}$. Hence by the Adem relation $(P^1)^n = n!P^n$ for $n < p$, since $p > n_l$, this would imply that $(x_l^p) \in \mathcal{J}$, which is impossible by the definition of $\mathcal{J}$

Consider now x a class such that $P^1(x) \notin \mathcal{J}$ with minimal degree, then

$$|P^1(x)| = |x| + 2(p-1) > 2p > 2(k-1)n_l$$
$$|P^1(x)| = |x| + 2(p-1) < |x| + 2p < 2n_l + 2kn_l < 2(k+1)n_l$$
$$\text{thus} \quad 2(k-1)n_l < |P^1(x)| < 2(k+1)n_l$$

and hence $P^1(x) \in F^k$.

Furthermore if $P^1(x) \in F^{k+1}$ since $P^1(x) \notin \mathcal{J}$, then $P^1(x)$ has monomials formed by x_i where $|x_i| = 2n_l$, but this is impossible by the right hand inequality before.

Moreover if $P^1(x) \in P^1F^k$ by the same reason as before and that x has minimal degree between the classes such that $P^1(x) \notin \mathcal{J}$ we obtain that it is also impossible.

Hence x is in the conditions of Proposition 4.1., and consequently X is not a C_k-space.

Proof of Theorem A. (1) It is a consequence of Theorem B.

(2) Also by the Theorem B we know that for $n_l < p < kn_l$ X is not a C_k-space.

Hence only remains the case $p < n_l$, but by the Theorem 1.1. of McGibbon we know that the only cases which are homotopy commutative are $S_p(2)$ at $p = 3$ and G_2 at $p = 5$, and these are the only cases that we must study.

And it is known that $H^*(BS_p(2), \mathbf{F}_3) = \mathbf{F}_4[x_4, y_8]$ and by the Adem relations $P^2(x_4) = x_4^3$, and as consequence of Propositon 4.1. we obtain that $S_p(2)$ at $p = 3$ is not a C_3-space.

For G_2 at $p = 5$, we can compute that only two cases are possible: $G_{2_{(5)}}$ is a C_2-space and is not a C_3-space, or is a C_4-space but is not a C_5-space.

References

1. Ewing, J.: Some examples of sphere bundles over spheres which are loop spaces mod p, Bull. Amer. Math. Soc. **80** (1974), p. 935–938
2. Hemmi, Y.: Higher homotopy commutativity of H-spaces and the mod p torus theorem, Pacific J. of Math. **149**(1) (1991), p. 95–111
3. James, I.M. and Thomas E.: Homotopy abelian topological groups, Topology **1** (1962), p. 237–240
4. McCleary, J.: Mod p decompositions of H-spaces; Another approach, Amer. Lecture Notes in Math **763** (1978), p. 70–87
5. McGibbon, C.A.: Some properties of H-spaces of rank 2, Proc. Amer. Math. Soc. **81** (1981), p. 121–124
6. McGibbon, C.A.: Homotopy commutativity in localized groups, Amer J. Math **274** (1984), p. 665–678

7. McGibbon, C.A.: Higher forms of homotopy commutativity and finite loop spaces, Math Z. **201** (1989), p. 363–374
8. Mimura, M and Toda, H.: Cohomology operations and the homotopy of compact Lie groups I, Topology **9** (1979), p. 317–336
9. Porter, G.J.: Higher order Whitehead products, Topology **3** (1965), p. 123–135
10. Toda, H.: "Composition methods in homotopy groups of spheres," Ann. of Math. Studies n. **49**. Princenton Universtiy Press, 1962
11. Williams, F.D.: Higher homotopy commutativity, Trans Amer. Math. Soc. **139** (1969), p. 191–206
12. Williams, F.D.: Higher Somelson products, J. Pure and Applied Alg. **2** (1972), p. 249–260