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On zero-dimensional subschemes of a complete intersections

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0 Introduction

The study of the postulation of a zero-dimensional scheme in $\mathbf{P}^n$ is a classical problem in Algebraic Geometry. One of the very useful ways of studying curves in the projective space, is to consider the set of points which arise as a general hyperplane section of the curve. This is usually called the *Castelnuovo method* after the crucial work made by G. Castelnuovo on the problem of the classification of all the possible genera of a projective curve in $\mathbf{P}^3$ in terms of its degree (see [C]).

These ideas have been reconsidered later for example by P. Dubreil in [Du] and more recently by D. Eisenbud and J. Harris in [EH], where it was definitely explained how a deep analysis of the postulation of a finite set can give very strong results in the theory of projective curves. Interesting developments of this method can be found in [GP], [MR] and [GM].

Unfortunately, most of the work was done for a finite set of points in the projective plane and only recently sporadic results appeared for points in $\mathbf{P}^n$, $n \geq 3$ (see [PPR], [R1], [R2], [CCD] and [BGM]).

The impulse to consider zero-dimensional schemes in $\mathbf{P}^n$, $n \geq 3$, came also from the relevance of this problem to the more general question of the possible Hilbert Functions of a graded domain. This point of vew, started with the fundamental work of Macaulay in [M], found a basic and crucial elaboration in the beautiful paper of R. Stanley [St1], who first realized strong connections with problems from combinatorics. Important contributions along this line have been given in [Hi], [Go] and [Gr].

This paper was motivated by an attempt to extend some of the classical results on the postulation of a finite set of points in $\mathbf{P}^2$, to $\mathbf{P}^n$, $n \geq 3$, where the

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Hilbert-Burch structure theorem, which was so crucial in codimension two, is no more available.

To do that, we present here a method which is based on a well-known property of a graded artinian Gorenstein ring which gives a sort of duality between the Hilbert Function of an ideal and that of its annihilator. The formula has already appeared elsewhere, see for example [DGO], but the novelty here is to use it for the study of the postulation of a zero-dimensional scheme X in $\mathbf{P}^n$, by means of the degrees of the hypersurfaces of a complete intersection passing through X.

The main result of this paper (Theorem 2.1) gives strong information on the behaviour of the Hilbert function $h_X(t)$ of X. More precisely, we can control the tail of the h-vector of X and this enables us to present large classes of zero-dimensional schemes X in $\mathbf{P}^n$ for which the h-vector is unimodal or of decreasing type. These classes include all zero-dimensional schemes in $\mathbf{P}^2$ or those in $\mathbf{P}^3$ which lie on an irreducible quadric or cubic.

As a consequence of the main theorem, we also obtain upper bounds for the minimal number of generators of homogeneous ideals in the polynomial ring $k[X_1, \dots, X_n]$ (Theorem 3.2 and 3.4). This extends a classical theorem of Dubreil to zero-dimensional schemes in $\mathbf{P}^n$.

1 Basic facts

Let k be any infinite field. By a graded ring A we always mean a standard graded k-algebra of finite type, that is, A is the quotient of a polynomial ring over k by an homogeneous ideal. We denote by

$$h_M(t) := dim_k M_t$$

the *Hilbert function* of any finitely generated graded A-module M. The generating function of this numerical function is the formal power series $P_M(z) := \sum_{t\geq 0} h_M(t)z^t$. In the case $M = A$, as a consequence of the Hilbert-Serre theorem, we can write $P_A(z) = H_A(z)/(1-z)^d$, where $H_A(z) \in \mathbf{Z}[z]$ is a polynomial with integer coefficients such that $H_A(1) \neq 0$. The natural number $H_A(1)$ is the *multiplicity* $e(A)$ of A while the degree of $H_A(z)$ is the *socle degree* of A. We will write $s(A)$ to indicate the socle degree of A. From the definition we get that the socle degree of a graded artinian ring A is s if $h_A(s) > 0$ and $h_A(s+1) = 0$. For example, the socle degree of an artinian complete intersection $k[X_1, \dots, X_n]/(F_1, \dots, F_n)$ is $\sum_{i=1}^n deg(F_i) - n$.

In the case A is Cohen-Macaulay with $P_A(z) = (\sum_{i=0}^s a_i z^i)/(1-z)^d$, the vector $(a_0, \dots, a_s)$ is the Hilbert Function of any artinian reduction of A. Hence $(a_0, \dots, a_s)$ is a sequence of positive integers which, following Stanley, is called the *h-vector of A*.

We shall often use the fact (see [St1]) that if A is a graded artinian Gorenstein ring, then the Hilbert function of A is symmetric, which means that if we let $s := s(A)$ then

$$h_A(t) = h_A(s-t)$$

for every $t = 0, \ldots, s$.

The basic result for our investigation is the following well-known property of the Hilbert function of an ideal in a graded artinian Gorenstein ring.

Theorem 1.1 *Let A be a graded artinian Gorenstein ring and let I be an homogeneous ideal of A. Then for every $t = 0, \ldots, s(A)$*

$$h_A(t) = h_{A/I}(t) + h_{A/(0:I)}(s(A) - t).$$

A proof for Theorem 1.1 can be found for example in [DGO].

The following consequence of Theorem 1.1 is more or less a well-known result in the ideal theory of Gorenstein rings.

For an homogeneous ideal I of A let $v(I)$ denote the minimal number of generators of I. Let $\tau(A)$ denote the *Cohen-Macaulay type* of A, which is by definition the dimension of the k-vector space $0 : A_1$. Note that if A is an artinian graded ring, $\tau(A)$ is the dimension of the socle of A.

Corollary 1.2 *Let A be a graded artinian Gorenstein ring and I an homogeneous ideal of A. Then for every $t = 0, \ldots, s(A)$*

$$h_{I/A_1I}(t) = h_{(0:A_1I)/(0:I)}(s(A) - t).$$

In particular

$$v(I) = \tau(A/(0 : I))$$

Proof. By Theorem 1.1 we have $h_A(t) = h_{A/I}(t) + h_{A/(0:I)}(s(A) - t)$, hence $h_I(t) = h_{A/(0:I)}(s(A) - t)$. In the same way we get $h_{A_1I}(t) = h_{A/(0:A_1I)}(s(A) - t)$. Hence

$$h_{I/A_1I}(t) = h_{A/(0:I)}(s(A) - t) - h_{A/(0:A_1I)}(s(A) - t).$$

This proves the first assertion. Since $v(I) = dim_k(I/A_1I)$ and

$$\tau(A/(0 : I)) = dim_k((0 : I) : A_1)/(0 : I)) = dim_k((0 : A_1I)/(0 : I)),$$

the second assertion follows as well.

2 Hilbert function

The main result of this paper is the following theorem which gives strong information on the Hilbert function of the graded ring R/I in terms of the degrees of the elements of a regular sequence in I.

Theorem 2.1 *Let $R = k[X_1, \ldots, X_n]$ and I be a zero-dimensional homogeneous ideal of R such that I contains a regular sequence $F_1, \ldots, F_n$ of forms of degrees $d_1 \leq \ldots \leq d_n$. Set $d := \sum_{i=1}^n d_i - n$.*

(a) If i is an integer, $1 \leq i \leq n$, then

$$h_{R/I}(t) \geq h_{R/I}(t + 1) + n - i$$

for $d - d_i + 1 \le t < s(R/I)$.

(b) If i is an integer, $1 \le i \le n-1$, *such that* $(F_1, \dots, F_{i-1})$ *is a prime ideal, then*

$$h_{R/I}(t) \ge \min\{nh_{R/I}(t+1),\ h_{R/I}(t+1) + 2n - i - 2\}$$

for $d - d_i + 1 \le t < s(R/I)$.

We will give a proof of this theorem at the end of this section. We illustrate first the main applications of the above result.

The most interesting cases of Theorem 2.1 are for $i = n, n-1$. In these cases, we have the following statements on the behaviour of the tail of the Hilbert function of R/I.

Corollary 2.2 *Let I be an ideal as in Theorem 1.1. Then*

(a) $h_{R/I}(t) \ge h_{R/I}(t+1)$ *for* $t \ge d - d_n + 1$. *Moreover,*

$$h_{R/I}(d - d_n) \ge h_{R/I}(d - d_n + 1)$$

if and only if for some $t \le d_1 + \cdots + d_{n-1} - n + 1$ *we have* $I_t \ne (F_1, \dots, F_{n-1})_t$.

(b) $h_{R/I}(t) > h_{R/I}(t+1)$ *for* $d - d_{n-1} + 1 \le t \le s(R/I)$.

(c) If $(F_1, \dots, F_{n-2})$ *is a prime ideal, then*

$$h_{R/I}(t) \ge h_{R/I}(t+1) + n - 1$$

for $d - d_{n-1} + 1 \le t < s(R/I)$.

This corollary covers some interesting results on the postulation of zero-dimensional schemes.

Let X be a zero-dimensional scheme in $\mathbf{P}^n = \mathbf{P}^n(k)$, where k is an algebraically closed field. We denote by $\Delta_X(t)$ the first difference of the Hilbert function of X, which is defined as follows:

$$\Delta_X(t) = \begin{cases} 1 & \text{if } t = 0, \\ h_X(t) - h_X(t-1) & \text{if } t > 0. \end{cases}$$

Since X is zero-dimensional, $\Delta_X(t) = 0$ for $t >> 0$. If A is the homogeneous coordinate ring of X, then $P_A(z) = \sum_{t=0}^{s} \Delta_X(t) z^t/(1-z)$ and $(\Delta_X(0), \Delta_X(1), \dots, \Delta_X(s))$ is the h-vector of A or of X.

Following [St2] and [MR] we say that the h-vector of X is *unimodal* if for some j we have

$$\Delta_X(0) \le \Delta_X(1) \le \dots \le \Delta_X(j) \ge \Delta_X(j+1) \ge \dots \ge \Delta_X(s) > 0.$$

and that it is of *decreasing type* if for some j we have

$$\Delta_X(0) \le \Delta_X(1) \le \dots \le \Delta_X(j) > \Delta_X(j+1) > \dots > \Delta_X(s) > 0.$$

The first notion has been studied mainly from the combinatorial point of view, but also it is very important in the problem of classification of all the possible Hilbert Functions of a graded domain (see [St2] and [Hi]). It was conjectured that the h-vector of a Gorenstein graded domain is unimodal, but we do not even know any Cohen-Macaulay graded domain whose h-vector is not unimodal.

The notion of h-vector of decreasing type plays an important role in the characterization of the general hyperplane sections of a codimension two reduced irreducible arithmetically Cohen-Macaulay and normal projective variety. See [Ha], [GP], [MR] , [GM] and [HTV] for a deeper investigation of this notion. Similar results on the postulation of zero-dimensional schemes on smooth quadrics in $\mathbf{P}^3$ have been recently discovered by Raciti, Paxia and Ragusa (see [R1], [R2] and [PRR]). The above Theorem and Corollary extend some of these results to a more general situation.

For example the main Theorem 3.4 in [R1] is our Corollary 2.2, (b) in the case $n = 3$. Further, part (a) of Theorem 2.1 for $i = n$ and part (b) for $i = n - 1$, have been proved in [R2], Theorem 2.2, under the restrictive assumption that R/I is an artinian reduction of the homogeneous coordinate ring of a zero-dimensional subscheme of an irreducible quadric in $\mathbf{P}^3$. Finally Theorem 3.1 in [R2] is a trivial consequence of the second part of (a) in Corollary 2.2.

In the sequel we will denote by $a_1 \leq a_2 \leq \ldots \leq a_r$ the degrees of the elements of a homogeneous minimal basis of the defining ideal of X, arranged in non-decreasing order.

Theorem 2.3 *Let X be a non-degenerate zero-dimensional scheme in $\mathbf{P}^n$. Assume that X lies on a complete intersection of $n-1$ hypersurfaces of degree $a_1, \ldots, a_{n-1}$ and moreover that $a_n \geq a_1 + \cdots + a_{n-1} - n$. Then the h-vector of X is unimodal.*

Proof. Let $I(X)$ be the defining ideal of X in $S = k[X_0, \ldots, X_n]$. Without restriction we may assume that X_0 is a non-zerodivisor of $I(X)$. Let I denote the artinian reduction $I(X) + (X_0)/(X_0)$ of $I(X)$ in $R = k[X_1, \ldots, X_n]$. Then $h_{R/I}(t) = \Delta_X(t)$. Moreover, there exists in I a regular sequence $F_1, \ldots, F_n$ with

$$d_1 = \deg F_1 = a_1, \ldots, d_{n-1} = degF_{n-1} = a_{n-1}, d_n = degF_n \geq a_n.$$

With these notations we have $d - d_n = a_1 + \ldots + a_{n-1} - n$, hence $a_n \geq d - d_n$. Further it is clear that for $t \leq a_n - 1$, $h_{R/I}(t) = h_{R/J}(t)$, where $J = (F_1, \ldots, F_{n-1})$. Since R/J is Cohen-Macaulay of positive dimension, $h_{R/J}(t) \leq h_{R/J}(t+1)$ for every $t \geq 0$. Hence, if $a_n \geq d - d_n + 1$, then $a_n - 1 \geq d - d_n$ so that

$$h_{R/I}(t) = h_{R/J}(t) \leq h_{R/J}(t+1) = h_{R/I}(t+1)$$

for every $t \leq d - d_n - 1$; since by the first part of Corollary 2.2, (a), $h_{R/I}(t)$ is not increasing for $t \geq d - d_n + 1$, the conclusion follows.

If $a_n = d - d_n$, then $a_n - 1 = d - d_n - 1$, so that

$$h_{R/I}(t) = h_{R/J}(t) \leq h_{R/J}(t+1) = h_{R/I}(t+1)$$

for every $t \leq d - d_n - 2$; by the second part of Corollary 2.2, (a), $h_{R/I}(t)$ is not increasing for $t \geq d - d_n$. As before we are left with a unique gap, so the conclusion follows.

In Theorem 2.3 we can replace the condition that X lies on a complete intersection of $n-1$ hypersurfaces of degree $a_1, \dots, a_{n-1}$ by the stronger condition that X lies on an *irreducible* complete intersection C of $n-2$ hypersurfaces of degree $a_1, \dots, a_{n-2}$. Moreover, the condition $a_n \geq a_1 + \cdots + a_{n-1} - n$ is satisfied in the following cases:

$$\begin{aligned} n &= 2; \\ n &= 3,\ a_1 \leq 3; \\ n &= 4,\ a_1 = a_2 = 2. \end{aligned}$$

Now we will present some cases where the h-vector of X is of decreasing type. Let us assume that X lies on a complete intersection of n hypersurfaces of degree $a_1, \dots, a_n$. As in the proof of Theorem 2.3, $\Delta_X(t) \leq \Delta_X(t+1)$ for $t < a_n - 1$. Hence we have that the h-vector of X is of decreasing type if $\Delta_X(t)$ is decreasing for $t \geq a_n$. Using the statement (b) of Corollary 2.2 we can see that this is true if $d - a_{n-1} + 1 \leq a_n$. It is easy to check that this condition is satisfied in the following cases:

$$\begin{aligned} n &= 2; \\ n &= 3,\ a_1 = 2; \end{aligned}$$

Combining the above observations with Theorem 2.3 and Corollary 2.2 (c) we obtain the following results on the postulation of zero-dimensional schemes in $\mathbf{P}^n$, $n = 2, 3, 4$.

Corollary 2.4 (cf. [H]) *Let X be a zero-dimensional scheme in $\mathbf{P}^2$. Then*

(a) The h-vector of X is unimodal.
(b) The h-vector of X is of decreasing type if X lies on a complete intersection C of two curves of degrees $a \leq b$ such that there is no curve of degree $< b$ passing through X but not C.

The assumption of Corollary 2.4 (b) is satisfied if X arises as an hyperplane section of a reduced irreducible curve of $\mathbf{P}^3$. According to [Sau] and [GM] (see also [HTV]) any reduced irreducible curve V in $\mathbf{P}^3$ lies on a complete intersection C of two surfaces of degrees $a \leq b$ such that there is no surface of degree $< b$ passing through V but not C.

Corollary 2.5 (cf. [R2], [PRR]) *Let X be a non-degenerate zero-dimensional scheme in $\mathbf{P}^3$. Then*

(a) The h-vector of X is unimodal if X lies on an irreducible quadric or cubic.
(b) The h-vector of X is of decreasing type if X lies on a complete intersection C of a quadric with two surfaces of degree $a \leq b$ such that there is no surface of degree $< b$ passing through X but not C. Moreover, if the quadric is irreducible, then $\Delta_X(t) \geq \Delta_X(t) + 2$ for $t \geq b$.

By the above remark, the assumption of Corollary 2.5 (b) is satisfied if X is the intersection of a quadric with a reduced irreducible curve in $\mathbf{P}^3$.

Corollary 2.6 *Let X be a non-degenerate zero-dimensional scheme in $\mathbf{P}^4$.*

The h-vector of X is unimodal if X lies on an irreducible complete intersection of two quadrics.

Now we want to show that from Theorem 2.1 we easily get a strong version of a classical result which, accordingly to [D], was first noted by Castelnuovo and reborn a number of times since: see [Hn], [GP], [Hs], [DGM] and especially [D] where a new and elementary proof is presented. This classical result is a special case, namely the case $n = 2$, of the following corollary.

Corollary 2.7 *Let $R = k[X_1, \ldots, X_n]$ and V be a proper subspace of R_s whose elements generate an ideal of height n. If t is an integer such that $t \geq ns-2s-n+1$ and $codim(R_t V) > 0$, then*

$$codim_k(R_{t+1}V) < codim_k(R_t V).$$

Proof. Let I be the ideal generated by a vector base of V. Then $I_{t+s+1} = R_{t+1}V$ and $I_{t+s} = R_t V$. Hence $codim_k(R_{t+1}V) = h_{R/I}(t+s+1)$ and $codim_k(R_t V) = h_{R/I}(t+s)$. The conclusion now follows from Corollary 2.1 (b) because $sn - s - n + 1 \leq t + s \leq s(R/I)$, where the last inequality follows from the assumption that $codim_k(R_t V) > 0$.

Another easy consequence of Theorem 2.1 is the following result which has been proved in [DGM, Theorem 2.4] by using an anusual genericity argument. Here we write, as before, Δh_A for the first difference of the Hilbert function of a graded ring A, which is defined as follows:

$$\Delta h_A(i) = \begin{cases} 1 & \text{if } i = 0, \\ h_A(i) - h_A(i-1) & \text{if } i > 0. \end{cases}$$

Corollary 2.8 *Let $R = k[X, Y]$ and $J \subseteq I$ be homogeneous ideals of R such that $J = (F, G)$ where F, G is a regular sequence of degree $a \leq b$. Then*

$$\Delta h_{R/J}(i) \geq \Delta h_{R/I}(i)$$

for every $i = 0, \ldots, s(R/I) + 1$.

Proof. It is well known that

$$\Delta h_{R/J}(i) = \begin{cases} 1 & \text{if } 0 \leq i \leq a-1 \\ 0 & \text{if } a \leq i \leq b-1 \\ -1 & \text{if } b \leq i \leq a+b-1. \end{cases}$$

Since $h_{R/I}(n+1) - h_{R/I}(n) \leq 1$ for every $n \geq 0$, the conclusion follows immediately in the interval $0 \leq i \leq a-1$, while in the interval $a \leq i \leq s(R/I) + 1$, it is a trivial consequence of Corollary 2.2.

The following example shows that this result does not hold if R has dimension > 2.

Example Let $R = k[X,Y,Z]$, $J = (X^3,Y^4,Z^5)$ and $I = (X^3,X^2Y,XZ^3,Y^4,Z^5)$. Then $s(R/I) = 7$ but $\Delta h_{R/J}(7) = -3$ while $\Delta h_{R/I}(7) = -2$.

We come now to the proof of Theorem 2.1 and its corollary.

For the proof we need the following crucial lemma where we use this well known property of regular sequences (see [RV], Lemma 1.3).

Let A be a graded ring and I an homogeneous ideal. If I/I^2 is a free A/I module and $\{r_1,\dots,r_t\}$ are homogeneous elements which form a regular sequence on A/I, then $\{r_1,\dots,r_t\}$ is a regular sequence on A/I^p for every $p \geq 1$.

Lemma 2.9 *Let A be a graded ring of depth $g \geq 1$ and embedding dimension n. If V is a subspace of A_t of dimension $r > 0$, then*

$$dim_k(A_1V) \geq r + g - 1.$$

Further, if $g \geq 2$ and V contains an element which is a non zero divisor in A, then

$$dim_k(A_1V) \geq \min\{rn, r + n + g - 3\}.$$

Proof. Let $F_1,\dots,F_r$ be a vector base of V and let $x_1,\dots,x_g$ be a regular sequence of linear forms in A. Set $x := (x_1,\dots,x_g)$.

Claim 1 The vector space xV generated by the rg vectors $\{x_iF_j\}$, $i = 1,\dots,g$ and $j = 1,\dots,r$, cannot be generated by a set of vectors $\{x_iF_j\}$ involving only $x_1,\dots,x_m$ with $m < g$.

Assume the contrary. Then we have $x_gF_j \in xV \subset (x_1,\dots,x_m)$ for every j, hence $F_j \in (x_1,\dots,x_m)$ for every j. Thus $xV \subset (x_1,\dots,x_m)^2$, and, for every j, we get $x_gF_j \in xV \subset (x_1,\dots,x_m)^2$. Since $x_1,\dots,x_g$ is a regular sequence, this implies that for every j, $F_j \in (x_1,\dots,x_m)^2$ so that $xV \subset (x_1,\dots,x_m)^3$. Going on in this way, we get $xV \subset (x_1,\dots,x_m)^{t+2}$, which is impossible. This proves Claim 1.

Now it is clear that $x_1F_1,x_1F_2,\dots,x_1F_r$ are vectors in xV which are linearly independent. We can use $g-1$ times Claim 1 to find vectors $x_2F_{i_2},\dots,x_gF_{i_g}$ in xV such that

$$x_1F_1,x_1F_2,\dots,x_1F_r,x_2F_{i_2},\dots,x_gF_{i_g}$$

are linearly independent. This proves the first part of Lemma 2.9.

As for the second assertion, let $x_1,\dots,x_n$ be the linear forms which are a k-basis of A_1. We may assume that $F_1,x_1,\dots,x_{g-1}$ form a regular sequence in A.

Claim 2 $F_1x_1,F_1x_2,\dots,F_1x_n,F_2x_1,\dots,F_rx_1$ are linearly independent vectors in A_1V.

Let

$$\sum_{i=1}^{n} \lambda_i F_1 x_i + \sum_{i=2}^{r} \mu_i F_i x_1 = 0$$

for some $\lambda_i, \mu_i \in k$. Then we get

$$F_1 \left(\sum_{i=1}^{n} \lambda_i x_i \right) + x_1 \left(\sum_{i=2}^{r} \mu_i F_i \right) = 0.$$

This implies $\sum_{i=2}^{r} \mu_i F_i \in (F_1)$. Since $F_1, \dots, F_r$ share the same degree and are linearly independent, we get $\mu_2, \dots, \mu_r = 0$. Since F_1 is a non zero divisor in A, we get $\sum_{i=1}^{n} \lambda_i x_i = 0$ and this implies $\lambda_1 = \dots = \lambda_n = 0$ by the linear independence of $x_1, \dots, x_n$. This proves Claim 2.

Claim 2 implies that we have $n + r - 1$ vectors in $A_1 V$ which are linearly independent. Hence, if $r = 1$, then $n + r - 1 = rn$ and the conclusion follows.

Claim 3 Let $r \geq 2$ and $x := (x_1, \dots, x_{g-1})$. Then xV cannot be generated by

$$F_1 x_1, F_1 x_2, \dots, F_1 x_n, F_2 x_1, \dots, F_r x_1$$

plus a set of vectors $\{x_i F_j\}$ involving only $x_1, \dots, x_m$ with $m < g - 1$.

Assume the contrary. Then we have

$$x_{g-1} F_j \in xV \subset (F_1, x_1, \dots, x_m),$$

hence $F_j \in (F_1, x_1, \dots, x_m)$ for every j. Thus $xV \subset (F_1) + (x_1, \dots, x_m)^2$ and, for every j, we get

$$x_{g-1} F_j \in xV \subset (F_1) + (x_1, \dots, x_m)^2.$$

Since $F_1, x_1, \dots, x_m$ is a regular sequence, this implies that for every j, $F_j \in (F_1) + (x_1, \dots, x_m)^2$ so that $xV \subset (F_1) + (x_1, \dots, x_m)^3$. Going on in this way we get $xV \subset (F_1) + (x_1, \dots, x_m)^{t+2}$ so that $xV \subset (F_1)$. This implies for example $x_1 F_2 = F_1 L$ for some linear form L in A. Hence $F_2 = \alpha F_1$, a contradiction. This proves Claim 3.

We can use $g - 2$ times Claim 3 to find vectors $x_2 F_{i_2}, \dots, x_{g-1} F_{i_{g-1}}$ in xV such that

$$F_1 x_1, F_1 x_2, \dots, F_1 x_n, F_2 x_1, \dots, F_r x_1, x_2 F_{i_2}, \dots, x_{g-1} F_{i_{g-1}}$$

are linearly independent. This proves the second part of Lemma 2.9.

Proof of Theorem 2.1 Let $A := R/J$ and $\bar{I} := I/J$, where $J = (F_1, \dots, F_n)$. Using Theorem 1.1 we get

$$\begin{aligned} h_{R/I}(t) = h_{A/\bar{I}}(t) &= h_A(t) - h_{A/0:\bar{I}}(d - t) \\ &= h_A(t) - h_A(d - t) + h_{0:\bar{I}}(d - t) = h_{0:\bar{I}}(d - t) \end{aligned}$$

for every $t \leq d$. For convenience we set $m = d - t$.

To prove the first assertion of Theorem 2.1 (a) we need to verify that

$$h_{0:\bar{I}}(m) \geq h_{0:\bar{I}}(m-1) + n - i$$

for $d - s(R/I) < m \leq d_i - 1$. Set $B = R/(F_1, \ldots, F_{i-1})$ and $Q = (J : I)/(F_1, \ldots, F_{i-1})$. For $m \leq d_i - 1$, we have $J_m = (F_1, \ldots, F_{i-1})_m$ so that we may identify $(0 : \bar{I})_m = (J : I/J)_m$ with the subspace Q_m of B_m. For $m > d - s(R/I)$, we have $t < s(R/I)$ so that

$$dim_k Q_{m-1} = h_{0:\bar{I}}(m-1) = h_{R/I}(t+1) > 0.$$

Therefore we can apply the first part of Lemma 2.9 to Q_{m-1}. Note that B has depth $n - i + 1 > 0$. Then we get

$$\begin{aligned} h_{0:\bar{I}}(m) = dim_k Q_m &\geq dim_k Q_{m-1}B_1 \\ &\geq dim_k Q_{m-1} + n - i = h_{0:\bar{I}}(m-1) + n - i, \end{aligned}$$

which proves Theorem 2.1 (a). The same arguments also show that the second part of Lemma 2.9 gives Theorem 2.1 (b).

Proof of Corollary 2.2 All the statements of Corollary 2.2 follow from Theorem 2.1 except the second assertion in (a). We need to prove that

$$h_{R/I}(d - d_n) < h_{R/I}(d - d_n + 1) \qquad (*)$$

if and only if $I_t = (F_1, \ldots, F_{n-1})_t$ for every $t \leq d - d_n + 1$.

If we let $B = R/(F_1, \ldots, F_{n-1})$ and $Q = (J : I)/(F_1, \ldots, F_{n-1})$, then as in the proof of Theorem 2.1, we get

$$h_{R/I}(d - d_n + 1) = h_Q(d_n - 1).$$

Since it is clear that $dim(JB)_{d_n} = dim(J/(F_1, \ldots, F_{n-1}))_{d_n} = 1$, from the exact sequence $0 \to JB \to Q \to (J : I)/_J \to 0$ we also get

$$h_{R/I}(d - d_n) = h_Q(d_n) - 1$$

so that $(*)$ is equivalent to

$$h_Q(d_n) - 1 < h_Q(d_n - 1)$$

or to

$$h_Q(d_n) \leq h_Q(d_n - 1).$$

Since Q is an ideal in a one dimensional Cohen-Macaulay graded ring, we clearly have $h_Q(d_n) \geq h_Q(d_n - 1)$. Hence $(\star)$ is equivalent to

$$h_Q(d_n) = h_Q(d_n - 1).$$

Let us assume that $h_Q(d_n) = h_Q(d_n - 1)$. If L is any linear form which is a non zero divisor in B, this implies $Q_{d_n} = LQ_{d_n - 1}$. Let P be the ideal generated by the forms of degree $d_n - 1$ in Q. Then $P \subseteq Q$, $P_{d_n-1} = Q_{d_n-1}$ and it is easy to see that $P_t = L^{t-d_n+1}P_{d_n-1}$ for every $t \geq d_n - 1$. Since $\bar{F}_n \in Q_{d_n} \subseteq P$, B/P is an artinian graded ring . Hence $B_t = P_t$ for $t >> 0$. This implies that for $t >> 0$

$$e(B) = h_B(t) = h_P(t) = h_P(d_n - 1) = h_Q(d_n - 1) \leq h_B(d_n - 1) \leq e(B).$$

Hence $h_{B/Q}(d_n - 1) = 0$ and $h_B(d_n - 1) = e(B)$. We get $d_n - 1 \geq s(B) = d - d_n + 1$ and

$$0 = h_{B/Q}(d_n - 1) = h_{R/(J:I)}(d_n - 1) = h_{R/J}(d - d_n + 1) - h_{R/I}(d - d_n + 1),$$

hence $d_n > d - d_n + 1$ and $I_{d-d_n+1} = J_{d-d_n+1}$. If, by contradiction, $t \leq d - d_n + 1$, and there exists a form $G \in I_t$, $G \notin (F_1, \ldots, F_{n-1})_t$, then $GL^{d-d_n+1-t} \in I_{d-d_n+1}$, $GL^{d-d_n+1-t} \notin (F_1, \ldots, F_{n-1})_{d-d_n+1}$. Since $d_n > d - d_n + 1$, $(F_1, \ldots, F_{n-1})_{d-d_n+1} = J_{d-d_n+1}$ and we get a contradiction to the equality $I_{d-d_n+1} = J_{d-d_n+1}$.

Conversely, since $F_n \in I$, $F_n \notin (F_1, \ldots, F_{n-1})$, the assumption $I_t = (F_1, \ldots, F_{n-1})_t$ for every $t \leq d - d_n + 1$, implies $d_n > d - d_n + 1 = s(B)$. This means that $h_B(d_n) = h_B(d_n - 1)$ and $I_t = (F_1, \ldots, F_{n-1})_t = J_t$ for every $t \leq d - d_n + 1$. By Theorem 1.1 we get

$$h_{B/Q}(d_n - 1) = h_{R/(J:I)}(d_n - 1) = h_{R/J}(d - d_n + 1) - h_{R/I}(d - d_n + 1) = 0,$$

hence $h_{B/Q}(d_n) = 0$. From this we get $h_Q(d_n) = h_Q(d_n - 1)$, as wanted.

We end this section with an application related to the classical Cayley-Bacharach theorem.

For a zero-dimensional scheme X in $\mathbf{P}^n$ the Hilbert function $h_X(t)$ is strictly increasing until it reaches the degree $\deg(X)$ of X, at which it stabilizes. The number $\omega_X(t) := \deg(X) - h_X(t)$ is called the *superabundance* of the linear system of hypersurfaces of degree t passing through X. Also $\omega_X(t) = h^1\Im_X(t)$, where $\Im_X$ is the ideal sheaf of X.

Let X be a zero-dimensional scheme in $\mathbf{P}^n$ which is the complete intersection of n hypersurfaces of degree $d_1, \ldots, d_n$ and $d := \sum_{i=1}^n d_i - n$. For a given subscheme Y of X, let Z be Z be the residual scheme of Y on X. Then using Theorem 3b) in [DGO] one can easily prove for every integer t, $0 \leq t \leq d - 1$,

$$\omega_Z(t) = h_X(d - t - 1) - h_Y(d - t - 1).$$

As a trivial corollary of this formula, one can proof the classical Cayley-Bacharach theorem in projective n-space, as commented in [DGO].

Here we present another application of this formula, thus giving a new way to think about Cayley-Bacharach property.

Corollary 2.10 [B] *Let X be a zero-dimensional scheme in $\mathbf{P}^n$ which is the complete intersection of n hypersurfaces of degree $d_1, \ldots, d_n$. Let $d := \sum_{i=1}^n d_i - n$, and Y a subscheme of X with $\omega_Y(d - 1) > 0$, then $Y = X$.*

Proof. By Proposition 3.1 we have $\omega_Z(0) = h_X(d-1) - h_Y(d-1)$. This implies

$$|X| - |Y| - h_Z(0) = |X| - 1 - (|Y| - \omega_Y(d-1))$$

which gives $h_Z(0) = 1 - \omega_Y(d-1)$ for $Z = X \setminus Y$. Thus we get $h_Z(0) = 0$, hence $I(Z) = (1)$ which implies $Z = \emptyset$ and $Y = X$.

3 Number of generators

In this last section we apply our methods to bound the minimal number of generators $v(I)$ of a homogeneous ideal I of a polynomial ring $R = k[X_1, \dots, X_n]$.

Our idea is to combine Theorem 1.1 with a simple result of J. Sally [Sa] which says that the minimal number of generators of any ideal in a one-dimensional Cohen-Macaulay local (or homogeneous) ring A is bounded above by the multiplicity $e(A)$ of A. For this we shall need the following easy lemma.

Lemma 3.1 *Let A be a graded ring. Then*

$$\max_{I \subseteq A} v(I) = \max_{J \subseteq A} \tau(A/J).$$

Proof. For every homogeneous ideal I we let J to be the ideal $A_1 I$. Then we have

$$v(I) = dim_k(I/J) \le dim_k(J : A_1/J) = \tau(A/J).$$

On the other hand for every homogeneous ideal J in A we have

$$\tau(A/J) = dim_k(J : A_1/J) \le dim_k(J : A_1/A_1(J : A_1)) = v(J : A_1).$$

Now the conclusion is immediate.

The main result of this section is a generalization of a classical theorem of Dubreil [Du, Theoreme II] which says that for any homogeneous ideal I of height 2 in $k[X_1, X_2]$, $v(I) \le a + b - s$, where a is the least degree of forms in I, b is the least number such that I contains a regular sequence of two forms of degree a and b, and s is the socle degree of $k[X_1, X_2]/I$.

Theorem 3.2 *Let I be a height n homogeneous ideal of $R := k[X_1, \dots, X_n]$, $n \ge 2$. Assume that there exists in I a regular sequence $F_1, \dots, F_n$ of forms of degrees $d_1 \le \dots \le d_n$ such that $(F_1, \dots, F_{n-2})$ is a prime ideal. Let $d = \sum_{i=1}^{n} d_i - n$, $d_0 = 1$ and s be the socle degree of R/I. Then*

$$v(I) \le (d - s) \prod_{i=0}^{n-2} d_i + n.$$

Proof. Let $A = R/(F_1,\dots,F_n)$ and $\bar{I} = I/(F_1,\dots,F_n)$. Then $s(A) = d$ and

$$v(I) \le v(\bar{I}) + n.$$

By Theorem 1.1 we get $h_{0:\bar{I}}(d - s) = h_{A/\bar{I}}(s) > 0$. Hence there exists a form $F \in R_{d-s}$ such that the image of F in A is a non-zero element of $0 : \bar{I}$. By Corollary 1.2 we may write

$$v(\bar{I}) = \tau(A/0 : \bar{I}) = \tau(B/Q),$$

where $B = R/(F_1,\dots,F_{n-2},F)$ and $Q = (J : I)B$. Since $(F_1,\dots,F_{n-2})$ is a prime ideal and $F \notin (F_1,\dots,F_n)$, the sequence $F_1,\dots,F_{n-2},F$ is regular, hence B is a one-dimensional Cohen-Macaulay ring. Using Lemma 4.1 and Sally's bound for the number of generators of ideals in B we get

$$\tau(B/Q) \le e(B) = (d - s)\prod_{i=0}^{n-2} d_i$$

which gives the conclusion.

Note that Theorem 3.2 does not hold if we drop the assumption that $(F_1,\dots,F_{n-2})$ is a prime ideal. If $d - s > d_{n-1}$, one should use instead of Theorem 3.2 the following trivial application of Sally's bound.

Lemma 3.3 (cf. [G, Proposition 3.7]) *Let I be an homogeneous ideal of $R := k[X_1,\dots,X_n]$, $n \ge 2$. If there exists a regular sequence $F_1,\dots,F_{n-1}$ in I of degree $d_1,\dots,d_{n-1}$, then*

$$v(I) \le \prod_{i=1}^{n-1} d_i + n - 1.$$

Proof. Let $B = R/(F_1,\dots,F_{n-1})$ and $Q = I/(F_1,\dots,F_{n-1})$. Then $v(I) \le v(Q)+ n - 1$. Since $e(B) = \prod_{i=1}^{n-1} d_i$, from Sally's bound we get $v(Q) \le \prod_{i=1}^{n-1} d_i$.

Note that for $n = 2$, Lemma 3.3 gives another classical result of Dubreil [Du, Theoreme I] (see also [G] and [DGM]). Now we will use Lemma 3.3 to prove a modified version of Theorem 3.2 which sometimes gives a better bound for $v(I)$.

Theorem 3.4 *Let I be a height n homogeneous ideal of $R := k[X_1,\dots,X_n]$, $n \ge 2$. Assume that there exists in I a regular sequence $F_1,\dots,F_n$ of forms of degrees $d_1 \le \dots \le d_n$ such that $(F_1,\dots,F_{n-2})$ is a prime ideal. Let $d = \sum_{i=1}^{n} d_i - n$, $d_0 = 1$ and m be the largest degree of the elements of a homogeneous minimal basis for I. Then*

$$v(I) \le (d - m)\prod_{i=0}^{n-2} d_i + n + 1.$$

Proof. Consider the artinian Gorenstein ring $A := R/J$ where $J := (F_1, \ldots, F_n)$. If $m \le d_n$, then

$$\begin{aligned}(d-m)\prod_{i=0}^{n-2} d_i + n + 1 &\ge (d-d_n)\prod_{i=0}^{n-2} d_i + n + 1\\ &= (\sum_{i=1}^{n-1} d_i - n)\prod_{i=0}^{n-2} d_i + n + 1\\ &= 2 + (\sum_{i=1}^{n-2} d_i - n)\prod_{i=0}^{n-2} d_i + \prod_{i=0}^{n-1} d_i + n - 1\\ &\ge \prod_{i=0}^{n-1} d_i + n - 1,\end{aligned}$$

Therefore, if $m \le d_n$, the conclusion follows from Lemma 3.3.

Let $m > d_n$ and $\bar{I} = I/J$. By Corollary 1.2 we have

$$h_{(0:A_1\bar{I})/(0:\bar{I})}(d-m) = h_{\bar{I}/A_1\bar{I}}(m) > 0.$$

This means that we can find an element $F \in R_{d-m}$ such that the image f of F in A belongs to $0 : A_1\bar{I}$ but $f \notin 0 : \bar{I}$. Using Corollary 1.2 we also get

$$\begin{aligned}v(I) &\le v(\bar{I}) + n = \tau(A/0:\bar{I}) + n\\ &\le v((0:\bar{I}) : A_1) + n = v(0 : A_1\bar{I}) + n.\end{aligned}$$

If $m = d$, then $0 : A_1\bar{I} = A$ and the conclusion follows. If $m < d$, then since $(F_1, \ldots, F_{n-2})$ is a prime ideal which does not contain F, we get that $F_1, \ldots, F_{n-2}, F$ form a regular sequence and the ring $B := R/(F_1, \ldots, F_{n-2}, F)$ is Gorenstein of dimension 1. Therefore

$$\begin{aligned}v(0 : A_1\bar{I}) &\le v(0 : A_1\bar{I}/f) + 1 = v((J : R_1 I)/(J, F)) + 1\\ &\le v((J : R_1 I)/(F_1, \ldots, F_{n-2}, F)) + 1 \le (d-m)\prod_{i=0}^{n-2} d_i + 1,\end{aligned}$$

where the last inequality follows again by the quoted Sally's bound for the number of generators of ideals in B. The proof of Theorem 3.4 is now complete.

Since $m \le s+1$ the bound given in Theorem 3.4 is better than the one given in Theorem 3.2 only if $m = s+1$.

The following example shows that in Theorem 3.2 and Theorem 3.4 we can not delete the assumption that $(F_1, \ldots, F_{n-2})$ is a prime ideal in R.

Example Let $R = k[X, Y, Z]$ and $I = (X^2, XY^2, XZ^2, XYZ, Y^3, Z^4, Y^2Z^3)$. Then $d_1 = 2$, $d_2 = 3$, $d_3 = 4$, $d = 6$, $s = 4$, and $m = 5$. But $v(I) = 7$, while

$$(d-s)d_1 + 2 - 1 = (d-m)d_1 + 3 + 1 = 6.$$

References

[B] Briancon, J.: Sur le degré des relations entre polynomes. C. R. Acad. Sc. Paris 297: 553–556 (1983)

[BGM] Bigatti, A.M., Geramita, A. V., Migliore, J. C.: Geometric consequences of extremal behaviour in a Theorem of Macaulay (Preprint)

[C] Castelnuovo, G.: Ricerche di geometria sulle curve algebriche. Zanichelli, Bologna (1937)

[CCD] Chiantini, L., Ciliberto, C., Di Gennaro, V.: The genus of projective curves. Duke Math. J. 70: 229–245 (1993)

[D] Davis, E.D.: Complements to a paper of Dubreil. (Preprint)

[DGM] Davis, E.D., Geramita, A.V., Maroscia, P.: Perfect homogeneous ideals: Dubreil's theorems revisited. Bull. Sc. Math. 108: 143–185 (1984)

[DGO] Davis, E.D., Geramita, A.V., Orecchia, F.: Gorenstein algebras and the Cayley-Bacharach theorem. Proc. A. M. S. 93: 593–597 (1985)

[Du] Dubreil, P.: Sur quelques propriétés des systèmes de points dans le plan et des courbes algébraiques. Bull. Soc. Math. France 61: 258–283 (1933)

[EH] Eisenbud, D., Harris, J.: Curves in Projective space. Sem. Mat. Sup. 85, Le Presse de l'Université de Montreal, Montreal (1982)

[G] Geramita, A.: Remarks on the number of generators of some homogeneous ideals. Bull. Sc. Math. 107: 193–207 (1983)

[GM] Geramita, A., Migliore, J.: Hyperplane sections of a smooth curve in $\mathbf{P}^3$.Comm. Algebra 17: 3129–3164 (1989)

[Go] Gotzmann: Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes. Math. Z. 158: 61–70 (1978)

[GP] Gruson, L., Peskine, C.: Genre de courbes de l'espace projectif. Lect. Notes in Math. 687: 31–59 (1978)

[Gr] Green, M.: Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann. SLN 1389, Algebraic Curves and Projective Geometry. Proceedings Trento. Berlin Heidelberg New York: Springer-Verlag 1988

[Hi] Hibi, T.: Flawless O-sequences and Hilbert Functions of Cohen-Macaulay integral domains. J. Pure Appl. Algebra 60: 245–251 (1989)

[Hn] Hartshorne, R.: Stable vector bundles of rank 2. Math. Ann. 238: 229–280 (1978)

[HTV] Herzog, J., Trung, N.V., Valla, G.: On hyperplane sections of reduced irreducible varieties of low codimension. J. Math. Kyoto Univ. (to appear)

[M] Macaulay, F.S.: Some properties of enumeration in the theory of modular systems. Proc. Lond. Math. Soc. 26: 531–555 (1927)

[MR] Maggioni, R., Ragusa, A.: The Hilbert function of generic plane sections of curves of $\mathbf{P}^3$. Invent. Math. 91: 253–258 (1988)

[PRP] Paxia, G., Raciti, G., Ragusa, A.: Uniform properties and Hilbert functions for points on a smooth quadric. J. Algebra 149: 102–121 (1992)

[R1] Raciti, G.: Sulla Funzione di Hilbert di un sottoschema zero-dimensionale di $\mathbf{P}^3$. Ann. Univ. Ferrara 35: 99–112 (1989)

[R2] Raciti, G.: Hilbert Function and geometric properties for a closed zero-dimensional subscheme of a quadric $Q \subset \mathbf{P}^3$. Comm. in Alg. 18: 3041-3053 (1990)

[RV] Robbiano, L., Valla, G.: On Normal Flatness and Normal Torsion-freeness. J. Algebra 43: 552–560 (1976)

[Sa] Sally, J.: Numbers of generators of ideals in local rings. Lect. Notes in Pure and Appl. Math. 35, Marcel Dekker (1978)

[Sau] Sauer, T.: Smoothing projectively Cohen-Macaulay space curves. Math. Ann. 272: 83–90 (1985)

[St1] Stanley, R.: Hilbert functions of graded algebras. Adv. in Math. 28: 57–83 (1978)

[St2] Stanley, R.: On the Hilbert function of a Cohen-Macaulay domain. J. Pure and Appl. Al. 73: 307–314 (1991)