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Quasiconformal homeomorphisms on CR 3-manifolds with symmetries

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1 Introduction

Given an oriented, compact, smooth surface R of genus > 1, divide all complex structures on R into equivalence classes so that two structures are in the same class if and only if there is a conformal homeomorphism between them which is homotopic to the identity. Teichmüller's theorem says that for any two complex structures S_1 and S_2 on R, among all quasiconformal homeomorphisms homotopic to the identity, there is an unique homeomorphism which minimizes the conformal distortion with respect to S_1 and S_2, and this extremal quasiconformal homeomorphism can be characterized in terms of certain holomorphic quadratic differentials [2]. The maximal dilatation of extremal quasiconformal homeomorphism measures how different the class $[S_1]$ is from the class $[S_2]$. Since these fundamental results have been established, Teichmüller space, the space of all equivalence classes, became one of the most important objects of research in complex analysis. Comprehensive literatures on Teichmüller theory include Abikoff's [1], Zhong Li's [13] and Nag's [16].

Lempert proposed an analogous problem in the setting of Cauchy-Riemann (CR) manifolds as follows [12]. Given two CR structures on a 3-dimensional contact manifold, describe the quasiconformal homeomorphisms that have the least conformal distortion with respect to these two CR structures. These homeomorphisms, if exist, are said extremal. Their maximal dilatation measures the nonisomorphism of the two CR structures. A Teichmüller type distance between the two CR manifolds is defined by the infimum of the logarithms of the maximal dilatations of all quasiconformal homeomorphisms between them. This can be regarded as a variational approach to the embeddability of an abstract CR structure. If the distance between an abstract CR structure and an embeddable CR structure is zero and is also realized, then the abstract CR structure is conformally equivalent to the embedded one. We were able to prove that with a very weak regularity assumption conformal equivalence implies CR equivalence for embeddable CR structures, and we conjecture this holds for general CR structures. Otherwise, one would like to know how far this CR structure is from the space of all embeddable structures.

The concept of quasiconformality is classically given on Riemann surfaces and Riemannian manifolds. It is a major machinery applied in Teichmüller theory. Mostow introduced it for symmetric spaces of real rank one, which include the Heisenberg groups [15]. Later Korányi and Reimann generalized notion of quasiconformality to strongly pseudoconvex CR manifolds [9].

We will study extremal quasiconformal homeomorphisms between smooth, compact, strongly pseudoconvex CR manifolds of dimension 3. In this paper, we shall mostly consider CR manifolds that admit a transversal CR action of S^1, in particular, the 3-sphere S^3 with the standard circle action. We remark that these CR structures are always embeddable ([6] [11]); if the underlying contact manifold is S^3, they can even be embedded into $\mathbb{C}^2$ as circular hypersurfaces [6].

There are two basic questions here. The first question is whether an extremal quasiconformal homeomorphism between two S^1-invariant CR structures is S^1-equivariant. The second question is what is the characterization of equivariant quasiconformal homeomorphisms.

The space of S^1-orbits of an invariant CR manifold is a surface with a complex structure induced from the CR structure. An equivariant homeomorphism between two S^1-invariant CR manifolds defines a quotient homeomorphism between the corresponding Riemann surfaces. In this paper we prove that an equivariant K-quasiconformal homeomorphism is characterized by an area-preserving property and K-quasiconformality of its quotient homeomorphism (Theorem 3.5, 3.7). This answers the second question. We also develop the first and second variation of the conformal distortion on S^3 (Proposition 5.1, 5.3). The method to compute the variation on S^3 works on any CR 3-manifolds. Then we construct a family of smooth S^1-invariant CR structures on S^3 so that no extremal quasiconformal homeomorphism between these CR structures and the standard CR structure is S^1-equivariant (Theorem 6.1). Thus we show that circular symmetry is broken for extremal quasiconformal homeomorphisms between these S^1-invariant CR structures.

Recently we found that in certain situations an extremal quasiconformal homeomorphism in a homotopy class must be equivariant. There the extremal homeomorphisms have behaviors analogous to Teichmüller mappings on Riemann surfaces. Details will appear in a forthcoming paper.

2 Quasiconformal homeomorphisms and contact flows

Let M be a 3-dimensional, connected, smooth, contact manifold with a smooth non-degenerate contact form η. Denote the contact bundle by $HM \triangleq \operatorname{Ker} \eta$. Let $J_0 : HM \longrightarrow HM$ be a smooth endomorphism such that $J_0^2 = -\mathrm{id}$. Thus J_0 is a smooth complex structure on HM which defines a strongly pseudoconvex CR structure on M. The corresponding CR manifold is denoted by M_0.

Call the orientation of M given by $d\eta \wedge \eta \neq 0$ positive and the orientation of HM given by $d\eta|_{HM}$ positive. Note if $\eta' = \lambda\eta$ with a function $\lambda \neq 0$ is another contact form, the orientation of M given by $d\eta' \wedge \eta' = \lambda^2 d\eta \wedge \eta$ is positive. The orientation of HM given by $d\eta'|_{HM} = \lambda d\eta|_{HM}$ is either positive when $\lambda > 0$ or negative when $\lambda < 0$.

Let $X \neq 0$ be a local section of HM, then X and J_0X are linearly independent. $d\eta$ is nondegenerate on HM, so $\langle d\eta, X \wedge J_0X \rangle \neq 0$. We say the CR structure of M_0 is positively (or negatively) oriented with respect to η if $\langle d\eta, X \wedge J_0X \rangle > 0$ (or < 0). Note

$$\langle d\eta \wedge \eta, X \wedge J_0X \wedge [J_0X, X] \rangle = (\langle d\eta, X \wedge J_0X \rangle)^2 > 0. \tag{2.1}$$

Hence $X, J_0X, [J_0X, X]$ is always a positively oriented frame no matter the CR structure is positively oriented or not.

A differentiable curve on M is called Legendrian if its tangent vector at each point is in the contact bundle HM. Let $U \subset M$ be an open set, Γ be a contact fibration of U, i.e., Γ is a smooth fibration of U consisting of smooth Legendrian curves. A subfamily Γ_1 of a contact fibration Γ of U is said to be of measure zero if for any smooth surface S which is transversal to each $\gamma \in \Gamma$ and any smooth area form ω on S

$$\int_{\{S \cap \gamma | \gamma \in \Gamma_1\}} \omega = 0. \tag{2.2}$$

Assume that M_1 is another smooth, strongly pseudoconvex CR manifold with the same underlying contact manifold M and a complex structure J_1 on HM. A homeomorphism $f : M_1 \to M_0$ is said to be ACL (absolutely continuous on lines) if for any open set $U \subset M$ and contact fibration Γ of U, f is absolutely continuous along all curves in Γ except for a subfamily of Γ of measure zero.

For $j = 0, 1$, let HM_j denote HM endowed with the CR structure J_j. Take any Hermitian metric on HM_j with respect to J_j. Denote by $|\cdot|_j$ the corresponding norm on HM_j.

Definition 2.1

(i) A homeomorphism $f : M_1 \to M_0$ is K-quasiconformal if

- *f is ACL;*
- *f is differentiable almost everywhere and its differential f_* preserves the contact bundle; and*
- *the maximal dilatation $K = K(f) = \operatorname{ess\,sup}_{q \in M_1} K(f)(q) < \infty$, where*

$$K(f)(q) = \frac{\max_{X \in H_qM_1, |X|_1 = 1} |f_*X|_0}{\min_{X \in H_qM_1, |X|_1 = 1} |f_*X|_0}. \tag{2.3}$$

is the dilatation of f at $q \in M_1$.

(ii) A 1-quasiconformal homeomorphism $f : M_1 \to M_0$ is called conformal. If such a conformal homeomorphism exists, M_1 and M_0 are said to be conformally equivalent.

Remark 2.2 (1) For any $q \in M$, $j = 0, 1$, $\dim_{\mathbb{C}} H_q M_j = 1$, so any two Hermitian metrics on $H_q M_j$ are scalar multiples of each other. Hence the value of $K(f)(q)$ is independent of the choices of the Hermitian metrics.

(2) A C^1 homeomorphism is conformal if and only if it is CR. When both M_0 and M_1 are smooth and embeddable into $\mathbb{C}^2$, a homeomorphism $f : M_1 \to M_0$ with L^1_{loc} horizontal derivatives is conformal if and only if it is smooth and CR. A proof to this will be given in a forthcoming paper.

(3) This definition is a generalization of the one given by Korányi and Reimann in [9]. On Heisenberg groups, Korányi and Reimann also gave a generalized analytic definition of quasiconformal homeomorphism in [8]. It is not clear yet how our definition is related to theirs in this case.

By the non-degeneracy of the contact structure of M, i.e., $d\eta \wedge \eta \neq 0$ on M, there is an unique smooth vector field T on M, such that $T \lrcorner d\eta = 0$, $\langle \eta, T \rangle = 1$ on M. T is called the characteristic vector field for η.

Let $T^{1,0}M_0$ denote the subbundle $\{X - iJ_0X \mid X \in HM_0\}$ of $\mathbb{C} \otimes TM_0$. Its elements are called $(1,0)$ vectors on M_0. $T^{0,1}M_0 \triangleq \overline{T^{1,0}M_0}$ is called $(0,1)$ tangent bundle of M_0. Denote by $\wedge^{0,1}M_0$ the space of complex linear functionals α on $\mathbb{C} \otimes HM$ so that $\alpha(Z) = 0, \forall\, Z \in T^{1,0}M_0$. An $\alpha \in \wedge^{0,1}M_0$ is called a $(0,1)$ form on M_0. Denote also $\overline{\wedge^{0,1}M_0}$ by $\wedge^{1,0}M_0$.

With two CR structures M_0 and M_1 on M with the same orientation, we associate a global section μ of $T^{1,0}M_0 \otimes \wedge^{0,1}M_0$ as follows. Let $\overline{W} \neq 0$ be a smooth $(0,1)$ vector field on an open set $U \subset M$ with respect to M_0, then μ is a section of $T^{1,0}M_0 \otimes \wedge^{0,1}M_0$ on U so that $\overline{W}_1 = \overline{W} - \mu(\overline{W})$ is a (0,1) vector with respect to M_1 on U. Let ψ be a smooth $(1,0)$ form on U with respect to M_0 such that $\{\psi, \overline{\psi}\}$ is the dual basis to $\{W, \overline{W}\}$. With these conventions, $\mu = \nu W \otimes \overline{\psi}$ for a function ν on U. The tensor μ is globally well defined and is called *the deformation tensor* of M_1 with respect to M_0. $|\mu|$ ($\triangleq |\nu|$ on U) is also a globally defined real valued function. Since M_0 and M_1 have the same orientation, $|\mu| < 1$ everywhere.

Definition 2.3 *If $f : M_1 \to M_0$ is a C^1 contact mapping which preserves the orientation of HM, let $f^{-1}(M_0)$ be a new CR structure on M so that $T^{0,1}f^{-1}(M_0) = f_*^{-1}(T^{0,1}M_0)$. Define the Beltrami tensor of f by the deformation tensor of $f^{-1}(M_0)$ with respect to M_1.*

Remark 2.4 Locally, since

$$f_*(\overline{W}_1) = \langle \psi, f_*(\overline{W}_1) \rangle W + \langle \overline{\psi}, f_*(\overline{W}_1) \rangle \overline{W}, \tag{2.4}$$

we have

$$\mu_f = \frac{\langle f_*\psi, \overline{W}_1 \rangle}{\langle f_*\psi, W_1 \rangle} W_1 \otimes \overline{\psi}_1, \tag{2.5}$$

where $\overline{\psi}_1 \in \wedge^{0,1}M_1$ with $\langle \overline{\psi}_1, \overline{W}_1 \rangle = 1$. Since f preserves the orientation of HM and the CR structures M_0 and M_1 have the same orientations, $\langle f_*\psi, W_1 \rangle \neq 0$ and $|\mu_f| < 1$. Hence (2.5) and (2.6) below are meaningful.

Theorem 2.5 *If $f : M_1 \to M_0$ is a C^1 quasiconformal homeomorphism and preserves the orientation of HM, then for $q \in M_1$, the dilatation at the point q is given by*

$$K(f)(q) = \frac{1 + |\mu_f(q)|}{1 - |\mu_f(q)|}. \tag{2.6}$$

In particular, the maximal dilatation is

$$K(f) = \sup_{M_1} \frac{1 + |\mu_f|}{1 - |\mu_f|} = \frac{1 + \sup_{M_1} |\mu_f|}{1 - \sup_{M_1} |\mu_f|}. \tag{2.7}$$

The proof of this theorem is simple linear algebra and is the same as the proof of an analogous fact on $\mathbb{C}$ (see [16]).

We now turn our attention to contact flows. First recall that the non-degeneracy of the contact structure of M shows that the mapping

$$\imath : HM \to \mathrm{Null}(T), \qquad X \mapsto X \lrcorner d\eta \tag{2.8}$$

is a bundle isomorphism. Here the space

$$\mathrm{Null}(T) = \{\omega \in \wedge^1 M \mid \langle \omega, T \rangle = 0\} \tag{2.9}$$

is a real rank 2 subbundle of $\wedge^1 M$. Denote the inverse of $\imath$ by $\sharp$.

Let V be a vector field on a contact manifold M which generates a smooth flow of contact transformations. For such a vector field V the real valued function $u = \langle \eta, V \rangle$ is called *the contact Hamiltonian function of V*.

Theorem 2.6 (Liebermann) *Suppose M is a smooth compact contact manifold with a smooth contact form η. If V is a smooth vector field which generates a flow of contact transformations of M, then*

$$V = uT + \sharp((Tu)\eta - du), \tag{2.10}$$

here u is the contact Hamiltonian of V.

(ii) *Conversely, if V is a vector field defined by (2.10) for a real valued smooth function u on M, then V generates a flow of contact transformations of M and the Hamiltonian of V is u.*

The part (i) is Théorème 3 in [14], a proof was given there. The sufficiency (ii) can be proved by straightforward computations.

On the 3-sphere $S^3 = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1|^2 + |w_2|^2 = 1\}$, the contact structure is defined by the contact form

$$\eta = -\mathrm{Im}(w_1 d\overline{w}_1 + w_2 d\overline{w}_2). \tag{2.11}$$

The characteristic vector field for η is

$$T = -2\,\mathrm{Im}(w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}). \tag{2.12}$$

Let S_0^3 be the sphere with the CR structure inherited from the standard complex structure of $\mathbb{C}^2$. Let us denote

$$W = \overline{w}_2 \frac{\partial}{\partial w_1} - \overline{w}_1 \frac{\partial}{\partial w_2}, \tag{2.13}$$

$$\psi = w_2 dw_1 - w_1 dw_2. \tag{2.14}$$

Then $W, \overline{W}$ are $(1,0), (0,1)$ vector fields on S_0^3 respectively, and $\psi, \overline{\psi}$ are $(1,0), (0,1)$ forms on S_0^3 respectively. Moreover $\{W, \overline{W}, T\}$ is dual to $\{\psi, \overline{\psi}, \eta\}$. Direct computations yield the commutator relations among these basis vectors of $\mathbb{C} \otimes TS^3$:

$$[W, \overline{W}] = -iT, \qquad [T, W] = -2iW, \qquad [T, \overline{W}] = 2i\overline{W}. \tag{2.15}$$

The vector fields $X \triangleq 2\mathrm{Re}W$, $Y \triangleq -2\mathrm{Im}W$ form a basis of the real contact space HS^3. We have

$$[X, Y] = -2T, \qquad [X, T] = 2Y, \qquad [Y, T] = -2X. \tag{2.16}$$

The forms $\sigma \triangleq \mathrm{Re}\psi, \tau \triangleq \mathrm{Im}\psi$ and η form a basis of the cotangent space $\wedge^1 S^3$. The commutator relations (2.16) imply that $\imath(X) = 2\tau, \imath(Y) = -2\sigma$, or, equivalently, $\sharp(\tau) = \frac{1}{2}X, \sharp(\sigma) = -\frac{1}{2}Y$. So for any real valued function u on S^3

$$\sharp((Tu)\eta - du) = \sharp(-(Xu)\sigma - (Yu)\tau) = -\frac{1}{2}(Yu)X + \frac{1}{2}(Xu)Y.$$

Hence we have proved the following corollary of Theorem 2.4.

Corollary 2.7 *A vector field on S^3 generates a smooth 1-parameter group of contact transformations if and only if*

$$V = -\frac{1}{2}(Yu)X + \frac{1}{2}(Xu)Y + uT, \tag{2.17}$$

or, equivalently,

$$V = i(\overline{W}u)W - i(Wu)\overline{W} + uT, \tag{2.18}$$

for a smooth real valued function u on S^3.

Remark 2.8 An equivalent theorem in the setting of the 3-dimensional Heisenberg group was given by Korányi and Reimann ([10], Theorem 5).

3 S^1-equivariant quasiconformal homeomorphisms

Let M be a smooth, compact 3-manifold. An S^1-action $\{U_\phi \mid \phi \in \mathbb{R} \bmod 2\pi\}$ on M is said to be free if no $U_\phi \not\equiv \mathrm{id}$ has a fixed point. M is called a regular contact manifold if M is contact and has a contact form η so that the characteristic vector field T for η generates a free S^1-action $\{U_\phi \mid \phi \in \mathbb{R} \bmod 2\pi\}$ on M. Here ϕ is the parameter of the contact flow. Obviously the action is transversal to the contact structure. Let $\Sigma = M/S^1$ be the space of orbits. Then Σ is a smooth compact surface and the natural projection $p : M \to \Sigma$ is open and smooth.

Theorem 3.1 *If M is a regular contact manifold, then*

(i) M is a principal fiber bundle over Σ with structure group S^1;
(ii) the contact structure HM defines a connection in this bundle; and
(iii) Σ has an oriented area form ω such that the structure equation of the connection is given by

$$d\eta = p^*\omega.$$

Later we will simply call such a manifold M *a contact circle bundle.*

A curve on a smooth compact manifold is said to be rectifiable if it is rectifiable with respect to a (hence any) smooth Riemannian metric on the manifold.

Lemma 3.2 *Let $\gamma : I \to \Sigma$ be a rectifiable curve starting at $q \in \Sigma$ with an interval $I = [0, l] \subset \mathbb{R}$, and $\tilde{q} \in p^{-1}(q)$. Then there is a unique curve $\tilde{\gamma} : I \to M$ starting at $\tilde{q}$ so that $p \circ \tilde{\gamma} = \gamma$, $\tilde{\gamma}$ is rectifiable, and the tangent vectors at its regular points are in HM.*

The curve $\tilde{\gamma}$ is called the horizontal lift of γ starting at $\tilde{q}$.

Proof. If γ is C^1, the lemma follows from Proposition II 3.1 in [7]. The following is a modification of the proof given there.

By the local triviality of the circle bundle, we have a rectifiable curve $\tilde{\alpha} : I \to M$ starting at $\tilde{q}$ so that $p \circ \tilde{\alpha} = \gamma$. We construct an absolutely continuous function $\phi : I \to \mathbb{R}$ such that the curve given by

$$\tilde{\gamma}(t) = U_{\phi(t)}(\tilde{\alpha}(t)), \qquad t \in I, \tag{3.1}$$

satisfies the requirement. Note that if T denotes the generator of the circle action,

$$\tilde{\gamma}'(t) = \phi'(t) T|_{\tilde{\gamma}(t)} + U_{\phi(t)_*}(\tilde{\alpha}'(t)). \tag{3.2}$$

This vector is in HM if and only if

$$0 = \langle \eta, \tilde{\gamma}'(t) \rangle = \phi'(t) + \langle \eta, U_{\phi(t)_*}(\tilde{\alpha}'(t)) \rangle. \tag{3.3}$$

The expression on the right hand side of the ordinary differential equation in the initial value problem

$$\begin{aligned} \phi' &= -\langle \eta, U_{\phi_*}(\tilde{\alpha}'(t)) \rangle, \\ \phi(0) &= 0, \end{aligned} \tag{3.4}$$

is smooth in ϕ and L^1 in t. So, by Theorem II 3.5 in [17], (3.4) has a unique solution ϕ on I which is absolutely continuous. Then the curve given by (3.1) with this ϕ is the horizontal lift starting at $\tilde{q}$ of γ. □

Let Ω be a simply connected domain on Σ with a rectifiable boundary $\gamma = \partial\Omega$. As a 1-chain γ has an orientation induced from that of Ω regarded as a 2-chain. For $q \in \gamma, \tilde{q} \in p^{-1}(q)$, let $\tilde{\gamma}$ be the horizontal lift of γ starting at $\tilde{q}$. The end point of $\tilde{\gamma}$ is $U_\phi(\tilde{q})$ for some $\phi \in [0, 2\pi)$. We call ϕ the *phase shift* from $\tilde{q}$ to $U_\phi(\tilde{q})$. The structure equation in Theorem 3.1 (iii) is the infinitesimal version of the following.

Proposition 3.3 *The ω-area of Ω satisfies* $\displaystyle\int_\Omega \omega = -\phi \bmod 2\pi.$

Proof. Without loss of generality, we assume that $\Omega \subset\subset \Omega'$ for a simply connected open set $\Omega' \subset \Sigma$ where the bundle M is trivial. That is, $p^{-1}(\Omega')$ is S^1-equivariantly diffeomorphic to $\Omega' \times S^1$. Note $d\omega = 0$ on Σ, so $\omega = d\alpha$ on Ω' for some 1-form α. Then

$$\int_\Omega \omega = \int_\gamma \alpha = \int_{\tilde{\gamma}} p^*\alpha. \tag{3.5}$$

Here the first equality is due to the Stokes formula for rectifiable γ which can be proved by exhausting Ω with C^1 bounded domains. Notice the homology group $H_1(p^{-1}(\Omega')) \cong \mathbb{Z}$. Let β be an S^1-fiber with the orientation given by T. Then regarded as a 1-chain, β generates $H_1(p^{-1}(\Omega'))$. If $\tilde{\gamma}_0$ is the oriented trajectory of T from $\tilde{q}$ to $U_\phi(\tilde{q})$, then $\tilde{\gamma} - \tilde{\gamma}_0$ is homologous to $m\beta$ for some $m \in \mathbb{Z}$. Because

$$\int_\beta p^*\alpha = \int_{p(\beta)} \alpha = 0$$

and

$$d(\eta - p^*\alpha) = d\eta - p^*d\alpha = d\eta - p^*\omega = 0, \tag{3.6}$$

$$\int_{\tilde{\gamma}-\tilde{\gamma}_0} \eta - p^*\alpha = \int_{m\beta} \eta - p^*\alpha = \int_{m\beta} \eta = 0 \bmod 2\pi. \tag{3.7}$$

Note also $\displaystyle\int_{\tilde{\gamma}} \eta = 0$ since $\tilde{\gamma}$ is Legendrian and $\displaystyle\int_{\tilde{\gamma}_0} p^*\alpha = \int_{p(\tilde{\gamma}_0)} \alpha = 0$. So (3.7) gives

$$\int_{\tilde{\gamma}_0} \eta + \int_{\tilde{\gamma}} p^*\alpha = 0 \bmod 2\pi,$$

or, by (3.5),

$$\int_\Omega \omega = -\int_{\tilde{\gamma}_0} \eta = -\phi \bmod 2\pi. \quad \square$$

If we start with an oriented, rectifiable, Legendrian curve $\tilde{\gamma}$ with the initial and end points on the same S^1-fiber, then the closed curve $\gamma = p(\tilde{\gamma}) \subset \Sigma$ may not bound a simply connected domain, and $\tilde{\gamma}$ may not be a single-sheeted cover of γ. However, when γ represents the null element of $H_1(\Sigma)$ it is easy to see that Proposition 3.3 can be generalized to

Corollary 3.4 *If $p(\tilde{\gamma}) = \partial\Omega$ for some 2-chain Ω on Σ, the ω-area of Ω has the same value as the phase shift from the end point of $\tilde{\gamma}$ to its initial point (mod 2π).*

A CR structure on M is S^1-invariant if each U_ϕ in the S^1-action is CR with respect to this CR structure. Assume M_0 is an S^1-invariant CR manifold with the underlying regular contact manifold M, then the CR structure induces a complex structure on the surface Σ so that $p : M \to \Sigma$ is CR. Equipped with this complex structure, Σ becomes a Riemann surface Σ_0 and $T^{1,0}\Sigma_0 = p_*(T^{1,0}M_0)$.

Moreover, when the CR structure of M_0 is positively oriented with respect to η, the area form ω and the complex structure on Σ_0 determine a Riemannian metric as follows. Let $J' : T\Sigma_0 \to T\Sigma_0$ be the endomorphism which defines the complex structure on Σ_0, then $\omega(X, J'X) > 0$ for nonzero $X \in T\Sigma_0$. Then for $X, Y \in T\Sigma_0$, define a Riemannian metric by $\langle X, Y\rangle = \omega(X, J'Y)$. This Riemannian metric has the oriented area form ω and induces the complex structure J' of Σ_0. Still use Σ_0 to denote the corresponding Riemannian 2-manifold.

Conversely, if there is a Riemannian metric on Σ whose oriented area form is ω, we can lift the complex structure determined by this Riemannian metric to an S^1-invariant CR structure on M by declaring $Z \in \mathbb{C} \otimes HM$ to be a $(1,0)$ tangent vector if $p_*(Z) \in T^{1,0}\Sigma$. This CR structure is positively oriented with respect to η.

A homeomorphism $f : M \to M$ is said S^1-equivariant if the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "U_\phi"'] & M \arrow[d, "U_\phi"] \\ M \arrow[r, "f"] & M \end{tikzcd} \tag{3.8}$$

commutes for each ϕ. Such a homeomorphism will induce a quotient homeomorphism $F : \Sigma \to \Sigma$ so that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ \Sigma \arrow[r, "F"] & \Sigma \end{tikzcd} \tag{3.9}$$

commutes.

Assume M_1 is another S^1-invariant CR manifold with the underlying contact manifold M. The corresponding quotient surface is $\Sigma_1 = M_1/S^1$ which has the area form ω too and the complex structure induced from the CR structure on M_1.

Theorem 3.5 *Let $M \xrightarrow{p} \Sigma$ be a contact circle bundle. Assume M_1, M_0 are two S^1-invariant CR manifolds with the same underlying contact manifold M, and $f : M_1 \to M_0$ is an S^1-equivariant quasiconformal homeomorphism. Then the quotient map $F : \Sigma_1 \to \Sigma_0$ is a quasiconformal homeomorphism in the classical sense and F preserves ω-area. Moreover $K(F) = K(f)$.*

Proof. Choose a region R on the Riemann surface Σ_1 corresponding to a rectangle in a conformal coordinate system. Let $\Gamma = \{\gamma\}$ be the family of all longest straight line segments in R which are parallel to a fixed side of R. Lifting each $\gamma \subset \Gamma$ to M_1 horizontally, we obtain a contact fibration $p^{-1}(\Gamma) = \{$all Legendrian lifts of $\gamma \mid \gamma \in \Gamma\}$ of $p^{-1}(R)$. Let $\Gamma_1 \subset \Gamma$ consist of lines γ so that f is absolutely continuous along a lift of γ. S^1-equivariance tells us if $\gamma \in \Gamma_1$, then along each lift of γ, f is absolutely continuous. Therefore if $\gamma \in \Gamma_1$, then F is absolutely continuous along it. By the ACL property of f, $p^{-1}(\Gamma \setminus \Gamma_1)$ is of measure zero. Therefore, F is absolutely continuous along almost every straight line segment $\gamma \in \Gamma$. Since R is arbitrary, F is ACL.

If f is differentiable at a point $\tilde{q}$, F is differentiable at $q = p(\tilde{q})$. Hence F is differentiable almost everywhere on Σ since so is f on M. The bounded distortion inequality for f at $\tilde{q}$ implies that for F with the same dilatation at q since p is CR. So F is a quasiconformal homeomorphism of Σ and $K(f) = K(F)$.

For $q \in \Sigma_1$, let D_r be a disc with radius r centered at q, for each positive small r. ACL regularity and S^1-equivariance of f implies that F is absolutely continuous along almost all circles ∂D_r, and f is absolutely continuous along all lifts of these circles. For those discs D_r along whose boundary F is absolutely continuous (equivalently, f is absolutely continuous along each lift of ∂D_r), $F(\partial D_r)$ is rectifiable. Hence Proposition 3.3 is valid for both such D_r and the corresponding $F(D_r)$. Then S^1-equivariance of f and Proposition 3.3 show that F preserves the ω-area of almost all discs D_r, hence of all discs. So F preserves the ω-area for q is arbitrary. □

Next we consider the converse to Theorem 3.5. That is, if we are given a K-quasiconformal homeomorphism $F : \Sigma_1 \longrightarrow \Sigma_0$ which preserves the ω-area, we want to know if we can lift it to an equivariant K-quasiconformal homeomorphism $f : M_1 \longrightarrow M_0$ in the sense that there exists such f so that F is the quotient map of f. Let us first give a regularity proposition which implies that an area-preserving K-quasiconformal mapping is $\sqrt{K}$-Lipschitz.

Let Σ_1 and Σ_0 be two Riemannian 2-manifolds. For $j = 0, 1$, denote area, curve length and distance on Σ_j by $|\cdot|_j$, l_j and d_j respectively. A mapping $F : \Sigma_1 \longrightarrow \Sigma_0$ is said to distort area by a factor $A > 0$ if $|F(D)|_0 \leq A\,|D|_1$, for all measurable sets $D \subset \Sigma_1$. For a constant $L > 0$, F is said to be L-Lipschitz if there exists a constant $\epsilon > 0$ such that $d_0(f(q_1), f(q_2)) \leq L d_1(q_1, q_2)$ whenever $q_1, q_2 \in \Sigma_1$ and $d_1(q_1, q_2) < \epsilon$.

Proposition 3.6 *Assume that Σ_1 and Σ_0 are two compact smooth Riemannian 2-mani- folds and $F : \Sigma_1 \longrightarrow \Sigma_0$ is K-quasiconformal and distorts area by a factor $A > 0$. Then F is $\sqrt{KA}$-Lipschitz.*

Proof. Since Σ_1 is compact, there exists a constant $\epsilon > 0$ such that two points $q_1, q_2 \in \Sigma_1$ with $d_1(q_1, q_2) < \epsilon$ can be joined by a length minimizing curve α. For $0 < \delta < \epsilon/3$, consider the δ-tubular neighborhood $Q \triangleq \{q \in \Sigma_1 \,|\, d_1(q, \alpha) < \delta\}$. Divide the boundary ∂Q of Q into four ordered sides $S_j, j = 1, 2, 3, 4$ so that $S_1 = \{q \in \partial Q \,|\, d_1(q, q_1) = \delta\}$ and $S_3 = \{q \in \partial Q \,|\, d_1(q, q_2) = \delta\}$. Then Q becomes a quadrilateral. Recall the module of the quadrilateral Q is defined by

$$\mathrm{Mod}(Q) = \sup_{\varrho \in A(Q)} \frac{(\inf\limits_{\gamma \in \Gamma_Q} \int_\gamma \varrho)^2}{\int_Q \varrho^2} = \inf_{\varrho \in A(Q)} \frac{\int_Q \varrho^2}{(\inf\limits_{\gamma \in \Gamma'_Q} \int_\gamma \varrho)^2}, \tag{3.10}$$

where $A(Q) = \{\varrho \geq 0 \,|\, \varrho$ is Borel-measurable on Q, $0 < \int_Q \varrho^2 < +\infty\}$ is the set of allowable measures, Γ_Q is the family of rectifiable curves in Q connecting the sides S_2 and S_4, and Γ'_Q is the family of rectifiable curves in Q connecting the sides S_1 and S_3. In particular

$$\frac{(\inf_{\gamma\in\Gamma_Q}\int_\gamma 1)^2}{|Q|_1} \leq \mathrm{Mod}(Q) \leq \frac{|Q|_1}{(\inf_{\gamma\in\Gamma'_Q}\int_\gamma 1)^2}. \tag{3.11}$$

Similar definitions and inequalities hold for the quadrilateral $F(Q)$. Since the homeomorphism $F : \Sigma_1 \to \Sigma_0$ is K-quasiconformal,

$$\mathrm{Mod}(Q) \leq K\,\mathrm{Mod}(F(Q)). \tag{3.12}$$

Combining (3.12) with (3.11) for both Q and $F(Q)$, we have

$$\frac{(\inf_{\gamma\in\Gamma_Q}\int_\gamma 1)^2}{|Q|_1} \leq K\frac{|F(Q)|_0}{(\inf_{\gamma\in\Gamma'_{F(Q)}} l_0(\gamma))^2}. \tag{3.13}$$

Denote $d(\delta) \triangleq \inf_{\gamma\in\Gamma'_{F(Q)}} l_0(\gamma)$. This is the distance between the side $F(S_1)$ and the opposite side $F(S_3)$ of $F(Q)$. Hence

$$\lim_{\delta\to 0} d(\delta) = d_0(F(q_1), F(q_2)). \tag{3.14}$$

Note also $\inf_{\gamma\in\Gamma_Q}\int_\gamma 1 = 2\delta$. Note $|F(Q)|_0 \leq A\,|Q|_1$ since F distorts area by the factor $A > 0$. Then (3.13) becomes

$$d(\delta) \leq \sqrt{KA}\,\frac{|Q|_1}{2\delta}. \tag{3.15}$$

Letting $\delta \to 0$, we obtain

$$d_0(F(q_1), F(q_2)) \leq \sqrt{KA}\,l_1(\alpha) = \sqrt{KA}\,d_1(q_1, q_2). \quad \square \tag{3.16}$$

Theorem 3.7 *Let $M \xrightarrow{p} \Sigma$ be a compact contact circle bundle with Σ homeomorphic to S^2. For $j = 0, 1$, let Σ_j be a Riemannian 2-manifold obtained by assigning to Σ a Riemannian metric whose area form is ω. In particular, Σ_j has a complex structure. Let M_j be an S^1-invariant CR manifold obtained by endowing M with the CR structure such that $p : M_j \to \Sigma_j$ is CR. Assume $F : \Sigma_1 \to \Sigma_0$ is a quasiconformal homeomorphism which preserves ω-area. Then there exists an equivariant quasiconformal homeomorphism $f : M_1 \to M_0$ such that $p \circ f = F \circ p$ and $K(F) = K(f)$.*

Proof. Fix a point $q_0 \in \Sigma_1$ and a points $\tilde{q}_0 \in p^{-1}(q_0)$. Define $f(\tilde{q}_0)$ to be any point in the fiber $p^{-1}(F(q_0))$. For any other $\tilde{q} \in M_1$, connect $\tilde{q}_0$ and $\tilde{q}$ by a rectifiable Legendrian curve $\tilde{\gamma}$. We can always do that by a theorem of Chow [5]. Project $\tilde{\gamma}$ onto a curve $\gamma \subset \Sigma_1$, then map it by F onto the curve $F(\gamma) \subset \Sigma_0$ which is rectifiable by Proposition 3.6. We define $f(\tilde{q})$ by the end point of the unique horizontal lift of $F(\gamma)$ starting at $f(\tilde{q}_0)$. The existence and uniqueness of the horizontal lift of $F(\gamma)$ is given by Lemma 3.2.

Assume $\tilde{\gamma}_1$ is another rectifiable Legendrian curve connecting $\tilde{q}_0$ and $\tilde{q}$, and γ_1 is its projection. Since Σ is simply connected, the 1-chain $\gamma_1 - \gamma = \partial\Omega$ for some 2-chain $\Omega \subset \Sigma_1$. Corollary 3.4 says that the ω-area of Ω is zero mod 2π, whence the same holds for the ω-area of $F(\Omega)$ since F preserves ω-area. By Proposition 3.3, the horizontal lifts of $F(\gamma)$ and $F(\gamma_1)$ initiated at $f(\tilde{q}_0)$ have the same end points. Therefore the mapping f is well-defined. Moreover, f is S^1-equivariant by an argument similar to the one given above based on Corollary 3.4.

Next we want to prove that f is Lipschitz with respect to some Riemannian metrics on M_1 and M_0 respectively. For $j = 0, 1$, there is a unique positive definite quadratic form on HM_j such that $p_{*|_{HM_j}} : HM_j \to T\Sigma_j$ is isometric. We extend this quadratic form on HM_j to TM_j by letting the generator T of the circle action to be a unit vector field orthogonal to HM_j. The distance and curve length with respect to this Riemannian metric are denoted by $\tilde{d}_j$ and $\tilde{l}_j$ respectively. For any two points $q_1, q_2 \in M_1$, there is a smooth length minimizing curve $\tilde{\alpha} : I \to M_1$ to connect them. Here $I = [0, r] \subset \mathbb{R}$. Let $\phi : I \to \mathbb{R}$ be the smooth solution of (3.4) with this curve $\tilde{\alpha}$. Then the curve $\tilde{\gamma} : I \to M_1, t \mapsto U_{\phi(t)}(\tilde{\alpha}(t))$ is either a single point or a Legendrian curve. We have

$$\tilde{l}_1(\tilde{\gamma}) = l_1(p(\tilde{\gamma})) \le \tilde{l}_1(\tilde{\alpha}), \tag{3.17}$$

by the construction of the Riemannian metric on M_1. Note q_1 is the starting point of $\tilde{\gamma}$. Let q_3 be the end point of $\tilde{\gamma}$. Denote the curve $I \to M_1, t \mapsto U_{\phi(t)}(q_3)$ by $\tilde{\beta}$. By the definition of the Riemannian metric on M_1,

$$\begin{aligned} \tilde{l}_1(\tilde{\beta}) &= \int_I |\phi'(t)|\, dt \\ &= \int_I |\langle U^*_{\phi(t)}\eta, \tilde{\alpha}'(t)\rangle|\, dt, \quad \text{by (3.4)}, \\ &\le c \int_I ||\tilde{\alpha}'(t)||_1\, dt, \quad \text{for some constant } c > 0, \\ &= c\, \tilde{d}_1(q_1, q_2), \end{aligned} \tag{3.18}$$

where $|| \cdot ||_1$ is the Riemannian norm on TM_1. Then

$$\begin{aligned}
&\tilde{d}_0(f(q_1), f(q_2)) \\
&\quad \leq \tilde{l}_0(f(\tilde{\gamma})) + \tilde{l}_0(f(\tilde{\beta})) \\
&\quad = l_0(F(p(\tilde{\gamma}))) + \tilde{l}_1(\tilde{\beta}), \text{since } f \text{ is equivariant,} \\
&\quad \leq \sqrt{K(F)}\, l_1(p(\tilde{\gamma})) + c\, \tilde{d}_1(q_1, q_2), \text{by Proposition 3.6 and (3.18),} \\
&\quad \leq (\sqrt{K(F)} + c)\, \tilde{d}_1(q_1, q_2), \text{by (3.17).} \qquad (3.19)
\end{aligned}$$

Therefore f is Lipschitz. Hence f is almost everywhere differentiable by Rademacher's Theorem (Theorem 3, page 250, [18]). At a point of differentiability, f_* preserves the contact structure by the construction of f, in particular, the proof of Lemma 3.2. Furthermore, f maps all rectifiable Legendrian curves to rectifiable Legendrian curves. Hence it is ACL. Its bounded distortion inequality follows from that of F, and f, F share the same value of dilatation since the S^1-action is CR. Therefore f is quasiconformal according to Definition 2.1. □

Remark 3.8

(1) The lift f of F constructed in the proof is unique up to composition with U_ϕ for some ϕ.

(2) When the base space Σ is not simply connected, a quasiconformal homeomorphism F on Σ preserving ω can be lifted to a quasiconformal homeomorphism f if and only if the monodromy representation of $\pi_1(\Sigma)$ in S^1 induced by F is trivial. In this case, the construction of f in the above proof applies. When Σ is homeomorphic to S^2, this obstruction to lifting does not exist.

4 Equivariantly extremal quasiconformal homeomorphisms on S^3

Here an equivariantly extremal quasiconformal homeomorphism refers to an equivariant quasiconformal homeomorphism with the least maximal dilatation among all equivariant homeomorphisms.

On $S^3 = \{|w_1|^2 + |w_2|^2 = 1\} \subset \mathbb{C}^2$ and the circle action is given by

$$U_\phi : (w_1, w_2) \to (e^{i\phi} w_1, e^{i\phi} w_2), \qquad (4.1)$$

we have the Hopf fibration $S^1 \to S^3 \to S^2$ of the 3-sphere. The projection is given by

$$p : S^3 \to S^2, \quad (w_1, w_2) \mapsto \frac{w_2}{w_1}. \qquad (4.2)$$

On S^2 the standard spherical metric is

$$ds = \frac{2\,|dz|}{1 + |z|^2} \qquad (4.3)$$

and $\omega_0 = \dfrac{4\, dx \wedge dy}{(1 + |z|^2)^2}$ is the spherical area form, where $z = x + yi$. Let η be the contact form of S^3 given by (2.11). Then direct computations prove

Proposition 4.1 *We have* $d\eta = p^*(\frac{1}{2}\omega_0).$

Given two smooth Riemannian metrics on S^2 which share the spherical area form, we lift the complex structures they determine to two smooth S^1-invariant CR structures on S^3 so that the projection p in (4.2) is CR. By results in the last section, if there is an extremal area-preserving quasiconformal homeomorphism on S^2 between these two Riemannian structures, then an S^1-equivariant lift of this homeomorphism is an equivariantly extremal quasiconformal homeomorphism on S^3 between the two lifted CR structures. This is the guideline for the rest of this section.

The spherical metric (4.3) on the unit Euclidean sphere S_0^2 is equivalently given by

$$ds_0^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{4.4}$$

where (θ, ϕ) are the spherical coordinates $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$. Let λ be a real valued smooth function on S^2 satisfying $1 \leq \lambda \leq \Lambda$ on S^2, $\lambda = 1$ near the poles where $\theta = 0, \pi$, λ attains its maximal value $\Lambda > 1$ at each point of the equator $E = \{\theta = \frac{\pi}{2}\}$, and $\lambda < \Lambda$ elsewhere. Define a new metric on S^2 by

$$ds_1^2 = \lambda^2 d\theta^2 + \frac{\sin^2\theta}{\lambda^2} d\phi^2. \tag{4.5}$$

S^2 equipped with the metric (4.5) is denoted by S_1^2. The metric on S_1^2 is obtained from the metric on S_0^2 by stretching in the meridian direction by the factor λ and shrinking in the parallel direction by the same factor. $\mathrm{id}_{S^2} : S_1^2 \to S_0^2$ is quasiconformal with maximal dilatation Λ^2 which occurs along the equator. Obviously, S_0^2 and S_1^2 have the area element $\sin\theta d\theta d\phi$.

A Jordan curve divides the sphere into two components. If these components have equal area, we call the curve area-halving curve. An area-halving curve on S_0^2 is also an area-halving curve on S_1^2. Let us give a folk lemma first. It is a very special case of the isoperimetric property on surfaces (Burago and Zalgaller [4], Theorem 2.2.1.). Our proof is very simple and intuitive.

Lemma 4.2 *The great circles on S_0^2 are the shortest area-halving curves.*

Proof. Any two area-halving curves on S_0^2 must intersect each other. Hence an area-halving curve intersects its antipodal image, and we conclude that an area-halving curve contains a pair of antipodal points. But the great semi-circles are the geodesics to connect two antipodal points. Therefore a Jordan curve is a shortest area-halving curve if and only if it is a great circle. □

Therefore the length of a shortest area-halving curve on S_0^2 is 2π. The construction of ds_1^2 shows that on S_1^2 the equator is the unique shortest area-halving curve and its length is $2\pi/\Lambda$.

Proposition 4.3 *The identity map $id_{S^2} : S_1^2 \to S_0^2$ has the least maximal dilatation among all area-preserving quasiconformal homeomorphism from S_1^2 to S_0^2.*

Proof. The equator $E = \{\theta = \frac{\pi}{2}\} \subset S_1^2$ is mapped by an area-preserving K-quasiconformal homeomorphism $F : S_1^2 \to S_0^2$ to a rectifiable curve $F(E)$ according to Proposition 3.6. More precisely,

$$l_0(F(E)) \leq \sqrt{K}\, l_1(E). \tag{4.6}$$

Here l_0 and l_1 stand for the curve lengths on S_0^2 and S_1^2 respectively. Note $F(E)$ is an area-halving curve on S_0^2 since E is an area-halving curve on S_1. Thus $l_0(F(E)) \geq 2\pi$ by Lemma 4.2. Then (4.6) implies $K \geq \Lambda^2 = K(\mathrm{id}_{S^2})$. □

The Riemannian metric on S_1^2 given by (4.5) can be written as

$$ds_1^2 = \frac{(\lambda^2+1)^2}{\lambda^2(|z|^2+1)^2}\,|dz + \frac{\lambda^2-1}{\lambda^2+1}\frac{z}{\bar{z}}d\bar{z}|^2. \tag{4.7}$$

Then on S_1^2, the (0,1) tangent space is spanned by

$$\frac{\partial}{\partial \bar{z}} - \frac{\lambda^2-1}{\lambda^2+1}\frac{z}{\bar{z}}\frac{\partial}{\partial z}, \tag{4.8}$$

which is annihilated by the (1,0) form

$$dz + \frac{\lambda^2-1}{\lambda^2+1}\frac{z}{\bar{z}}d\bar{z}.$$

Denote $\tilde{\lambda} = \lambda \circ p$ and $\overline{W}_1 = \overline{W} - \nu W$, where W is given by (2.13) and

$$\nu = \frac{\tilde{\lambda}^2-1}{\tilde{\lambda}^2+1}\frac{w_1 w_2}{\overline{w}_1\overline{w}_2}. \tag{4.9}$$

Then direct computations give

$$p_*(-\overline{w}_1{}^2\overline{W}_1) = \frac{\partial}{\partial \bar{z}} - \frac{\lambda^2-1}{\lambda^2+1}\frac{z}{\bar{z}}\frac{\partial}{\partial z}. \tag{4.10}$$

Use S_1^3 to denote S^3 equipped with the CR structure whose (0,1) vector space is spanned by $\overline{W}_1$. By Theorem 3.5, 3.7 and Proposition 4.1, 4.3, we have proved

Theorem 4.4 *With the above notation, $id_{S^3} : S_1^3 \to S_0^3$ is an equivariantly extremal quasiconformal homeomorphism, namely, it has the least maximal dilatation among all equivariant quasiconformal homeomorphism from S_1^3 to S_0^3.*

Remark 4.5

(1) The dilatation of $\mathrm{id}_{S^3} : S_1^3 \to S_0^3$ attains its maximum on the covering of the equator $E \subset S^2$, i.e., the Clifford torus

$$T_C = \{(w_1, w_2) \mid |w_1|^2 = |w_2|^2 = \frac{1}{2}\}$$

and its maximal value is Λ^2.

(2) $\mathrm{id}_{S^3} : S_1^3 \to S_0^3$ is not the only equivariantly extremal quasiconformal homeomorphism. Any small S^1-equivariant perturbation of id_{S^3} away from T_C will give another equivariantly extremal mapping.

5 Variation of the conformal distortion

As before, we denote the 3-sphere endowed with the canonical CR structure by S_0^3. Assume S_1^3 is the 3-sphere endowed with a new smooth, strongly pseudo-convex CR structure whose $(0,1)$ tangent space is spanned by $\overline{W}_1 = \overline{W} - \mu(\overline{W})$, where $\mu = \nu W \otimes \overline{\psi}$ is a global section of $T^{1,0}S_0^3 \otimes \wedge^{0,1}S_0^3$ with a smooth function ν with $|\nu| < 1$ on S^3.

Let g_s be a flow of contact transformations generated by a vector field V with Hamiltonian function u. Then the maximal dilatation of $g_s : S_1^3 \to S_0^3$, by Theorem 2.3, is measured by the magnitude of the Beltrami tensor μ_{g_s}.

In this section we will give an asymptotic formula for $|\mu_{g_s}|$ as $s \to 0$ up to the first order for a general CR structure on S_1^3 and then up to the second order when the CR structure on S_1^3 is S^1-invariant and the first variation vanishes.

According to (2.5)

$$|\mu_{g_s}| = \left|\frac{\langle g_s^*\psi, \overline{W}_1\rangle}{\langle g_s^*\psi, W_1\rangle}\right| = \left|\frac{\nu_s - \nu}{1 - \overline{\nu}\nu_s}\right|, \tag{5.1}$$

where

$$\begin{aligned}
\nu_s &\triangleq \frac{\langle g_s^*\psi, \overline{W}\rangle}{\langle g_s^*\psi, W\rangle} \\
&= \frac{\langle L_V\psi, \overline{W}\rangle s + \frac{1}{2}\langle L_V L_V\psi, \overline{W}\rangle s^2 + \mathcal{O}(s^3)}{1 + \langle L_V\psi, W\rangle s + \mathcal{O}(s^2)}\,. \\
&= \langle L_V\psi, \overline{W}\rangle s + \left(\frac{1}{2}\langle L_V L_V\psi, \overline{W}\rangle - \langle L_V\psi, \overline{W}\rangle\langle L_V\psi, W\rangle\right) s^2 + \mathcal{O}(s^3) \\
&\triangleq as + bs^2 + \mathcal{O}(s^3),
\end{aligned} \tag{5.2}$$

for small $s \in \mathbb{R}$. Then on the set where $\nu \neq 0$,

$$\begin{aligned}
&|\mu_{g_s}| \\
&\quad = |(\nu - \nu_s)\left(1 + \overline{\nu}\nu_s + \overline{\nu}^2{\nu_s}^2 + \mathcal{O}(s^3)\right)| \\
&\quad = |\nu| - \frac{1 - |\nu|^2}{|\nu|}\mathrm{Re}(\overline{\nu}a)s + \frac{1 - |\nu|^2}{2|\nu|}\left((1 - |\nu|^2)|a|^2 - 2\mathrm{Re}(\overline{\nu}^2 a^2) - 2\mathrm{Re}(\overline{\nu}b)\right)s^2 \\
&\quad +\mathcal{O}(s^3).
\end{aligned} \tag{5.3}$$

Now we compute the coefficients appearing in (5.2) and (5.3).

$$\begin{aligned}
L_V\overline{W} &= [V, \overline{W}] \\
&= [i(\overline{W}u)W - i(Wu)\overline{W} + uT, \overline{W}] \qquad \text{by (2.18)} \\
&= -i(\overline{W}^2 u)W + i(\overline{W}Wu + 2u)\overline{W},
\end{aligned} \tag{5.4}$$

and so

$$L_V W = i(W^2u)\overline{W} - i(W\overline{W}u + 2u)W. \tag{5.5}$$

Hence

$$a = \langle L_V\psi, \overline{W}\rangle = V\langle\psi, \overline{W}\rangle - \langle\psi, L_V\overline{W}\rangle = i(\overline{W}^2u). \tag{5.6}$$

Combining (5.3) with (5.6), we have proved the following proposition about the first variation of the absolute value of Beltrami tensor.

Proposition 5.1 *If $g_s : S_1^3 \to S_0^3$ is a flow of contact transformations generated by a vector field with Hamiltonian u, then for small $s \in \mathbb{R}$*

$$|\mu_{g_s}| = |\nu| + \frac{1-|\nu|^2}{|\nu|} Im(\overline{\nu}\overline{W}^2u)s + \mathcal{O}(s^2) \qquad \textit{where } \nu \neq 0\textit{; and} \tag{5.7}$$

$$|\mu_{g_s}| = |\overline{W}^2u|\cdot|s| + \mathcal{O}(s^2) \qquad \textit{where } \nu = 0.. \tag{5.8}$$

We will go on to compute the second order term in (5.2) and (5.3). By (5.5)

$$\begin{aligned}\langle L_V\psi, W\rangle &= V\langle\psi, W\rangle - \langle\psi, L_V W\rangle \\ &= i(W\overline{W}u + 2u),\end{aligned} \tag{5.9}$$

$$\begin{aligned}\langle L_V L_V\psi, \overline{W}\rangle &= V\langle L_V\psi, \overline{W}\rangle - \langle L_V\psi, L_V\overline{W}\rangle \\ &= \Big(i(\overline{W}u)W - i(Wu)\overline{W} + uT\Big)(i\overline{W}^2u) \quad \text{by (2.18)} \\ &- \langle L_V\psi, -i(\overline{W}^2u)W + i(\overline{W}Wu + 2u)\overline{W}\rangle \quad \text{by (5.4)}, \\ &= -(\overline{W}u)(W\overline{W}^2u) + (Wu)(\overline{W}^3u) \qquad (5.10) \\ &+ iu(T\overline{W}^2u) - (\overline{W}^2u)([W, \overline{W}]u) \quad \text{by (5.6),(5.9)}, \\ &= -(\overline{W}u)(W\overline{W}^2u) + (Wu)(\overline{W}^3u) \\ &+ iu(T\overline{W}^2u) + i(\overline{W}^2u)(Tu), \quad \text{by (2.15)}.\end{aligned}$$

So we finally get the expression of b in (5.2).

$$\begin{aligned}b &= \frac{1}{2}\langle L_V L_V\psi, \overline{W}\rangle - \langle L_V\psi, \overline{W}\rangle\langle L_V\psi, W\rangle \\ &= -\frac{1}{2}(\overline{W}u)(W\overline{W}^2u) + \frac{1}{2}(Wu)(\overline{W}^3u) + \frac{1}{2}iu(T\overline{W}^2u) \\ &+ \frac{1}{2}i(\overline{W}^2u)(Tu) + (\overline{W}^2u)(W\overline{W}u) + 2(\overline{W}^2u)u.\end{aligned} \tag{5.11}$$

If on the set where $\mu \neq 0$, $\mathrm{Im}(\overline{\nu}\overline{W}^2u) = 0$, i.e., the first variation of the absolute value of Beltrami tensor vanishes, then Proposition 5.1 is not enough to analyse the behavior of the perturbation. We will need to study the second variation of $|\mu_{g_s}|$ in this case.

Next we will compute the second order term in (5.3) on the set where

$$\operatorname{Im}(\overline{\nu}\,\overline{W}^2 u) = 0 \qquad \text{and} \qquad \nu \neq 0 \tag{5.12}$$

holds. Note one term in the second order coefficient in (5.3) is

$$\begin{aligned} 2\operatorname{Re}(\overline{\nu} b) &= \operatorname{Re}\left(-\overline{\nu}(\overline{W}u)(W\overline{W}^2 u) + \overline{\nu}(Wu)(\overline{W}^3 u) + 2\overline{\nu}(\overline{W}^2 u)(W\overline{W}u)\right) \\ &\quad +\operatorname{Re}\left(iu\overline{\nu}(T\overline{W}^2 u) + 4\overline{\nu}(\overline{W}^2 u)\right) \\ &\quad +\operatorname{Re}\left((i\overline{\nu}\,\overline{W}^2 u)(Tu)\right) \\ &\triangleq I_1 + I_2 + I_3. \end{aligned} \tag{5.13}$$

To simplifiy I_1, let $c = \dfrac{\overline{\nu}(\overline{W}^2 u)}{|\nu|^2}$. With the assumption (5.12), c is real valued.

$$\begin{aligned} I_1 &= \operatorname{Re}\left(-\overline{\nu}(\overline{W}u)W(\nu c) + \overline{\nu}(Wu)\overline{W}(\nu c) + 2\overline{\nu}(\overline{W}^2 u)(W\overline{W}u)\right) \\ &= \overline{\nu}(\overline{W}^2 u)\left(\Delta u + \operatorname{Re}\left(\frac{1}{\nu}(\overline{W}\nu)(Wu) - \frac{1}{\nu}(W\nu)(\overline{W}u)\right)\right), \end{aligned} \tag{5.14}$$

where $\Delta u = (W\overline{W} + \overline{W}W)u$.

For simplicity and for later applications, we will assume in the rest of this section that the CR structure of S_1^3 is S^1-invariant. Then S^1-invariance of the CR structure on S_1^3 implies that $L_T(\overline{W} - \nu W)$ is a multiple of $\overline{W} - \nu W$. But

$$\begin{aligned} L_T(\overline{W} - \nu W) &= [T, \overline{W} - \nu W] \\ &= 2i\overline{W} + (2i\nu - T\nu)W, \quad \text{by (2.15).} \end{aligned} \tag{5.15}$$

Therefore, we have proved

Proposition 5.2 *On S^3, $\mu = \nu W \otimes \overline{\psi}$ defines an invariant CR structure if and only if*

$$L_T \mu = 4i\mu \qquad \text{or} \qquad T\nu = 4i\nu. \tag{5.16}$$

With this simple fact, we have

$$\begin{aligned} I_2 &= \operatorname{Re}\left(iuT(\overline{\nu}\,\overline{W}^2 u) - iu(T\overline{\nu})\overline{W}^2 u + 4\overline{\nu}(\overline{W}^2 u)u\right) \\ &= uT\left(\operatorname{Re}(i\overline{\nu}\,\overline{W}^2 u)\right) + \operatorname{Re}\left(-4u\overline{\nu}\,\overline{W}^2 u + 4u\overline{\nu}\,\overline{W}^2 u\right), \quad \text{by (5.16),} \\ &= 0, \quad \text{by (5.12).} \end{aligned} \tag{5.17}$$

Obviously $I_3 = 0$ by (5.12). Combining this with (5.3), (5.6), (5.13), and (5.14), we obtain

Proposition 5.3 *If the smooth CR structure on S_1^3 is S^1-invariant, the Beltrami tensor of $g_s : S_1^3 \to S_0^3$ satisfies*

$$\begin{aligned} |\mu_{g_s}| &= |\nu| + \frac{1-|\nu|^2}{2|\nu|}\Big\{(1+|\nu|^2)|\overline{W}^2 u|^2 - (\bar{\nu}\overline{W}^2 u)\Big[\Delta u \\ &\quad + Re\Big(\frac{1}{\nu}(\overline{W}\nu)(Wu) - \frac{1}{\nu}(W\nu)(\overline{W}u)\Big)\Big]\Big\}s^2 + \mathcal{O}(s^3), \qquad (5.18) \end{aligned}$$

for small $s \in \mathbb{R}$ on the set where $\nu \neq 0$ and $Im(\bar{\nu}\overline{W}^2 u) = 0$.

6 Symmetry breaking

In this section, we will use a contact perturbation of the equivariantly extremal quasiconformal homeomorphism $\mathrm{id}_{S^3} : S_1^3 \to S_0^3$ constructed in Section 4 to show id_{S^3} is not extremal among all quasiconformal homeomorphisms between S_1^3 and S_0^3. Namely, we will construct a nonequivariant quasiconformal homeomorphism near id_{S^3} with smaller maximal dilatation. That will prove the following

Theorem 6.1 *With S_1^3, S_0^3 denoting the S^1-invariant CR manifolds constructed in section 4, no extremal quasiconformal homeomorphism between S_1^3 and S_0^3 is equivariant.*

We call this phenomenon a symmetry breaking of the extremal quasiconformal homeomorphism between CR structures on S^3.

Proof. Assume an extremal quasiconformal homeomorphism $f : S_1^3 \to S_0^3$ is equivariant. By Theorem 4.4, $K(f) = K(\mathrm{id})$. We shall construct a contact flow g_s with a Hamiltonian u which satisfies

$$\begin{aligned} &\mathrm{Im}(\bar{\nu}\overline{W}^2 u) = 0, && \text{on } S^3, \\ &(1+|\nu|^2)|\overline{W}^2 u|^2 - (\bar{\nu}\overline{W}^2 u)\Delta u < 0, && \text{on the torus } T_C. \qquad (6.19) \end{aligned}$$

Here (6.1), by Proposition 5.1, makes the first variation of the absolute value of Beltrami tensor of $g_s : S_1^3 \to S_0^3$ zero, and Proposition 5.3 applies. Direct computations show that $W\nu = \overline{W}\nu = 0$ on T_C. So (6.2) gives that the the second order term in (5.18) is negative. This will contradict the extremality of f, since $K(g_s) < K(f)$ for small $s \in \mathbb{R}$.

For (6.2), we consider the equation

$$(1+|\nu|^2)W^2 u - \bar{\nu}\Delta u = -W^2 u$$

on T_C. By (4.9) this is equivalent to

$$\Delta u - H\frac{w_1 w_2}{\overline{w}_1 \overline{w}_2} W^2 u = 0 \qquad (6.3)$$

on T_C, where H is the constant value of $\frac{2+|\nu|^2}{|\nu|}$ on T_C. Hence to satisfy (6.1), (6.2), it suffices to find u satisfying the system

$$\begin{cases} \Delta u - H \operatorname{Re}\left(\frac{w_1 w_2}{\overline{w}_1 \overline{w}_2} W^2 u\right) = 0, & \text{on } T_C, \\ \operatorname{Re}\left(\frac{w_1 w_2}{\overline{w}_1 \overline{w}_2} W^2 u\right) \neq 0, & \text{on } T_C, \\ \operatorname{Im}\left(\frac{w_1 w_2}{\overline{w}_1 \overline{w}_2} W^2 u\right) = 0, & \text{on } S^3. \end{cases} \tag{6.4}$$

If u is independent of w_2, the system (6.4) simplifies to

$$\begin{cases} \frac{\partial^2 u}{\partial w_1 \partial \overline{w}_1} - \operatorname{Re}\left(2w_1 \frac{\partial u}{\partial w_1} + H w_1^2 \frac{\partial^2 u}{\partial w_1^2}\right) = 0, & \text{when } |w_1|^2 = \frac{1}{2}, \\ \operatorname{Re}\left(w_1^2 \frac{\partial^2 u}{\partial w_1^2}\right) \neq 0, & \text{when } |w_1|^2 = \frac{1}{2}, \\ \operatorname{Im}\left(w_1^2 \frac{\partial^2 u}{\partial w_1^2}\right) = 0, & \text{when } |w_1|^2 \leq 1. \end{cases} \tag{6.5}$$

In polar coordinates $w_1 = re^{i\vartheta}$, (6.5) becomes

$$\begin{cases} (1 - r^2 H)\frac{\partial^2 u}{\partial r^2} + (\frac{1}{r} - 2r + rH)\frac{\partial u}{\partial r} + (\frac{1}{r^2} + H)\frac{\partial^2 u}{\partial \vartheta^2} = 0, & \text{when } r = \frac{\sqrt{2}}{2}, \\ r^2 \frac{\partial^2 u}{\partial r^2} - r\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial \vartheta^2} \neq 0, & \text{when } r = \frac{\sqrt{2}}{2}, \\ \frac{\partial u}{\partial \vartheta} - r\frac{\partial^2 u}{\partial \vartheta \partial r} = 0, & \text{when } 0 \leq r \leq 1. \end{cases} \tag{6.6}$$

Any real function u which is independent of ϑ and satisfies

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{H}{2} - 1, \\ \frac{\partial^2 u}{\partial r^2} = \frac{\sqrt{2}}{2} H, \end{cases} \quad \text{when } r = \frac{\sqrt{2}}{2} \tag{6.7}$$

solves the system (6.6). There are plenty of such real functions. For example,

$$u = (\frac{H}{2} - 1)(r - \frac{\sqrt{2}}{2}) + \frac{\sqrt{2}}{4} H (r - \frac{\sqrt{2}}{2})^2. \tag{6.8}$$

Therefore the proof is complete. □

Remark 6.2 No contact perturbation of $\mathrm{id}_{S^3} : S_1^3 \to S_0^3$ with smooth Hamiltonian u can reduce the magnitude of its Beltrami tensor on T_C at the level of the first variation. This fact becomes clear if polar coordinates $w_1 = re^{i\vartheta}$, $w_2 = \rho e^{i\varphi}$ are used to express

$$\operatorname{Im}(\overline{\nu} \overline{W}^2 u) = 2\frac{\lambda^2 - 1}{\lambda^2 + 1}\left(-r\frac{\partial^2 u}{\partial r \partial \vartheta} - \rho\frac{\partial^2 u}{\partial \rho \partial \vartheta} - r\frac{\partial^2 u}{\partial r \partial \varphi} - \rho\frac{\partial^2 u}{\partial \rho \partial \varphi} + \frac{\partial u}{\partial \vartheta} + \frac{\partial u}{\partial \varphi}\right). \tag{6.9}$$

In fact, the integral of the right hand side of (6.9) over $(\vartheta, \varphi) \in [0, 2\pi] \times [0, 2\pi]$ is zero for $u = u(\vartheta, \varphi)$ is doubly 2π-periodic in (ϑ, φ). So $\mathrm{Im}(\overline{\nu}\overline{W}^2 u)$ is neither positive nor negative on T_C. This is the reason we need to consider the second variation of $|\nu_{g_s}|$ to demonstrate the symmetry breaking.

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