Math. Z. 218, 25–42 (1995)

Mathematische Zeitschrift

Quadrics through a set of points and their syzygies

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Received 16 October 1992; in final form 23 August 1993

Introduction

Let X be a non degenerate projective variety in $\mathbf{P}^n = P_k^n$, where k is an algebraically closed field. Let $I = I(X)$ denote the defining ideal of X in the polynomial ring $R := k[X_0, \ldots, X_n]$ and $A := R/I$ its homogeneous coordinate ring.

The graded R-module A has a minimal free resolution

$$\mathbf{E}: 0 \to E_h \to \cdots \to E_1 \to R \to A \to 0$$

where $h = \mathrm{hd}_R(A)$ and $E_i = \bigoplus_{j=1}^{\beta_i} R(-d_{ij})$.

Several recent investigations and conjectures relate the numerical invariants of the resolution with the geometrical properties of X.

In this paper we are mainly concerned with the "linear part" of the resolution. Hence for every $i = 1, \ldots, h$, we let $a_i(X) = \dim_k [\mathrm{Tor}_i^R(A, k)]_{i+1}$ to be the multiplicity of the shift $i+1$ in E_i.

Since $\mathrm{depth}(A) \geqq 1$, then $h \leqq n$ so that $a_i = 0$ if $i \geqq n+1$. Also it is clear that $a_1 = \dim_k(I_2)$ and it is well known that if for some integer i we have $a_i = 0$ then $a_j = 0$ for every $j \geqq i$. Hence we are interested in projective varieties X which lie on some quadric and we want to study the syzygies of the quadrics passing through X.

The main idea coming from the pioneering work of Green (see [G2]) is that a long linear strand in the resolution has a uniform and simple motivation.

Following this approach we start by proving that $a_i \neq 0$ in the following geometric situations. Either X is contained on a variety of minimal degree and

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★ The authors are partially supported by MURST (Italy). This research was made in the framework of Europroj project

dimension $n-i$, or X is contained in the union of two linear subspaces of $\mathbf{P}^n$ of dimension k and r where $k, r<n$ and $r+k=2n-i-1$ (see Proposition 1.2).

A natural question is to what extent the converse of the above result is true.

In Sect. 1 we prove that this is the case if $i=1$ (trivial) or $i=n$ (see Proposition 1.5). To achieve this result we need a more sophisticated analysis of the quadrics passing through X, a theme which will be central in all this paper.

In Sect. 2 we consider the case $i=n-1$ and we restrict ourselves to reduced schemes which are zero dimensional.

A basic result by Green, the so called Strong Castelnuovo Lemma, says that for a set X of distinct points in linearly general position in $\mathbf{P}^n$, we have $a_{n-1} \neq 0$ if and only if the points are on a rational normal curve of $\mathbf{P}^n$ (see [G2, Theorem 3.c.6]).

Here we complete Green's result by proving that for a set X of distinct points in $\mathbf{P}^n$, such that $n-1$ are never on a linear subspace of dimension $n-3$, we have $a_{n-1} \neq 0$ if and only if either the points are on a rational normal curve of $\mathbf{P}^n$ or $X \subseteq \mathbf{P}^k \cup \mathbf{P}^r$ with $k+r=n$ (see Theorem 2.6).

We conjecture that this result should be true in its complete generality. However even this weaker form of the conjecture, unables us to solve one of the problems which motivated our present research. Namely in Sect. 4 we apply the above theorem to describe in a concrete geometric way the open set where the Minimal Resolution Conjecture holds for a given set of $n+4$ points spanning $\mathbf{P}^n$ (see Proposition 4.2).

In the case $i=n-2$ we remark that the analogous of the Strong Castelnuovo Lemma does not hold. For example 12 general points in $\mathbf{P}^7$ are not on a rational normal scroll of dimension 2 but have $a_5=4 \neq 0$. In Sect. 3 we can prove that they are on a threefold of minimal degree as a consequence of a general result which in particular asserts that if $n \geqq 3$ and p is an integer $1 \leqq p \leqq n-2$, then every set of $2n+1-p$ points in linearly general position lies on a rational normal scroll of dimension $n-p-1$ (see Theorem 3.7). This result extends a classical theorem of Bertini who considered the case $p=n-2$, i.e. the case of $n+3$ points in $\mathbf{P}^n$. Finally we want to remark that the proof of the theorem is entirely constructive, so that, given the coordinates of the points, one easily gets the matrix whose maximal minors are the defining equations of the rational normal scroll containing the given set of points.

1 The extremal cases: $i=1$ and $i=n$

Let X be a non degenerate projective variety in $\mathbf{P}^n$ and let I be the defining ideal of X in the polynomial ring $R=k[X_0, \ldots, X_n]$. The homogeneous coordinate ring $A=R/I$ of X is a graded R-module $A=\bigoplus_{t \geqq 0} A_t$. We define

$$a_i(X)=\dim_k[\operatorname{Tor}_i^R(A, k)]_{i+1}$$

for every $i=1, \ldots, h$ where $h=\operatorname{hd}_R(A)$. We write a_i instead of $a_i(X)$ when there is no confusion.

We can compute $\mathrm{Tor}_i^R(A, k)$ using a resolution of k which can be obtained by the Koszul complex of $X_0, \ldots, X_n$.

Let V be a fixed k-vector space of dimension $n+1$; then the Koszul resolution of k is given by

$$0 \to \overset{n+1}{\Lambda} V \otimes R(-n-1) \xrightarrow{\delta_{n+1}} \overset{n}{\Lambda} V \otimes R(-n) \to \cdots \to \Lambda V \otimes R(-1) \xrightarrow{\delta_1} R \to k \to 0$$

where the δ_i are the usual Koszul maps.

For any $j \geqq 1$ we shall denote by K_j the Kernel of δ_j in degree $j+2$; the following proposition is a particular case of a crucial result which is easy and proved in [CRV1, Proposition 1].

Proposition 1.1 *Let i be any integer, $1 \leqq i \leqq h$. With the above assumptions and notations, we have*

$$a_i = \dim_k\left[\left(\overset{i-1}{\Lambda} V \otimes I_2\right) \cap K_{i-1}\right].$$

From the above result, it follows immediately that if I and J are ideals of R with initial degree at least 2 and $J_2 \subseteq I_2$, then for every integer i we have

$$a_i(J) \leqq a_i(I).$$

By using the above proposition we can prove that, for a given integer i, $a_i \neq 0$ in two different geometric situations.

In the following we say that a projective variety $V \subseteq \mathbf{P}^n$ is a **variety of minimal degree,** if V is reduced, irreducible and has degree equal to the codimension plus one.

Varieties of minimal degree have been classified by Bertini and Del Pezzo. It turns out that they are either the Veronese surface in $\mathbf{P}^5$ or a quadric hypersurface or a rational normal scroll (see [DEP, Theorem 3.1]). In any case they are arithmetically Cohen Macaulay.

Proposition 1.2 *Let X be a non degenerate projective variety in $\mathbf{P}^n$, and i an integer, $1 \leqq i \leqq n$. Let us assume that either X is contained in $\mathbf{P}^r \cup \mathbf{P}^k$ for some integers $r<n$, $k<n$ with $r+k=2n-i-1$, or X lies on a projective variety of minimal degree and dimension $n-i$. Then $a_i \neq 0$.*

Proof. Let us first assume that $X \subset \mathbf{P}^r \cup \mathbf{P}^k$ with $r+k=2n-i-1$; since X is non degenerate we may assume, after a suitable change of coordinates, that $\mathbf{P}^k$ is the linear space $X_0 = \cdots = X_{n-k-1} = 0$ and $\mathbf{P}^r$ the linear space $X_{n-k} = \cdots = X_{2n-k-r-1} = 0$. This implies that $X_t X_j \in I_2$ for every $t = 0, \ldots, n-k-1$ and $j = n-k, \ldots, 2n-k-r-1$. Now let $\alpha := e_0 \wedge \cdots \wedge e_{n-k-1}$ and $\beta := e_{n-k} \wedge \cdots \wedge e_{2n-k-r-1}$. Since $i-1 = (n-k-1) + (n-r-1)$, we have $\delta\alpha \wedge \delta\beta \in \Lambda^{i-1} V \otimes I_2$; but $\delta\alpha \wedge \delta\beta = \delta(\alpha \wedge \delta\beta)$, hence $\delta\alpha \wedge \delta\beta \in K_{i-1}$. Since clearly $\delta\alpha \wedge \delta\beta \neq 0$, we have $a_i \neq 0$ as wanted.

Now let V_{n-i} be a projective variety of minimal degree and dimension $n-i$ and let B be the artinian reduction of the Cohen Macaulay homogeneous coordinate ring of V_{n-i}. The Hilbert function of B is $H_B(0)=1$, $H_B(1)=i$, $H_B(j)=0$ for $j\geqq 2$. It follows that the socle degree is equal to 1 and in particular $a_i=i$. If X lies on V_{n-i}, then $a_i(X)\geqq a_i(V_{n-i})=i$ and so $a_i=a_i(X)\neq 0$.

We prove now that the converse of the above proposition holds if $i=1$ or $1=n$.

If $a_1\neq 0$, then X lies on a quadric hypersurface, say Q. If Q is irreducible, it is a variety of minimal degree, otherwise is the union of two hyperplanes so that $X\subseteq \mathbf{P}^{n-1}\cup \mathbf{P}^{n-1}$.

For the case $i=n$ we need some further notations and remarks on the quadrics passing through X.

Let $e_0,\dots,e_n$ be a k-vector base of V; if α is an element in $(\Lambda^{i-1}V\otimes I_2)\cap K_{i-1}$, we may write

$$\alpha=\sum_{j=\{j_1,\dots,j_{i-1}\}} e_{j_1}\wedge\cdots\wedge e_{j_{i-1}}\otimes F_j$$

with $F_j\in I_2$ and $\delta_{i-1}(\alpha)=0$.

Since $X\subseteq \mathbf{P}^n$ is a non degenerate projective variety, after a suitable change of coordinates, we may assume that X contains the coordinate points $P_0,\dots,P_n$ where $P_0:=(1,0,\dots,0)$, $P_1:=(0,1,0,\dots,0),\dots,P_n:=(0,0,\dots,1)$.

We will refer to the **extra points** of X for the points of X which are different from $P_0,\dots,P_n$.

We have some important remarks.

Remark 1.3 (i) In F_j there is no term of the form X_p^2 for every $p=0,\dots,n$.

It follows from the fact that F_j must vanish on the coordinate points.

(ii) In F_j there is no monomial of the form X_pX_q with $p\in\{j_1,\dots,j_{i-1}\}$.

If for example $\alpha=(e_{j_1}\wedge\cdots\wedge e_{j_{i-1}}\otimes\lambda X_{j_1}X_q)+\cdots$, with $\lambda\neq 0$, then $\delta_{i-1}(\alpha)=(e_{j_2}\wedge\cdots\wedge e_{j_{i-1}}\otimes\lambda X_{j_1}^2X_q)+\cdots$ cannot be zero since to cancel $e_{j_2}\wedge\cdots\wedge e_{j_{i-1}}\otimes\lambda X_{j_1}^2X_q$ we need in α a term $\pm(e_{j_2}\wedge\cdots\wedge e_{j_{i-1}}\wedge e_q\otimes\lambda X_{j_1}^2)$. This is impossible by (i).

In the following if j is a k-uple $\{0\leqq j_1<\cdots<j_k\leqq n\}$, we denote by ε_j the element $e_{j_1}\wedge\cdots\wedge e_{j_k}$ of Λ^kV.

Hence if $\alpha=\sum_{|j|=i-1}\varepsilon_j\otimes F_j\in(\Lambda^{i-1}V\otimes I_2)\cap K_{i-1}$, then in F_j we have only monomials of the form X_pX_q with $p\neq q$ and $p,q\in\mathscr{C}_j:=\{0,\dots,n\}\setminus\{j\}$.

Thus every element $\alpha\in(\Lambda^{i-1}V\otimes I_2)\cap K_{i-1}$ can be written as

$$\alpha=\sum_{|j|=i-1}\varepsilon_j\otimes F_{\mathscr{C}_j}$$

where $F_{\mathscr{C}_j}\in I_2$ is a square free quadratic form in the variables X_h, $h\in\mathscr{C}_j$.

The following remark will be very useful for our approach.

Remark 1.4 Let $\alpha=\sum_{|j|=i-1}\varepsilon_j\otimes F_{\mathscr{C}_j}$ be an element of $(\Lambda^{i-1}V\otimes I_2)\cap K_{i-1}$. We remark that as j runs among the subsets of $i-1$ elements, $\mathscr{C}_j$ runs among the subsets of $n-i+2$ elements of $\{0,\dots,n\}$.

We claim that, for every $h=\{h_1, \ldots, h_{n-i+3}\}$ such that $0 \leqq h_1 < \cdots < h_{n-i+3} \leqq n$, we have

$$\sum_{r=1}^{n-i+3} (-1)^{h_r - r+1} X_{h_r} F_{h_1 \ldots \hat{h}_r \ldots h_{n-i+3}} = 0 \, .$$

In fact

$$0 = \delta_{i-1}(\alpha) = \sum_{|j|=i-1} \sum_{r=1}^{i-1} (-1)^{r+1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_r} \wedge \cdots e_{j_{i-1}} \otimes F_{c_j} X_{j_r}$$

$$= \sum_{|q|=i-2} \varepsilon_q \otimes \left(\sum_{r=1}^{n-i+3} (-1)^{h_r - r+1} X_{h_r} F_{h_1 \ldots h_r \ldots h_{n-i+3}} \right)$$

where $\{h_1 < \cdots < h_{n-i+3}\} = \{0, \ldots, n\} \setminus q$. The conclusion follows.

In the following we will use this remark very often. Hence, to avoid eavy notation, we will say that the polynomials $\{G_1, \ldots, G_t\}$ are **related** if $\alpha = \sum_{i=1}^{t} (-1)^{c_i} G_i = 0$ for some $(c_1, \ldots, c_t) \in \mathbf{Z}^t$.

Proposition 1.5 *Let $X \subseteq \mathbf{P}^n$ be a non degenerate projective variety.*
If $a_n \neq 0$, then X is contained in $\mathbf{P}^r \cup \mathbf{P}^k$ with $r+k=n-1$.

Proof. Let $\alpha \neq 0$ be an element in $(\Lambda^{n-1} V \otimes I_2) \cap K_{n-1}$; as we have seen in Remark 1.3 we can write

$$\alpha = \sum_{0 \leqq i < j \leqq n} e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_n \otimes \lambda_{ij} X_i X_j$$

for some $\lambda_{ij} \in k$.

First we claim that if $\lambda_{0i} = 0$ for every $i = 1, \ldots, n$, then $\alpha = 0$; in fact for every $0 < j < k \leqq n$, the polynomials $\{X_k \lambda_{0j} X_0 X_j, X_j \lambda_{0k} X_0 X_k, X_0 \lambda_{jk} X_j X_k\}$ are related by Remark 1.4 so that $\{\lambda_{0j}, \lambda_{0k}, \lambda_{jk}\}$ are related too. This means that $\lambda_{jk} = 0$ for every $0 < j < k \leqq n$, hence $\alpha = 0$.

Thus we may assume that $\lambda_{01} = \cdots = \lambda_{0r} = 0$ and $\lambda_{0r+1}, \ldots, \lambda_{0n} \neq 0$ for some $1 \leqq r \leqq n-1$. Since for every $1 \leqq i \leqq r, r+1 \leqq j \leqq n$, $\{\lambda_{0i}, \lambda_{0j}, \lambda_{ij}\}$ are related, we get $\lambda_{ij} \neq 0$.

Then $X_0 X_{r+1}, \ldots, X_0 X_n, X_i X_j \in I$ for every $1 \leqq i \leqq r$ and $r+1 \leqq j \leqq n$, hence $(X_0, \ldots, X_r) \cap (X_{r+1}, \ldots, X_n) \subseteq I$. This implies that if Y is the $\mathbf{P}^r$ defined by $X_{r+1} = \cdots = X_n = 0$ and Z is the $\mathbf{P}^{n-r-1}$ defined by $X_0 = \cdots = X_n = 0$ and Z is the $\mathbf{P}^{n-r-1}$ defined by $X_0 = \cdots = X_r = 0$, then we get $X \subset Y \cup Z$ as desired.

2 The case $i=n-1$

In this section we restrict ourselves to a finite set of distinct points in $\mathbf{P}^n$ and we deal with the following question: when is $a_{n-1} \neq 0$?

By the previous section, we may assume $n \geqq 3$.

A very important result on this question is the so called **Strong Castelnuovo Lemma** (SCL for short) proved by Green in [G2]. It says that if X is a set of points in linearly general position, then $a_{n-1} \neq 0$ if and only if the points are on a rational normal curve of $\mathbf{P}^n$. Here we recall that a set of points in $\mathbf{P}^n$ is said to be in **linearly general position** if $n+1$ of the points are never on an hyperplane. Thus for points in linearly general position the converse of Proposition 1.2 holds for $i=n-1$.

Here we want to complete the SCL by considering sets of points which are not necessarily in linearly general position. We propose the following conjecture.

For a set of points spanning $\mathbf{P}^n$, one has $a_{n-1} \neq 0$ if and only if either the points are on a rational normal curve or on $\mathbf{P}^k \cup \mathbf{P}^r$ for some positive integers k and r such that $k+r=n$.

We cannot prove this conjecture in its wide generality; we need an additional assumption.

Theorem 2.1 *Let X be a set of points spanning $\mathbf{P}^n$ such that $n-1$ points of X are never on a $\mathbf{P}^{n-3}$.*

Then $a_{n-1} \neq 0$ if and only if either X lies on a rational normal curve or $X \subset \mathbf{P}^r \cup \mathbf{P}^k$ for some positive integers k and r with $k+r=n$.

Proof. We will prove that if the points are not in linearly general position and not on $\mathbf{P}^r \cup \mathbf{P}^k$ for positive integers k and r with $k+r=n$, then $a_{n-1}=0$. By Proposition 1.2 and the SCL this gives the conclusion.

Let s be the number of points in X. We remark that if $X \not\subset \mathbf{P}^r \cup \mathbf{P}^k$ with $k+r=n$, then in particular $s-2$ points of X are never on a $\mathbf{P}^{n-1}$. Hence we may assume $s \geqq n+4$.

First we prove the result with a stronger assumption.

Step 1. Let us assume that n points of X are never on a $\mathbf{P}^{n-2}$.

In this case the following conditions are satisfied:

(a) $s \geqq n+4$
(b) there exist $n+1$ points on a $\mathbf{P}^{n-1}$
(c) $s-2$ points are never on a $\mathbf{P}^{n-1}$
(d) n points are never on a $\mathbf{P}^{n-2}$

and we need to prove that $a_{n-1}=0$.

Since $n+1$ points are on an hyperplane, they must span it, otherwise we get $n+1$ points on a $\mathbf{P}^{n-2}$, a contradiction. This means that, after a suitable change of coordinates, we may assume that X contains the coordinate points and that the hyperplane $X_n=0$ contains $n+1$ points of X. Hence we can find an extra point, say $Q=(y_0, \ldots, y_{n-1}, 0)$, which lies on this hyperplane. We remark that $y_i \neq 0$ for every $i=0, \ldots, n-1$, otherwise, if $y_i=0$, then the n points $Q, P_0, \ldots, \hat{P}_i, \ldots, P_{n-1}$ would be on a $\mathbf{P}^{n-2}$.

Claim 1 Let L, M, P be non zero linear forms involving r variables such that L and M are linearly independent and $r \leqq 4$. Then

$$(LP, MP) \not\subset I .$$

By assumption $L=M=0$ is a $\mathbf{P}^{n-2}$ which contains at most $n-1$ points of X. Let k be the number of variables involved in L and M and t the number of variables involved in P. Then our $\mathbf{P}^{n-2}$ contains $n+1-k$ coordinate points, hence at most $n-1-(n+1-k)=k-2$ extra points. If $(LP, MP)\subseteq I$, the remaining $s-(n+1)-(k-2)$ extra points are on the hyperplane $P=0$ which contains also $n+1-t$ coordinate points. All together we have $s-k-t+2$ points on $P=0$. Since $k+t=r\leqq 4$ we get $s-2$ points on a $\mathbf{P}^{n-1}$, a contradiction.

Let

$$\alpha=\sum_{|j|=n-2}\varepsilon_j\otimes F_{abc}$$

be an element of $(\Lambda^{n-2}V\otimes I_2)\cap K_{n-2}$, where $\{0\leqq a<b<c\leqq n\}=\mathscr{C}_j$.

Claim 2 For every $a, b<n$ we have $F_{abn}=\mu X_aX_n+\sigma X_bX_n$ for some $\mu, \sigma\in k$.

In fact $F_{abn}=\lambda X_aX_b+\mu X_aX_n+\sigma X_bX_n$ with $\lambda, \mu, \sigma\in k$; hence $0=F_{abn}(Q)=\lambda y_ay_b$. This implies $\lambda=0$, as wanted.

Claim 3 If $F_{abn}=0$ for every $a, b<n$ then $\alpha=0$.

If $F_{abn}=0$ for every $a, b<n$, then it is clear that

$$\alpha=\sum_{\substack{|j|=n-2\\ n\in j}}\varepsilon_j\otimes F_{\mathscr{C}_j}\,.$$

By Remark 1.4, for any $0\leqq a<b<c<n$ the polynomials

$$\{X_aF_{bcn}, X_bF_{acn}, X_cF_{abn}, X_nF_{abc}\}$$

are related so that $X_nF_{abc}=0$. It follows that $F_{abc}=0$ and so $\alpha=0$.

Conclusion. By Claim 3 we may assume $F_{abn}\neq 0$ for some $0\leqq a<b<n$. By Claim 2 we have $F_{abn}=X_nL$ with $L\in(X_a, X_b)$, $L\neq 0$. Let $c\neq a, b, n$. By Remark 1.4 the polynomials

$$\{X_aF_{bcn}, X_bF_{acn}, X_cF_{abn}, X_nF_{abc}\}$$

are related, hence, using again Claim 2 for F_{bcn} and F_{acn}, we get

$$F_{abc}=X_aP+X_bM+X_cL$$

for suitable linear forms $M\in(X_a, X_c)$, $P\in(X_b, X_c)$.

If $P=M=0$, then $(X_cL, X_nL)\subseteq I$, a contradiction to Claim 1. If for example $M\neq 0$, then $(X_nL, X_nM)\subseteq I$, so that by Claim 1, L and M are linearly dependent. This means that $L=\lambda X_a$ and $M=\mu X_a$ for some $\lambda, \mu\in k^\star$. This implies $F_{abc}=X_a(P+\mu X_b+\lambda X_c)=X_aT$ with $T\in(X_b, X_c)$. If $T\neq 0$, then $(X_aT, X_aX_n)\subseteq I$, while if $T=0$, then $P\neq 0$ and $(X_nX_a, X_nP)\subseteq I$. In both cases we get a contradiction again to Claim 1.

Step 2 Let us assume that there exist n points of X on a $\mathbf{P}^{n-2}$.

In this case the following conditions are satisfied:

(a) $s\geqq n+4$
(b) there exist n points on a $\mathbf{P}^{n-2}$

(c) $n-1$ points are never on a $\mathbf{P}^{n-3}$
(d) $X \not\subset \mathbf{P}^r \cup \mathbf{P}^k$ with $k+r=n$

and we need to prove that $a_{n-1}=0$.

Since n points are on a $\mathbf{P}^{n-2}$, they must span it, otherwise we get n points on a $\mathbf{P}^{n-3}$, a contradiction. This means that, after a change of coordinates, we may assume that the coordinate points are on X and that the linear space defined by $X_{n-1}=X_n=0$, contains n points of X. Hence we can find an extra point, say $Q:=(y_0,\ldots,y_{n-2},0,0)$ which lies on it. It is clear that $y_i \neq 0$ for every $i=0,\ldots,n-2$, otherwise the $n-1$ points $Q, P_0,\ldots,\hat{P}_i,\ldots,P_{n-2}$ would be on a $\mathbf{P}^{n-3}$.

Claim 4 Let L, M, N, T be non zero linear forms involving t variables, such that L, M, N are linearly independent and $t \leqq 5$. Then $(TL, TM, TN) \not\subset I$.

The proof of this claim is the same as that of Claim 1, hence we omit it.

As before let

$$\alpha=\sum_{|j|=n-2} \varepsilon_j \otimes F_{abc}$$

be an element of $(\Lambda^{n-2}V \otimes I_2) \cap K_{n-2}$, where $\{0 \leqq a<b<c \leqq n\}=\mathscr{C}_j$. In this case we can give a complete description of the quadrics which occur in α.

Claim 5 (a) For every $a<b \leqq n-2$, we have

$$F_{abn-1}=X_{n-1}L_{ab} \quad \text{and} \quad F_{abn}=X_nM_{ab}$$

with $L_{ab}, M_{ab} \in (X_a, X_b)$.

(b) For every $0 \leqq a<b<c \leqq n-2$, the polynomials

$$\{F_{abc}, X_aL_{bc}, X_bL_{ac}, X_cL_{ab}\}$$

are related.

The same for

$$\{F_{abc}, X_aM_{bc}, X_bM_{ac}, X_cM_{ab}\}\ .$$

(c) For every $i=0,\ldots,n-2$, there exist $P \in (X_{n-1}, X_n)$ and $\alpha_i \in k$ such that

$$F_{i,n-1,n}=(-1)^i X_i P+\alpha_i X_{n-1}X_n\ .$$

(d) For every $0 \leqq a<b \leqq n-2$, the polynomials

$$\{L_{ab}, M_{ab}, \alpha_b X_a, \alpha_a X_b\}$$

are related.

Claim 6 If $W_1+W_2=0$, then $\alpha=0$.

If $W_1=W_2=0$, then by Claim 5(d) and (c), we have $F_{i,n-1,n}=(-1)^i X_i P$ for every $i=0,\ldots,n-2$, hence $(P) \cap (X_0,\ldots,X_{n-2}) \subseteq I$. Since $X \not\subset \mathbf{P}^{n-1} \cup \mathbf{P}^1$, we get $P=0$ which, together with the assumption $L_{ab}=M_{ab}=0$ for every $0 \leqq a<b \leqq n-2$, implies $\alpha=0$.

Claim 7 (a) If $L_{ab} \neq 0$ and $X_nL_{ab} \in I$, then $(L_{ac}, L_{bc}) \neq (0,0)$ for every $c \leqq n-2$.

(b) If $L_{ab} \neq 0$ then $\dim_k \langle L_{ai}, L_{bj}\rangle_{i \neq a,\, j \neq b} \leqq 2$.

(c) If $L_{ab}L_{ac}L_{ad} \neq 0$ for some $0 \leqq a < b < c < d \leqq n$, at least two of them are monomial in X_a.

If for some c we have $L_{ac} = L_{ab} = 0$, then by Claim 5(b), we have $F_{abc} = \pm X_c L_{ab}$ so that $\{X_a L_{ab}, X_b L_{ab}, X_c L_{ab}\} \subseteq I$, a contradiction to Claim 4. This proves (a).

As for (b), if $\dim_k \langle L_{ai}, L_{bj} \rangle_{i \neq a, j \neq b} \geqq 3$ and for example L_{ab}, L_{ai}, L_{bj} are linearly independent, since $(X_{n-1}L_{ab}, X_{n-1}L_{ai}, X_{n-1}L_{bj}) \subseteq I$, we get a contradiction to Claim 4.

Finally let us consider the matrix of the coefficients of X_a, X_b, X_c, X_d in L_{ab}, L_{ac}, L_{ad} respectively:

$$\begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & 0 & \cdot & 0 \\ \cdot & 0 & 0 & \cdot \end{pmatrix}$$

By (b) this has rank $\leqq 2$. The vanishing of its 3×3 minors gives the conclusion.

Claim 8 If $W_1 \neq 0$, then $X_n W_n \nsubseteq I$.

If, for example, we have $L_{ab} \neq 0$ and $X_n L_{ab} \in I$, then by Claim 7(a), $(L_{ac}, L_{bc}) \neq (0, 0)$ for every $c \leqq n-2$. If $n \geqq 6$ so that $n-2 \geqq 4$, this implies that at least three elements in $\{L_{ai}\}_{0 \leqq i \leqq n-2}$ or in $\{L_{bj}\}_{0 \leqq j \leqq n-2}$ are not zero. Hence by Claim 7(c), we may assume for example that $L_{ab} = \lambda X_a$ $L_{ac} = \mu X_a$ for some $\lambda, \mu \in k^\star$. By Claim 5(b), the polynomials

$$\{F_{abc}, X_a L_{bc}, X_b L_{ac} = \mu X_a X_b, X_c L_{ab} = \lambda X_a X_c\}$$

are related, hence $F_{abc} = X_a T$, where T is a linear form in (X_b, X_c). Since $\lambda X_a X_n = X_n L_{ab} \in I$, and $\lambda X_a X_{n-1} = X_{n-1} L_{ab} = F_{a,b,n-1} \in I$, we get

$$(X_a X_n, X_a X_{n-1}, X_a T) \subseteq I\ .$$

If $T \neq 0$, this is a contradiction by Claim 4. If $T = 0$, then $L_{bc} = \pm \mu X_b \pm \lambda X_c \neq 0$. Let $d \neq a, b, c$; since $L_{ab} = \lambda X_a \neq 0$ is not a monomial in X_b, by part (c) we must have $L_{bd} = 0$. But also $L_{ac} = \mu X_a \neq 0$ is not a monomial in X_c, hence $L_{cd} = 0$ too. By part (a) this implies $X_n L_{bc} \notin I$, as desired.

If $3 \leqq n \leqq 5$, the proof is even easier, hence we omit it.

We recall that by Claim 5(c), we have for every $i = 0, \ldots, n-2$

$$F_{i,n-1,n} = (-1)^i X_i P + \alpha_i X_{n-1} X_n .$$

Claim 9 (a) If $(\alpha_0, \ldots, \alpha_{n-2}) = (0, \ldots, 0)$ then $\alpha = 0$.
(b) If $X_{n-1} X_n \notin I$, then $\alpha = 0$.

First we prove (a). If $\alpha_i = 0$ for every $i \leqq n-2$, then by Claim 5(d), we get $W_1 = W_2$ so that $X_n W_1 \subseteq I$. By Claim 8 this implies $W_1 = W_2 = 0$ hence $\alpha = 0$ by Claim 6. This proves (a).

By Claim 6 we may assume that $W_1 + W_2 \neq 0$, hence we let, for example, $W_1 \neq 0$. By Claim 8 there exist $a, b \leqq n-2$ such that $X_n L_{ab} \notin I$.

Let Q' be a point of X with $X_n(Q') \neq 0$ and $L_{ab}(Q') \neq 0$; since $0 \neq X_{n-1}L_{ab} = F_{a,b,n-1} \in I$ we have $X_{n-1}(Q')=0$ and also Q' must be an extra point, otherwise $Q'=P_n$ and $L_{ab}(Q')=0$.

By imposing $F_{i,n-1,n}(Q')=0$ for every $i=0,\ldots,n-2$, we get $P(Q')=0$, hence $P=\rho X_{n-1}$ for some $\rho \in k$. If $W_2 \neq 0$, by repeating the same procedure we also obtain $P \in (X_n)$ and hence $P=0$. Since $X_{n-1}X_n \notin I$, this implies $(\alpha_0,\ldots,\alpha_{n-2})=(0,\ldots,0)$, hence $\alpha=0$ by part (a). So let $W_2=0$ and let $i \leqq n-2$ be an integer such that $\alpha_i \neq 0$; by Claim 5(d), for every $j \neq i$

$$\{L_{ij}, \alpha_i X_j, \alpha_j X_i\}$$

are related, hence the $n-2$ vectors $\{L_{ij}\}_{j \neq i}$ are linearly independent. This implies

$$n-2 = \dim_k \langle L_{ij} \rangle_{j \neq i} \leqq \dim_k(W_1)\ .$$

Now we have

$$F_{i,n-1,n} = (-1)^i \rho X_i X_{n-1} + \alpha_i X_{n-1} X_n = X_{n-1}((-1)^i \rho X_i + \alpha_i X_n) = X_{n-1} T\ ,$$

where $T=(-1)^i \rho X_i + \alpha_i X_n$. Since $\alpha_i \neq 0$, $T \notin W_1$, hence $\dim_k(W_1 + \langle T \rangle) \geqq n-1$. Further $(X_{n-1}W_1, X_{n-1}T) \subseteq I$, hence $X \subset \mathbf{P}^1 \cup \mathbf{P}^{n-1}$, a contradiction.

Conclusion. By Claim 9, we may assume that $X_{n-1}X_n \in I$ and $\alpha_i \neq 0$ for some $0 \leqq i \leqq n-2$. Since $X_{n-1}W_1 \subseteq I$, $X_n W_2 \subseteq I$, and $X_{n-1}X_n \in I$ by Claim 4 we get $\dim_k \langle L_{ij} \rangle_{j \neq i} \leqq 1$ and $\dim_k \langle M_{ij} \rangle_{j \neq i} \leqq 1$. If $n \geqq 5$ the set $\{L_{ij}\}_{j \neq i}$ has at least three elements; this together with $\dim_k \langle L_{ij} \rangle_{j \neq i} \leqq 1$ implies that at least two elements, say L_{ij}, L_{ik} are monomials in X_i, perhaps the zero monomial. But by Claim 5(d)

$$\{M_{ij}, L_{ij}, \alpha_j X_i, \alpha_i X_j\}$$

are related as well as

$$\{M_{ik}, L_{ik}, \alpha_k X_i, \alpha_i X_k\}\ .$$

This implies that M_{ij}, M_{ik} are linearly independent, a contradiction to $\dim_k \langle M_{ij} \rangle_{j \neq i} \leqq 1$.

If $3 \leqq n \leqq 4$ the proof is easier, hence we omit it.

We remark that if $n=3$, $n-1$ points of X are never on a $\mathbf{P}^{n-3}$, so that, in this case, we get a proof of our conjecture.

3 Points on a variety of minimal degree

In this section we are dealing with the case $i \leqq n-2$. In the above mentioned paper (see [G2]), Green stated the following problem which can be considered as the natural generalisation of the Strong Castelnuovo Lemma:

Given a set of points in linearly general position such that $a_{n-2} \neq 0$ is it true that the points are on a rational normal scroll of dimension two?

The answer to this question is negative as the following example, suggested to us by Eisenbud, shows. Let us consider a set X of twelve random points in $\mathbf{P}^7$. By using the computer algebra system Macaulay, one can see that $a_5(X)=4$. Now a rational normal scroll S of dimension two in $\mathbf{P}^7$ has degree 6 so that, as in the proof of Proposition 1.2, we get $a_5(S)=5$. Hence the points cannot lie on such a surface.

Here we want to find suitable conditions under which a set of points in linearly general position with $a_i \neq 0$ for some integer $i<n$, lies on a rational normal scroll of dimension $n-i$.

The following result gives a partial answer for $i=2$ and it is a considerable extension of Proposition 3.3 in [CRV2] even if one needs essentially the same idea.

Proposition 3.1 *Let* $n \geqq 3$ *and* $X \subseteq \mathbf{P}^n$ *be a set of points in linearly general position. If* $\dim_k(I_2) \leqq 3$ *and* $a_2 \neq 0$, *then the points of* X *are on a rational normal scroll of dimension* $n-2$.

Proof. First we remark that all the quadrics passing through the points of X are irreducible. In fact we have $s \geqq H_X(2) = \binom{n+2}{2} - \dim_k(I_2) \geqq \binom{n+2}{2} - 3 \geqq 2n+1$ since $n \geqq 3$. Now it is clear that a quadric passing through $2n+1$ points in linearly general position must be irreducible.

Since $a_2 \neq 0$ we can find a non zero element $\alpha \in (\Lambda V \otimes I_2) \cap K_1$. We write $\alpha = \sum_{i=0}^{n} e_i \otimes F_i$ with $F_i \in I_2$ and $\sum_{i=0}^{n} X_i F_i = 0$. Since $\alpha \neq 0$ and $\dim_k(I_2) \leqq 3$, we may assume that $F_1 \neq 0$ and $F_i \in \langle F_0, F_1, F_2 \rangle$ for every $i \geqq 3$. Hence $F_i = \sum_{j=0}^{2} \lambda_{ij} F_j$ with $\lambda_{ij} \in k$; this gives $\sum_{i=0}^{n} X_i F_i = \sum_{i=0}^{2} X_i F_i + \sum_{i=3}^{n} X_i (\sum_{j=0}^{2} \lambda_{ij} F_j) = \sum_{j=0}^{2} (X_j + \sum_{i=3}^{n} \lambda_{ij} X_i) F_j = 0$.

Now let $m_0 := X_0 + \sum_{i=3}^{n} \lambda_{i0} X_i$, $m_1 := X_1 + \sum_{i=3}^{n} \lambda_{i1} X_i$, and $m_2 := X_2 + \sum_{i=3}^{n} \lambda_{i2} X_i$. It is clear that they form a regular sequence in R, so we must have $F_0 = \ell_2 m_1 - \ell_1 m_2$, $F_1 = -\ell_2 m_0 + \ell_0 m_2$, $F_2 = \ell_1 m_0 - \ell_0 m_1$, for some $\ell_0, \ell_1, \ell_2 \in R_1$. It follows that our points lie on he locus

$$V_{n-2} : \left\{ \operatorname{rank} \begin{pmatrix} m_0 & m_1 & m_2 \\ \ell_0 & \ell_1 & \ell_2 \end{pmatrix} \leqq 1 \right\}.$$

In order to prove that V_{n-2} is a rational normal scroll of dimension $n-2$, we need only to prove that for any $(t, u) \neq (0, 0)$ the linear forms $tm_0 + u\ell_0$, $tm_1 + u\ell_1$, $tm_2 + u\ell_2$ are linearly independent (see [H, p. 104]).

This can be proved as in [CRV2, Proposition 3.3].

We remark that the assumption of the above proposition is verified if X is a set of s points in linearly general position in $\mathbf{P}^n$ such that

$$\binom{n+2}{2} - 3 \leqq s \leqq \binom{n+2}{2} - 1$$

and X imposes independent conditions on the hypersurfaces of degree two.

Now it is well known, by a classical result of Bertini, that $n+3$ points of $\mathbf{P}^n$ in linearly general position are on a rational normal curve. On the other hand, if p is an integer, $p=1, \ldots, n-2$, let us consider a set X of $s=2n+1-p$ points of $\mathbf{P}^n$ in linearly general position. Green and Lazarsfeld in [GL] proved that X has a resolution which is linear at least for the first p steps. This implies $a_{p+1} \neq 0$ (see Proposition 2.1 and Proposition 2.5 in [CRV2]). Hence one can ask the following question: do these points lie on a rational normal scroll of dimension $n-p-1$? A positive answer is given by the following theorem.

Theorem 3.2 *Let $n \geqq 3$ and p an integer, $p=1,\ldots,n-2$. If $X \subseteq \mathbf{P}^n$ is a set of s points in linearly general position such that $s<2n+1-p+\frac{n-p-1}{p+1}$, then the points of X are on a rational normal scroll of dimension $n-p-1$.*

Proof. As usual, without loss of generality, we may assume that X contains the coordinate points. Also it is clear that it is enough to prove our assertion for

$$s=\max\left\{t\in\mathbb{N}\;\middle|\; t<2n+1-p+\frac{n-p-1}{p+1}\right\}.$$

Hence $s \geqq 2n+1-p$.

Claim 1 Let L and M be non zero linear forms involving k variables. If $k \leqq n-p+1$, them $LM \notin I$.

Let t be the number of variables involved in L an r the number of variables involved in M so that $k=t+r$. Since the hyperplane $L=0$ contains $n+1-t$ coordinate points, it can contain at most $t-1$ extra points. The remaining $s-(n+1)-(t-1)=s-n-t$ extra points must be on $M=0$ which contains also $n+1-r$ coordinate points. All together we have at least $s-n-t+n+1-r=s-k+1$ points on the hyperplane $M=0$. Since

$$s-k+1 \geqq 2n+1-p-(n-p+1)+1=n+1$$

we get a contradiction.

Claim 2 There exist $p+2$ linear forms $L_0,\ldots,L_{p+1}$, such that

(a) $L_0 \in (X_{p+2},\ldots,X_n)$, $L_i \in (X_i, X_{p+2},\ldots,X_n)$ for every $i=1,\ldots,p+1$
(b) $(L_0,\ldots,L_{p+1}) \neq (0,\ldots,0)$
(c) $X_0L_i-X_iL_0 \in I$ for every $i=1,\ldots,p+1$.

It is clear that if we consider $p+2$ linear forms as in (a), then for every $i=1,\ldots,p+1$, $X_0L_i-X_iL_0$ is a square free quadratic form, hence a quadric passing through the coordinate points. If we impose that these $p+1$ quadrics pass through the $s-n-1$ extra points, we get an homogeneous system of $(p+1)(s-n-1)$ equations in $(n-p-1)+(p+1)(n-p)$ unknowns, the coefficients of $L_0,\ldots,L_{p+1}$. By our assumption we have

$$(p+1)(s-n-1)<(p+1)\left[2n+1-p+\frac{n-p-1}{p+1}-n-1\right]$$
$$=(n-p)(p+1)+(n-p-1).$$

The conclusion follows.

Claim 3 X is contained on the locus

$$V_{n-p-1}:\left\{\operatorname{rank}\begin{pmatrix} X_0 & X_1 & \dots & X_{p+1} \\ L_0 & L_1 & \dots & L_{p+1} \end{pmatrix} \leqq 1\right\}.$$

Let $0<i<j\leqq p+1$; we have

$$X_0(X_iL_j-X_jL_i)=X_i(X_0L_j-X_jL_0)-X_j(X_0L_i-X_iL_0);$$

hence $X_0(X_iL_j-X_jL_i)\in I$. Since the points are in linearly general position, $X_0(Q)\neq 0$ for every extra point Q of X, hence $X_iL_j-X_jL_i\in I$, as desired.

In the following it will be useful to put $\lambda_0=0$ and for every $i=0,\dots,p+1$, to let

$$L_i:=\lambda_iX_i+\sum_{j=p+2}^{n}\lambda_{ij}X_j$$

and

$$M_i:=L_i-\lambda_iX_i=\sum_{j=p+2}^{n}\lambda_{ij}X_j .$$

Claim 4 (a) $L_i\neq 0$ for every $i=0,\dots,p+1$.

(b) $M_i\neq 0$ for every $i=0,\dots,p+1$

(c) If for some $r\geqq 1$ we have $\lambda_{j_1}=\dots=\lambda_{j_r}=\lambda\in k$, then $M_{j_1},\dots,M_{j_r}$ are linearly independent.

By Claim 2(b), there exists $0\leqq i\leqq p+1$ such that $L_i\neq 0$. If $L_j=0$ for some $j\neq i$, then, by Claim 3, X_jL_i is a non zero element of I. Since L_i involves at most $n-p$ variables, we get a contradiction to Claim 1. This proves (a).

Now by part (a) we have $M_0=L_0\neq 0$; if $M_i=0$ for some $1\leqq i\leqq p+1$, then $L_i=\lambda_iX_i$ with $\lambda_i\neq 0$ by (a). Hence $X_0L_i-X_iL_0=X_i(\lambda_iX_0-L_0)\in I$, where $\lambda_iX_0-L_0\neq 0$ and involves $n-p$ variables. The conclusion follows again by Claim 1

If $r=1$, (c) follows by (b). So we may assume $M_{j_1},\dots,M_{j_{r-1}}$ linearly independent and

$$M_{j_r}=\sum_{i=1}^{r-1}\alpha_iM_{j_i}$$

for some $\alpha_i\in k$. It is clear that the vector space W spanned by $M_{j_1},\dots,M_{j_{r-1}}$ is a subspace of dimension $r-1$ of the vector space V spanned by $X_{p+2},\dots,X_n$ which has dimension $n-p-1$. Since

$$(n-p-r+1)+r-1=n-p>n-p-1,$$

however we choose $n-p-r+1$ among these variables, the vector space they generate must intersect properly W. Hence we can find a linear form, say

$$M=\sum_{i=1}^{r-1}\beta_iM_{j_i},$$

which is not zero and involves $n-p-r+1$ variables.

Now if in the matrix

$$\begin{pmatrix} X_0 & X_1 & \dots & X_{p+1} \\ L_0 & L_1 & \dots & L_{p+1} \end{pmatrix}$$

we add to the second row the first multiplied by $-\lambda$ we get the matrix

$$\begin{pmatrix} \dots & X_{j_1} & \dots X_{j_i} & \dots & X_{j_r} & \dots \\ \dots & M_{j_1} & \dots M_{j_i} & \dots & M_{j_r} & \dots \end{pmatrix}.$$

Since $M_{j_r}=\sum_{i=1}^{r-1}\alpha_i M_{j_i}$, by the corresponding elementary operation we get the matrix

$$\begin{pmatrix} \dots & X_{j_1} & \dots X_{j_i} & \dots & X_{j_r}-\sum_{i=1}^{r-1}\alpha_i X_{j_i} & \dots \\ \dots & M_{j_1} & \dots M_{j_i} & & 0 & \dots \end{pmatrix}.$$

Finally, if for example $\beta_i \neq 0$, we can elementarily further operate to get the matrix

$$\begin{pmatrix} \dots & X_{j_1} & \dots \sum_{i=1}^{r-1}\beta_i X_{j_i} & \dots & X_{j_r}-\sum_{i=1}^{r-1}\alpha_i X_{j_i} & \dots \\ \dots & M_{j_1} & \dots M & \dots & 0 & \dots \end{pmatrix}.$$

Since a determinantal ideal does not change if we make elementary operations on the matrix, this implies that

$$M\left(X_{j_r}-\sum_{i=1}^{r-1}\alpha_i X_{j_i}\right)\in I .$$

But M involves $n-p-r+1$ variables and the other hyperplane r variables, so that we get the conclusion again by Claim 1.

Conclusion. In order to prove that V_{n-p-1} is a rational normal scroll of dimension $n-2$ we need only to prove that for any $(t,u)\neq(0,0)$ the linear forms $tX_0+uL_0, tX_1+uL_1,\dots,tX_{p+1}+uL_{p+1}$ are linearly independent.

If $u=0$, this is trivial. Let $u\neq 0$ and remark that $tX_i+uL_i=X_i(t+u\lambda_i)+uM_i$ for every $i=0,\dots,p+1$.

We have vectors $v_i:=X_i(t+u\lambda_i)+uM_i$ for every $i=0,\dots,p+1$. If $t+u\lambda_i=0$, then $\lambda_i=-\frac{t}{u}$ and $v_i=uM_i$ so that all the vectors with this property are linearly independent by Claim 4(c). Further, if we consider the lexicographic order

$$X_1>X_2>\cdots>X_n ,$$

these vectors $v_i=uM_i$ have maximal terms $\leqq X_{p+2}$. The other vectors with $t+u\lambda_i\neq 0$ have different maximal terms which are $\geqq X_{p+1}$. The conclusion easily follows.

We remark that the following 9 points in $\mathbf{P}^4$:

$$X=\{P_0,P_1,P_2,P_3,P_4,(2,-1,-3,4,3),(3,2,-1,1,-2),$$
$$(1,2,3,4,-1),(1,1,1,1,1)\} ,$$

where $P_0, \ldots, P_4$ are the coordinate points, dont lie on the locus

$$V_2 : \left\{ \operatorname{rank} \begin{pmatrix} X_0 & X_1 & X_2 \\ L_0 & L_1 & L_2 \end{pmatrix} \leqq 1 \right\}$$

for every choice of linear forms L_0, L_1, L_2 as in Theorem 3.2. This means that we cannot extend our result to a bigger number of points. On the other hand we can prove that these points are on a rational surface of minimal degree.

Remark 3.3 It is clear that if a set X of points lies on a rational normal scroll of dimension r, then X lies also on a rational normal scroll of dimension $r+1$.

But it is clear that we can always find $n+4$ points in linearly general position such that one of them does not lie on the unique rational normal curve passing through the others $n+3$ points. In this case the points are on a rational normal surface but not on a rational normal curve.

4 Applications

If we have a set X of s points in $\mathbf{P}^n$ which are generic, then it is clear that they have maximal Hilbert function, which means that

$$H_X(t) = \min \left\{ \binom{n+t}{t}, s \right\}.$$

By imposing maximal rank conditions, it is easy to guess the numerical resolution of such a set. This is known as **Minimal Resolution Conjecture** (MRC for short). This conjecture is due to Lorenzini who first worked out explicitly the expected Betti numbers (see [L]).

The MRC holds if $n=2, 3$ or $n+1 \leqq s \leqq n+4$ (see [GGR, BG, GEL, CRV1]).

Recently we have been told by Schreyer that a counterexample can be given for 12 points in $\mathbf{P}^7$.

The question which motivated our present research was to identify in some concrete geometric way the open set where the MRC holds, if it holds.

For example a set of $n+2$ points spanning $\mathbf{P}^n$ verifies MRC if and only if the points are in linearly general position (see [HSV, Theorem C]).

For a set of $n+3$ points spanning $\mathbf{P}^n$ we proved in [CRV3] that they verify MRC if and only if they are in GL position. Here a set of $s = 2n+1-p$ points is said to be in **GL** position if $2k+2-p$ points of X are never on a $\mathbf{P}^k$ for every $k = p, \ldots, n-1$.

This definition has been motivated by the work of Green and Lazarsfeld in the above mentioned paper [GL].

In the same paper they say that a non degenerate projective variety $X \subseteq \mathbf{P}^n$ verifies property (N_p) for a given integer $p \geqq 1$, if $I(X)$ is generated by quadrics and the resolution is linear for the first p steps.

Here we are dealing with the case $s=n+4$. For this number of points we can easily see that MRC holds if and only if $a_{n-1}=0$ and X verifies (N_{n-3}) (see [CRV2]).

We need the following Lemma.

Lemma 4.1 *Let $n \geqq 5$ and X be a set of s points in $\mathbf{P}^n$. If $s-2$ points of X are never on an hyperplane and $n-1$ never on a $\mathbf{P}^{n-3}$, then for every positive integers k and r such that $k+r=n$, X is not contained in $\mathbf{P}^k \cup \mathbf{P}^r$.*

Proof. If $X \subseteq \mathbf{P}^1 \cup \mathbf{P}^{n-1}$, since on a $\mathbf{P}^{n-1}$ we have at most $s-3$ points, the remaining 3 points must be on a $\mathbf{P}^1$, so that we can find $3+n-4=n-1$ points on a $\mathbf{P}^{n-3}$, a contradiction. If $X \subseteq \mathbf{P}^2 \cup \mathbf{P}^{n-2}$, since on a $\mathbf{P}^{n-2}$ we have at most $s-4$ points, the remaining 4 points must be on a $\mathbf{P}^2$, so that we can find $4+n-5=n-1$ points on a $\mathbf{P}^{n-3}$, a contradiction. If $X \subseteq \mathbf{P}^r \cup \mathbf{P}^{n-r}$ with $r \leqq n-3$, since on a $\mathbf{P}^r$ we have at most $r+1$ points, the remaining $s-r-1$ points must be on a $\mathbf{P}^{n-r}$, so that we can find $s-r-1+r-1=s-2$ points on a $\mathbf{P}^{n-1}$, a contradiction.

Proposition 4.2 *Let $n \geqq 5$ and X be a set of $n+4$ points spanning $\mathbf{P}^n$. Then X verifies the MRC if and only if the following conditions are satisfied:*

(a) *$n+2$ points of X are never on a $\mathbf{P}^{n-1}$.*

(b) *$n-1$ points of X are never on a $\mathbf{P}^{n-3}$.*

(c) *X does not lie on a rational normal curve.*

Proof. If X verifies MRC, then $a_{n-1}=0$ and (N_{n-3}) holds for X. Since $a_{n-1}=0$, X does not lie on a rational normal curve by Proposition 1.2. Also, $n+2$ points of X are never on an hyperplane, otherwise $X \subseteq \mathbf{P}^1 \cup \mathbf{P}^{n-1}$. Further since (N_{n-3}) holds, $n-1$ points of X are never on a $\mathbf{P}^{n-3}$ by a result of Nagel (see [N, Corollary 3.5]). Conversely it is clear that if $n+2$ points of X are never on a $\mathbf{P}^{n-1}$ and $n-1$ points of X are never on a $\mathbf{P}^{n-3}$, then the points are in GL position, so that (N_{n-3}) holds by the main result in [CRV4]. Finally $a_{n-1}=0$ follows by the above lemma and Theorem 2.1.

We can easily complete the above proposition by considering the case of 7 points in $\mathbf{P}^3$ and 8 points in $\mathbf{P}^4$. We get:

(a) A set of 7 points in $\mathbf{P}^3$ verifies the MRC if and only if the points are not on a rational normal curve and $X \nsubseteq \mathbf{P}^1 \cup \mathbf{P}^2$.

(b) A set of 8 points in $\mathbf{P}^4$ verifies the MRC if and only if the points are not on a rational normal curve, $X \nsubseteq \mathbf{P}^1 \cup \mathbf{P}^3$, $X \nsubseteq \mathbf{P}^2 \cup \mathbf{P}^2$ and 3 points of X are never on a line.

We remark that for 7 points in $\mathbf{P}^3$ the above proposition does not hold. Take for example four points on a plane and three points on a line but out of the plane. Then it is clear that the conditions of the proposition are satisfied but since $X \subseteq \mathbf{P}^1 \cup \mathbf{P}^2$ we have $a_2 \neq 0$ and the MRC does not hold. Similar examples can be given for 8 points in $\mathbf{P}^4$.

We recall now that in [TV] it is proved that given s and n, almost every set of s points in $\mathbf{P}^n$ have the last Betti number as predicted by the MRC. This was known as the Cohen Macaulay type conjecture. It turns out that if $n+1 \leqq s < \binom{n+2}{2}$, we expect to have $a_n = 0$.

Our Proposition 1.5 proves that a set X of $s < \binom{n+2}{2}$ points in $\mathbf{P}^n$ has the expected Cohen Macaulay type if and only $X \not\subseteq \mathbf{P}^r \cup \mathbf{P}^k$ with $k+r=n-1$.

We end the paper with the following result, a special case of which we proved in [CRV3] and [CRV4, Proposition 4.1].

Proposition 4.3 *Let $X \subseteq \mathbf{P}^n$ be a set of $2n+1-p$ points, $1 \leqq p \leqq n-1$. If the points are in* GL *position, then $a_n = 0$.*

Proof. By Proposition 1.5 it is enough to prove that X is not contained in $\mathbf{P}^r \cup \mathbf{P}^k$ with $r+k=n-1$.

Let

$$m := \max(k+1, 2k+1-p)$$

and let us assume by contradiction that $X \subseteq \mathbf{P}^k \cup \mathbf{P}^{n-k-1}$. By GL position we have at most m points on $\mathbf{P}^k$, hence at least $2n+1-p-m$ points on $\mathbf{P}^{n-k-1}$. Again by GL position, on a $\mathbf{P}^{n-k-1}$ we have at most m' points, where

$$m' := \max(n-k, 2(n-k-1)+1-p) = \max(n-k, 2n-2k-p-1) .$$

Hence we need only to prove that

$$2n+1-p-m > m'$$

or also

$$m + m' \leqq 2n - p .$$

This can be done by easy computation.

We remark that the above proposition does not hold for $2n+1$ points. Take for example two skew lines in $\mathbf{P}^3$ and four points on a line and four on the other. Then it is clear that they are in GL position since seven are never on a plane and five never on a line. But $a_3 \neq 0$ by Proposition 1.5.

Some of the results here were discovered or confirmed with the help of the computer algebra program CoCoA written by Giovini and Niesi (see [GN]).

Acknowledgement. The authors thank U. Nagel for some helpful discussion on the subject of this paper.

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