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K-theory of mapping class groups: general p-adic K-theory for punctured spheres

Luke Hodgkin

Department of Mathematics, King's College, London, U.K.

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1 Introduction

Let Γ^n denote the mapping class group of the n-punctured sphere, (where $n \geq 3$) and let p be a prime. Our main aim in this paper is to find the p-adic component in $K^*(B\Gamma^n)$ (ordinary topological K-theory). Note that for these surfaces, all the cohomology groups are finite [3], and hence the reduced K-theory is profinite. I had earlier hoped (see [8]) that it might be torsion-free, and hence simply a sum of p-adic groups, but this is untrue; a counter-example is given in an appendix. As we shall see, the set of primes p which can give rise to p-adic summands for any given n is quite restricted.

We start from a formula of A. Adem [1]. This, in its local version, describes the p-adic K-theory (coefficients in the p-adic closure $\mathbf{C}_p$ of $\mathbf{Q}$) of $B\Gamma$ when Γ has finite virtual cohomological dimension (vcd). Under these conditions there is an extension

(1) $$1 \to \Gamma' \to \Gamma \to G \to 1$$

with Γ' torsion-free and G finite. In such a situation, Adem's formula reads:

(2) $$K_p^*(B\Gamma) \otimes \mathbf{C}_p = \bigoplus_{\substack{(\gamma) \\ \gamma \in \Gamma(p)}} \left(K_p^*(B(\Gamma' \cap C(\gamma)))\right)^{H_\gamma} \otimes \mathbf{C}_p$$

where $\Gamma(p)$ is the set of elements of p-power order in Γ, and for any $\gamma \in \Gamma$, (γ) denotes its conjugacy class, $C(\gamma)$ its centralizer, and H_γ the finite group $C(\gamma)/(C(\gamma) \cap \Gamma')$. In our case, Γ' is the subgroup K^n of maps fixing the n points ('pure mapping class group'), and G is the symmetric group Σ_n. K^n is the group which is denoted K_n in [3]; it is a quotient of the pure braid group on n strings. Because in this case $\tilde{K}^*(B\Gamma^n)$ is profinite, $\tilde{K}_p^*(B\Gamma^n)$ mod torsion is a sum of copies of $\hat{\mathbf{Z}}_p$. The number of copies is simply the rank of $\tilde{K}_p^*(B\Gamma^n) \otimes \mathbf{C}_p$ as determined by formula (2). Hence the computation of $K^*(B\Gamma^n)$ is, up to torsion, given by the formula.

Our programme is: first, to identify the terms in Adem's formula for each n, p in terms of invariant submodules of suitable groups $K^*(BK^r)$, and second, to use the character theory of the symmetric groups to compute the ranks of these submodules. The general results, stated in sections 2, 3, are rather cumbersome, but there are particular cases where they can be easily computed. We give some examples — in particular the cases $n = 3, \ldots, 10$ — in section 3.6.

2 Computing Adem's formula

2.1 The elements of finite order

We begin by finding $\Gamma^n(p)$, the set of elements of p-power order in the mapping class group. As a guide, we have the result of Harvey and McLachlan [7]:

Fact 1. If $\gamma \in \Gamma^n$ has order r, then r divides one of n, $n-1$, $n-2$.

The reason for this fact is quite geometrical; it follows from another basic result, which we shall use constantly:

Fact 2 (adapted from [3]). If γ is any element of Γ^n of order r, then there is a representative diffeomorphism f in the class γ which is a rotation through $\frac{2m\pi}{r}$ where $(m,r) = 1$.

From this, it is clear that all of the n distinguished points which are not on the axis of rotation of f can be divided into cycles of r points permuted by f. Either 0,1 or 2 of the n points are at the poles of the rotation; hence r divides one of n, $n-1$, $n-2$. As a further help in picturing the action of f we shall suppose (reasonably, it is clear as far as the p-part is concerned) that:

(i) $n - \alpha = qp^k$, where α is 0,1 or 2, and $p \nmid q$.
(ii) qp^k of the n distinguished points are evenly spaced around the equator, and the other α points, if any, are at the poles.
(iii) f which represents a class γ of order p^s, acts as a rotation through an angle $\frac{2m\pi}{p^s}$, where $(m,p) = 1$, and $s \leq k$.

Clearly, under these conditions, a fundamental region for f can be taken to be a sector bounded by two meridians, of angle $\frac{2\pi}{p^s}$, which contains qp^{k-s} of the distinguished points on the equator.

Note. If p is odd, then at most one value of α is possible (if $p = 3$, exactly one). However, we shall try, so as to be general, to consider the more complicated case of $p = 2$ simultaneously.

We must now determine the conjugacy classes of such elements.If α is 0 or 2, there is a rotation which exchanges the poles and preserves the set of n points. Its class is therefore in Γ^n. Hence the dihedral group $D_{qp^k} \subset \Gamma^n$ and if γ is the class of a rotation as described above, then γ^t is conjugate to γ^{-t} for all t.

Proposition 2.1 *These are the only conjugacy relations for p-power order elements in Γ^n. More precisely, let p, n be such that Γ^n has p-power elements. Then:*

(i) *Suppose either p or n is odd. Then there is a unique value of α for which $n - \alpha$ is divisible by p, say equals qp^k. Let f be a rotation through $\frac{2m\pi}{p^k}$ as above, and γ its class in Γ^n. Then, any γ' of p-power order is conjugate to a power of γ; and two distinct powers of γ are conjugate if and only they are inverse and α is 0 or 2.*

(ii) *If $p = 2$ and n is even, both the choices $\alpha = 0$ and $\alpha = 2$ are possible. One gives $k = 1$, and an element (say γ_1) of order 2; the other gives $k \geq 2$ and a generator (say γ_2) of order 2^k. Every 2-power element is conjugate to a power of one of these; two distinct powers of γ_2 are conjugate if and only if they are inverse; and γ_1 is not conjugate to any power of γ_2.*

Proof. The statements about the possible values of α and p are obvious. The fact that any element of p-power order is conjugate to a power of γ follows from the corollary of [7], p.508, with adjustments for $p = 2$. (That is, an element of order m is a power of an element of order $n - \alpha$, where m divides $n - \alpha$, and there is exactly one conjugacy class of cyclic subgroups of order $n - \alpha$.) However, the statement also contained in [7] that all *elements* of order $n - \alpha$ are conjugate is untrue.[1] In fact, if f is a rotation and h any diffeomorphism of S^2 then $h^{-1}fh$ is also a diffeomorphism with two fixed points. It is an isometry with respect to the metric $h^*(\mu)$, and the jacobians at the fixed points of f and its conjugate must be the same by a standard argument. Hence, if the conjugate is homotopic to f^t, then $t = \pm 1$.

Notice further that if h is a conjugacy of f with f^{-1}, then h must interchange the poles. But if $\alpha = 1$, a map which does this cannot be in Γ^n, since one pole is a puncture and the other isn't. Thus part (i) is established. All of part (ii) now follows by the same methods, except for the statement that γ_1 is conjugate to no power of γ_2. This is true since even their images in Σ_n are of different cycle types, and so not conjugate there. In fact, one fixes two elements, the other fixes none.

It follows from this proposition that we can count the number of conjugacy classes of elements γ of order p^s. The result is a little complicated owing to the variety of cases; it is the following.

Proposition 2.2

(i) *If p is odd, $n = qp^k + \alpha$, and $0 < s \leq k$ then there are $\frac{1}{2}(p - 1)p^{s-1}$ classes of order p^s in Γ^n if $\alpha = 0, 2$, and $(p - 1)p^{s-1}$ if $\alpha = 1$.*

(ii) *If $n = q.2^k + 1$ and $0 < s \leq k$, then there are 2^{s-1} classes of order 2^s in Γ^n.*

(iii) *If $n = q.2^k + \alpha$ where $k > 1$ and $\alpha = 0, 2$, then there are two classes of order 2 and 2^{s-2} classes of order 2^s for $1 < s \leq k$.*

[1] This is confirmed by Bill Harvey.

Proof. This follows immediately from Proposition 2.1, and well-known facts about elements of p-power order.

2.2 *The centralizers*

We now have to find the centralizer $C(\gamma)$ in Γ^n of a rotation f in the isotopy class γ of order p^s. Suppose, then, that $h : S^2 \to S^2$ commutes with f, i.e. $f \circ h = h \circ f$. Let G be the cyclic group generated by f; then h defines a diffeomorphism $\bar{h}$ of the quotient S^2/G. We can regard S^2/G as a sphere with distinguished points at the two poles of the sphere.These may be of puncture type (if the pole is already a distinguished point), or elliptic of order p^s (if not). To these we must add the qp^{k-s} distinguished points of the fundamental region; the result is a sphere $\bar{S}$ with $qp^{k-s}+2$ such points. (For the basics of this theory — and in particular for a justification of the implied claim that an element in $C(\gamma)$ is represented by a genuinely commuting map — see Harvey and McLachlan [7].)

It is natural to look at the mapping class group of this surface. This is not exactly the centralizer, as we shall see; but for the moment we concentrate our attention on the subgroup $C(\gamma) \cap K^n$.

Proposition 2.3 *The intersection $C(\gamma) \cap K^n$ can be naturally identified with the group $K^{qp^{k-s}+2}$ of maps of $\bar{S}$ fixing the above distinguished points.*

Proof. First, given a map h which commutes with f and fixes all n points, the factored map $\bar{h}$ must fix all those distinguished points of $\bar{S}$ which come from S^2. In particular, it fixes all those points which are not at the poles (qp^{k-s} of them). If $\alpha = 2$, it also fixes both the poles; if $\alpha = 1$, it fixes one and hence necessarily the other.

If $\alpha = 0$, we must examine the possibility of $\bar{h}$ exchanging the poles of $\bar{S}$, which are of elliptic type. If this happens, choose a sector T of S^2 as a fundamental region containing points $(1, 2, \ldots, qp^{k-s})$. A map $\bar{h}$ which exchanges the poles can be thought of as exchanging right and left sides of this region, moving the boundary, say, anti-clockwise through a half turn with points on the right edge moving down and those on the left moving up. It is easily seen that such a map does not lift to S^2, since the right side of one fundamental region is the left side of the next.

Note moreover that:

(i) any $\bar{h}$ which fixes these points lifts to an h which commutes with f (and so defines an element of $C(\gamma)$); and although several such lifts exist, we can choose h uniquely so that it fixes all n points on S^2. This is standard covering theory.
(ii) similarly, any isotopy of $\bar{h}$'s, relative to the distinguished points, lifts to a similar isotopy of h's.

To find the general element of $C(\gamma)$ we need first to ask which maps $\bar{h}$ of the $qp^{k-s}+2$-punctured sphere lift to maps h commuting with f. The answer is given by:

Proposition 2.4

(i) Suppose $p^s \neq 2$ or $\alpha = 1$. Then a map $\bar{h} : \bar{S} \to \bar{S}$ lifts to an $h : S^2 \to S^2$ commuting with f — and hence defines an element of $C(\gamma)$ — if and only if $\bar{h}$ fixes the poles of the quotient sphere $\bar{S}$.

(ii) If $p^s = 2$ and $\alpha \neq 1$, then $\bar{h}$ defines an element of $C(\gamma)$ if and only if it either fixes or exchanges the poles of $\bar{S}$; and maps of both kinds exist.

Proof. First, for $\bar{h}$ to lift to a diffeomorphism of S^2, it must preserve the orbit structure (of the orbit space $\bar{S}$). In this case, this means that it either preserves the poles or interchanges them — the latter only being allowed if the poles are indistinguishable, i.e. $\alpha \neq 1$. Now, a map which interchanges the poles does lift to an h with the same property; but as in the proof of Proposition 2.1., such an h conjugates f to f^{-1}, and so does not commute with f unless $p^s = 2$. (Notice that h descends to $\bar{S}$ if $h \circ f = f^t \circ h$ for some t.) On the other hand, if $\bar{h}$ fixes the poles, we can again construct a lift h by a covering argument which commutes with rotation through $\frac{2\pi}{p^s}$, i.e. with f. This proves part (i).

For part (ii), suppose $p^s = 2$ and $\alpha \neq 1$. Then we can show as above that a map $\bar{h}$ which exchanges the poles lifts to an h which commutes with γ. It remains to show that such maps exist. This is easy; suppose $f \in \gamma$ is a rotation through an angle π about the polar axis, and let h be a rotation in the dihedral group which permutes the equatorial distinguished points and exchanges the poles. Then h, f commute so the class of h is in $C(\gamma)$.

The lift h which we have constructed is not unique, since it is easy to see that $h \circ f^t$ is another lift, for $t = 1, \ldots, p^s - 1$. On the other hand, an isotopy of $\bar{h}$'s does lift to an isotopy of h's. These two remarks make it possible to pass to the next stage, the precise identification of $C(\gamma)$. With this, we can simultaneously describe the group H_γ used in Adem's formula. To simplify notation, let Λ be the subgroup of $\Gamma^{qp^{k-s}+2}$ which fixes two given points (in our notation, the north and south poles). We can think of it as the inverse image, in the exact sequence analogous to (1), of $\Sigma_{qp^{k-s}} \subset \Sigma_{qp^{k-s}+2}$. Also, if $p^s = 2$, let $\tilde{\Lambda}$ be the subgroup which fixes the poles as a *subset.* This is the inverse image of $\Sigma_{q.2^{k-1}} \times \Sigma_2$.

Proposition 2.5

(i) Suppose either $p^s \neq 2$ or $\alpha = 1$. Then there is an exact sequence:

(3) $$1 \to \mathbf{Z}/p^s \to C(\gamma) \to \Lambda \to 1$$

This, when we quotient by $C(\gamma) \cap K^n$ gives a sequence of finite groups:

(4) $$1 \to \mathbf{Z}/p^s \to H_\gamma \to \Sigma_{qp^{k-s}} \to 1$$

(ii) If $p^s = 2$ and $\alpha \neq 1$, the same sequences hold; but in (3), Λ must be replaced by $\tilde{\Lambda}$, and in (4), $\Sigma_{qp^{k-s}}$ must be replaced by $\Sigma_{q.2^{k-1}} \times \Sigma_2$.

Proof. In fact, the map from $C(\gamma)$ to Λ is that which sends the isotopy class of h (in the preceding discussion) to that of $\bar{h}$. We have seen that this is an epimorphism, but that to every $\bar{h}$, there corresponds a $\mathbf{Z}/p^s$ of h's. This establishes sequence (3); and (4) follows immediately. The proof in the exceptional case is similar.

It is not too difficult to identify H_γ as the subgroup of Σ_{qp^k} consisting of permutation matrices of form $A \otimes B$, where A is a cyclic permutation in $\mathbf{Z}/p^s \subset \Sigma_{p^s}$, and B is arbitrary in $\Sigma_{qp^{k-s}}$.

2.3 The main result

We are now almost in a position to put everything together; that is, to translate Adem's formula (2) into our specific situation, using the descriptions we have obtained of the terms in it. One final point remains to be noted; that the subgroup $\mathbf{Z}/p^s \subset H_\gamma$ acts trivially on the K-theory of the classifying space $BK^{qp^{k-s}+2}$. In fact, the cyclic subgroup is generated by γ, and by definition, its elements commute with those of $K^{qp^{k-s}+2} \subset C(\gamma)$. Hence, the action of H_γ on the classifying space reduces to that of the quotient, i.e. of $\Sigma_{qp^{k-s}}$ or $\Sigma_{q.2^{k-1}} \times \Sigma_2$, at least as far as the K-theory is concerned. With this in mind, we can state:

Theorem 1 *Let Γ^n, p be as before. Then*

$$K_p^*(B\Gamma^n) \otimes \mathbf{C}_p = K_p^*(BK^n)^{\Sigma_n} \otimes \mathbf{C}_p \quad \oplus \ X$$

where

(i) If p is odd and $n = qp^k$ or $qp^k + 2$,

$$X = \bigoplus_{s=1}^{k} \frac{1}{2}(p-1)p^{s-1}.(K_p^*(BK^{qp^{k-s}+2})^{\Sigma_{qp^{k-s}}}) \otimes \mathbf{C}_p$$

(ii) If p is arbitrary and $n = qp^k + 1$, X is as in (i) but with each factor doubled

(iii) If $p = 2$ and $n = q.2^k, k > 1$,

$$X = \Big\{ \Big(\bigoplus_{s=2}^{k} 2^{s-2}.(K_2^*(BK^{q.2^{k-s}+2})^{\Sigma_{q.2^{k-s}}}) \Big)$$

$$\oplus (K_2^*(BK^{q.2^{k-1}+2})^{(\Sigma_{q.2^{k-1}} \times \Sigma_2)}) \oplus (K_2^*(BK^{q.2^{k-1}+1})^{(\Sigma_{q.2^{k-1}-1} \times \Sigma_2)}) \Big\} \otimes \mathbf{C}_2$$

(iv) If $p = 2$ and $n = q.2^k + 2, k > 1$,

$$X = \Big\{ \Big(\bigoplus_{s=2}^{k} 2^{s-2}.(K_2^*(BK^{q.2^{k-s}+2})^{\Sigma_{q.2^{k-s}}}) \Big)$$

$$\oplus (K_2^*(BK^{q.2^{k-1}+2})^{(\Sigma_{q.2^{k-1}} \times \Sigma_2)}) \oplus (K_2^*(BK^{q.2^{k-1}+3})^{(\Sigma_{q.2^{k-1}+1} \times \Sigma_2)}) \Big\} \otimes \mathbf{C}_2$$

(v) X is zero otherwise.

(Note the small but significant difference between cases (iii) and (iv), arising from the relative placing of the $4k$ part and the $4k + 2$ part.)

The proof of this result is a straightforward application of the results we have derived in Propositions 2.1 to 2.5 to Adem's formula (2). The interesting thing about it is that all of the modules of invariants which enter in the terms we have called 'X' are of the same type: essentially, the invariants of Σ_{n-2} or its product with Σ_2 in $K^*(BK^n)\otimes \mathbf{C}_p$ for $n = 3, 4, \ldots$. It is the ranks of these modules which we shall set out to find in section 3.

Note. The above terms account for the whole reduced K_p-theory of $B\Gamma^n$, which is the interesting part. The remaining term $K_p^*(BK^n)^{\Sigma_n}$ is just the unit. In fact, its rank over $\mathbf{Z}_p$ is independent of p, and so equals the rank of $K^*(BK^n;\mathbf{Q})^{\Sigma_n}$. Using the spectral sequence from $H^i(B\Sigma_n; K^j(BK^n;\mathbf{Q}))$ to $K^*(B\Gamma^n;\mathbf{Q})$, and the fact that the K-theory of $B\Gamma^n$ is profinite we find that this rank must be 1. It is then easy to identify the summand with the unit.

3 The invariants

3.1 Description of the classifying spaces

The main result of this section is a much simpler one than Theorem 1, and can be immediately stated, now that we know what we are looking for. It is as follows:

Theorem 2 *Let r_n, s_n be the ranks of $(K^0(BK^n))^{\Sigma_{n-2}}$, $(K^1(BK^n))^{\Sigma_{n-2}}$ respectively, where $n \geq 3$. Then $r_n + s_n = n - 2$; and $r_n - s_n$ is 0 (n even) and 1 (n odd). In other words:*

(i) If $n = 2m$, then $r_n = s_n = m - 1$;
(ii) If $n = 2m + 1$, then $r_n = m$, $s_n = m - 1$.
Correspondingly, let $\tilde{r}_n$, $\tilde{s}_n$ be the ranks of the invariants of $\Sigma_{n-2} \times \Sigma_2$. Then

$$\tilde{r}_n - \tilde{s}_n = \begin{cases} 1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{r}_n + \tilde{s}_n = \begin{cases} 2k & \text{if } n = 4k,\ 4k+1, \text{ or } 4k+2 \\ 2k+1 & \text{if } n = 4k+3 \end{cases}$$

In consequence we have:
(iii) If $n = 4k$, $4k + 1$, or $4k + 2$, then $\tilde{r}_n = k$, $\tilde{s}_n = k$
(iv) If $n = 4k + 3$, then $\tilde{r}_n = k + 1$, $\tilde{s}_n = k$.

The method of proof of Theorem 2 is quite standard; we find the fixed point sets of the elements of Σ_{n-2} and $\Sigma_{n-2} \times \Sigma_2$ on the space BK^n and deduce the Lefschetz numbers. From these the characters of the graded representation space $(K^*(BK^n))$ follow, and they in turn give the rank of the invariant subspace (the number of copies of the trivial character). The one problem is that the Lefschetz formula only gives us the difference of the ranks (and so $r_n - s_n$, etc.); to find the sum we have to twist the elements of the symmetric group.

We shall use as a starting point the description of the classifying space BK^n given in [3]. For any topological space X, (we shall consider only $X = S^2$ and $X = \mathbf{R}^2$) let $\tilde{F}_n(X)$ denote the configuration space of sequences of n distinct points of X, $(x_1, \ldots, x_n)$, topologized as a subspace of the product X^n. The group $SO(3)$ acts freely on $\tilde{F}_n(S^2)$ in particular, and the quotient space $\tilde{F}_n(S^2)/SO(3)$ is a model for BK^n. The action of Σ_n on this space is the obvious one which comes from permuting the n points.

Because we are mainly interested in subgroups of Σ_{n-2}, it is useful to simplify this particular model by fixing one of the points. In fact, let p_0 be a fixed basepoint on S^2, e.g. the north pole. $SO(3)$ acts transitively on S^2, so that any point of $\tilde{F}_n(S^2)$ is $SO(3)$-equivalent to one of form $(x_1, \ldots, x_{n-1}, p_0)$. Furthermore, the $SO(3)$-stabilizer of such a point is clearly $SO(2)$. Identifying $S^2 \setminus \{p_0\}$ with $\mathbf{R}^2$ by stereographic projection, we obtain:

Proposition 3.1 *$\tilde{F}_n(S^2)/SO(3)$ is homeomorphic to the quotient $\tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$, where $SO(2)$ acts in the obvious way, and the subgroup $\Sigma_{n-1} \subset \Sigma_n$ acts by permuting the $(n-1)$ points.*

Note, then, that while we have sacrificed some symmetry in obtaining this simpler model, we still have the symmetry which is usually needed. This is going to be useful. (As the referee has pointed out, if we leave the action of Σ_n out of the picture, $\tilde{F}_n(S^2)/SO(3)$ has a very simple description. In fact, $\tilde{F}_n(S^2)$ is homotopy equivalent to the product of $SO(3)$ with $\tilde{F}_{n-3}(\mathbf{R}^2 \setminus \{0, 1\})$, and from this we can deduce a homotopy equivalence from the quotient $BK^n = \tilde{F}_n(S^2)/SO(3)$ to $\tilde{F}_{n-3}(\mathbf{R}^2 \setminus \{0, 1\})$. The cohomology of configuration spaces for $\mathbf{R}^2 \setminus \{\text{points}\}$ has been studied, and we can derive the K-theory of BK^n from this.)

Note also that the space $\tilde{F}_{n-1}(\mathbf{R}^2)$ is well-known; it is the classifying space for the 'pure braid group' P_{n-1} [6]. Of course, this relationship is not accidental - see [3] for more details.

3.2 Fixed point sets for Σ_{n-2}

We next look at those elements of $\Sigma_{n-2} \subset \Sigma_{n-1}$ acting on $\tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$ which have non-empty fixed point sets. Let σ accordingly be a permutation in Σ_{n-2}, and denote by $[x]$ or $[x_1, \ldots, x_{n-1}]$ the $SO(2)$-orbit of (x) or $(x_1, \ldots, x_{n-1}) \in \tilde{F}_{n-1}(\mathbf{R}^2)$. Then the action of Σ_{n-2} is given by:

$$\sigma.[x_1, \ldots, x_{n-1}] = [x_{\sigma(1)}, \ldots, x_{\sigma(n-2)}, x_{n-1}]$$

If σ fixes such an orbit, it must be because there is a rotation ρ in $SO(2)$ such that $\rho(x_i) = x_{\sigma(i)}$ for $1 \le i \le n-2$; and $\rho(x_{n-1}) = x_{n-1}$. Hence,

(a) $x_{n-1} = 0$, or σ is the identity and so is ρ;
(b) ρ is a rotation of finite order k dividing $n-2$;

(c) σ is a product of $\frac{n-2}{k}$ disjoint k-cycles.

These conditions of course strictly limit the number of elements of Σ_{n-2} which can have non-trivial fixed point sets. Our next task is to find what these sets look like. For this, we need some notation. Let $E(q,k) \subset \tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$ be the fixed point set of a product of q distinct k-cycles as above, where $qk = n-2$. We call the product of cycles σ; since it is unimportant which σ we choose in its conjugacy class, we set:

$$\sigma = (1, 2, \ldots, k) \ldots ((q-1)k+1, \ldots, qk)$$

and $E(q,k)$ is the set $\{[x] = [x_1, \ldots, x_{qk}] : \sigma(x) = \rho(x)\}$ where ρ is some rotation of order exactly k. Now there are $\phi(k)$ such rotations (Euler's function); and a given $[x]$ can only belong to one ρ. Hence, $E(q,k)$ splits into $\phi(k)$ disjoint pieces — we shall see they are components — belonging to the different ρ's. Call one such piece $E(q,k)_0$. We shall also for convenience make the obvious identifications of $\mathbf{R}^2$ with $\mathbf{C}$ and $SO(2)$ with $U(1)$, so that ρ is multiplication by $e^{\frac{2m\pi i}{k}}$, where $(m,k) = 1$.

Lemma. *A point $[x]$ of $E(q,k)_0$ is completely determined by the sequence*

$$(\hat{x}) = (x_1, x_{k+1}, \ldots, x_{(q-1)k+1})$$

For $k > 1$ (the case where σ is non-trivial), we can make this determination unique by further requiring x_1 to lie on the positive real axis $\mathbf{R}^+$.

Proof. The key is in the formula (derived from $\sigma(x) = \rho(x)$)

$$x_{rk+s} = \rho^{s-1}(x_{rk+1}) = e^{\frac{2(s-1)m\pi i}{k}}.x_{rk+1}$$

valid when $0 \le r < q$, $1 \le s \le k$. From this, it is clear that $(\hat{x})$ determines (x). However, it is more than we need to determine $[x]$, which is a $U(1)$-orbit of (x)'s (note that if one of them satisfies $\sigma(x) = \rho(x)$ they all do). None of them can be 0, since x_{n-1} is by remark (a) above. We can accordingly choose x_1 arbitrarily to be on the positive real axis.

(Arguments of the above type, which one could describe as 'gauge fixing' by analogy with the physical procedure, will be frequently used in future, and I shall not go through a detailed justification in each case.) We can now — for $k > 1$ — determine $E(q,k)_0$ by induction on q. If $q = 1$, then $(\hat{x})$ consists of only x_1, so $E(1,k)_0 = \mathbf{R}^+$ is contractible. Given $(\hat{x})$ determining (x) and so $[x]$ in $E(q,k)_0$, we want to find out how many sequences of $q+1$ points $(\hat{y})$ in $E(q+1,k)_0$ begin with the q points of $(\hat{x})$. The only choice we can make is that of the last coordinate y_{qk+1}; but this is restricted by the condition that it must not coincide with any of the qk points in (x), or of course with zero. Hence, y_{qk+1} lies in $\mathbf{C}$ with $(qk+1)$ points removed, which has the homotopy type of a wedge of $(qk+1)$ circles. The consequence of this is:

Proposition 3.2 *For any $q \geq 0$, and for $k = 2, 3, \ldots$, there is a fibration up to homotopy*

$$\bigvee^{qk+1} S^1 \to E(q+1,k)_0 \to E(q,k)_0 \tag{5}$$

This has a cross section.

Proof. That the sequence (5) defines a fibration is standard; see [6] for the analogous result for 'braid spaces'. We only need to establish the existence of a cross section. Let, then, $(\hat{x}) = (x_1, \ldots, x_{(q-1)k+1})$ define a point of $E(q,k)_0$ as before. Set

$$y_{qk+1} = 1 + \sup\{|x_{rk+1}| : r = 1, \ldots, q-1\} \in \mathbf{R}$$

This is clearly continuous, and defines a point not equal to any of the points in (x).

Corollary. *For $k = 1, 2, 3, \ldots$,$E(q,k)$ has exactly $\phi(k)$ components. Each is homotopy equivalent to a finite CW-complex, with Poincaré polynomial equal to*

$$(1 + (k+1)t)(\ldots)(1 + ((q-1)k+1)t) \tag{6}$$

Consequently the Euler characteristic of the whole set $E(q,k)$ is $\phi(k).(-k)^{q-1}$ $(q-1)!$ The important and non-trivial point here is that I have included the case $k = 1$ in the general scheme. To check that this is valid, note that $E(n-2,1)$ is the whole of $\tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$, i.e. of BK^n. We can use slightly more complicated geometric arguments than those above to deduce that the formula (6) gives the Poincaré polynomial of BK^n when we set $k = 1$; or we can borrow the result from [6], for example (where it is found using braid groups). I shall do the latter, leaving the former as an interesting exercise.

3.3 The twisted version

As was mentioned above, the results of the previous section need to be supplemented by 'twisted' fixed point sets which will allow the calculation of the sum of even and odd invariants. The twisting we have in mind is simple. The group $O(2)$ acts on $\tilde{F}_{n-1}(\mathbf{R}^2)$, with $SO(2)$ as normal subgroup. Hence the quotient group, which is cyclic order two, acts on the orbit space $\tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$ by an involution which we shall call τ. The action of τ is easy to describe; given $[x]$ as above, represented by $(x) = (x_1, \ldots, x_{n-1})$, $\tau[x]$ is obtained by reflecting the points of (x) in an arbitrary line l through the origin. Different choices of the line l give representatives (x) which differ by a rotation, and hence define the same $[x]$. Our aim is now, given any σ in Σ_{n-2}, to find the fixed point set of the composite $\sigma\tau$ in $\tilde{F}_{n-1}(\mathbf{R}^2)/SO(2)$. Alternatively, using $\tau^2 = 1$, we are looking for the set of points which satisfy

$$\sigma[x] = \tau[x] \tag{7}$$

Lemma. *The set of points which satisfy (7) is non-empty if and only if σ is a product of disjoint 2-cycles.*

Proof. Let $[x]$ be any point in the set, and let $(x) = (x_1, \ldots, x_{n-1})$ be a representative. From the definition of τ, there is a line l such that $x_{\sigma(r)}$ is obtained from x_r by reflecting in l, for $r = 1, \ldots, n-1$. It follows immediately that σ is of order two, and so a product of disjoint 2-cycles.

If σ is the identity, then all the points $(x_1, \ldots, x_{n-1})$ are on the line l. We can rotate so that l is the real axis $\mathbf{R}$, but there is still an ambiguity in the choice of representative, since we could multiply all $(n-1)$ points by -1. A very simple analysis similar to the ones we have already done shows that the ordered sequences of $(n-1)$ distinct points in $\mathbf{R}$ form a space of $(n-1)!$ contractible components. Taking account of the ambiguity, we find $\frac{1}{2}(n-1)!$ contractible components for the set of solutions of (7) in this case. (We must suppose $n > 2$, but this is always true in applications.)

Next, let σ consist of $k > 0$ disjoint 2-cycles; for definiteness say

$$\sigma = (1,2)\ldots(2k-1, 2k)$$

Then, rotating so that l is the x-axis again, we have:

(i) x_{2r} is the reflection of x_{2r-1} in the real axis (its complex conjugate, if you like) for $r = 1, \ldots, k$;
(ii) x_r is in $\mathbf{R}$ for $r > 2k$.

We can now remove the remaining ambiguity by requiring x_1 to be in the upper half plane. Let $E(k)$ denote the set of solutions of (7) when σ is composed of k disjoint 2-cycles. Then we have the following description of the $E(k)$'s:

Proposition 3.3

(i) $E(0)$ consists of $\frac{1}{2}(n-1)!$ contractible components.
(ii) If $k > 0$, then $E(k)$ consists of $2^{k-1}.(n-1-2k)!$ components. Each of these components has the homotopy type of $\tilde{F}_k(\mathbf{R}^2)$.

Proof. We have already proved (i). For (ii), note that we have 2^{k-1} choices of how to dispose $x_3, x_5, \ldots, x_{2k-1}$ between upper and lower half planes; as usual, these give us a division of $E(k)$ into 2^{k-1} homeomorphic subsets. If we concentrate on the subset where $x_1, x_3, \ldots, x_{2k-1}$ are all in the upper half plane, we see that any point of $E(k)$ is determined uniquely by these k distinct points and by $x_{2k+1}, \ldots, x_{n-1}$ on $\mathbf{R}$. The first k points define an element of $\tilde{F}_k(\mathbf{R}^2)$ (since the half plane is homeomorphic to $\mathbf{R}^2$), and the last $(n-1-2k)$ give us a space of $(n-1-2k)!$ contractible components.

There is now an important simplification which can be made, from the point of view of the Lefschetz number. As before, we can construct the components $\tilde{F}_k(\mathbf{R}^2)$ inductively as fibrations with wedges of circles as fibres; indeed this is done in [6]. However, all we need to know is

Proposition 3.4 *$\tilde{F}_1(\mathbf{R}^2)$ is simply $\mathbf{R}^2$, and so contractible; while for $k > 1$, $\tilde{F}_k(\mathbf{R}^2)$ has a factor $(1+t)$ in its Poincaré polynomial. In particular, its Euler characteristic is* 0.

Proof. See [6].

3.4 The dihedral invariants — untwisted

In this section and the next, we follow through the preceding analysis for the group $\Sigma_{n-2} \times \Sigma_2$. Here, it is no longer possible in general to simplify by reducing from S^2 to $\mathbf{R}^2$, since we may have no fixed points. The problem is to determine the invariant subsets of elements of $\Sigma_{n-2} \times \Sigma_2$ on BK^{n-2}; and since we have done this for elements of type $(\sigma, 1)$, it remains to do it for elements of type (σ, ϵ), where ϵ exchanges the last two points x_{n-1}, x_n. Given any $\sigma \in \Sigma_{n-2}$, then, we must determine all ($SO(3)$-equivalence classes of) sequences $(x_1, \ldots, x_n)$ such that for some rotation ρ:

$$\rho(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n-2)}, x_n, x_{n-1})$$

Now, any rotation has an axis and an angle; and its angle determines its order. Since ρ exchanges two points, its angle must be π and its order 2, so σ also has order 2. In other words, it must again be a product of disjoint two-cycles. Also, ρ, which is not the identity, fixes exactly two points and so σ cannot fix more than two of the x_i's.

We now have to separate the cases of n even and odd. If n is odd, say $n-2 = 2k+1$, then σ must be a product of k two-cycles, say $\sigma = (1,2)\ldots(2k-1, 2k)$. Hence, ρ fixes x_{n-2}. Thinking of S^2 as the Riemann sphere, choose (x) in its $SO(3)$-orbit so that $x_{n-2} = \infty$, and x_{n-1} is on the positive real axis. (ρ is accordingly multiplication by (-1) in the plane.) We then, as usual, have to choose $x_1, x_3, \ldots, x_{2k-1}$ in $\mathbf{R}^2 - \{0\}$, at each stage missing the points already chosen. This means that x_1 is in the complement of 3 points ($\simeq$ wedge of three spheres), x_3 in the complement of 5, and so on. By an argument similar to that of proposition 3.2., we obtain a fixed point set with only one component, of Poincaré polynomial

$$P(t) = (1-3t)(1-5t)\ldots(1-(2k+1)\dot{t})$$

whose Euler characteristic is accordingly $\chi = (-2)^k.k!$

If n is even, we have two possibilities. Either two of the points (x_i) are the poles of ρ (and so fixed by σ), or none are. If $n-2 = 2k$, the first case corresponds to $\sigma = (1,2)\ldots(2k-3, 2k-2)$, the second to $\sigma = (1,2)\ldots(2k-1, 2k)$. In the first case, we can take $x_{2k-1} = \infty$ and $x_{2k} = 0$, with ρ acting on the plane as before; and we can then fix the last two points as before on the real axis. By a similar argument to that above, we have $\chi = (-2)^{k-1}.(k-1)!$

In the second case, we need more care (essentially because we have no obvious way of choosing between the poles of the rotation). We proceed differently;

let $E(k)$ be the fixed point set of $(1,2)\dots(2k-1,2k)$ in $\tilde{F}_{2k}(S^2)/SO(3)$, and consider the maps $p_k : E(k+1) \to E(k)$ (fibrations as before) which arise from leaving out the last pair of points. If $k > 1$, $SO(3)$ acts freely on $\tilde{F}_{2k}(S^2)$ and we can find the fibre of p_k simply by taking the fibre of the corresponding map of fixed point sets *before* quotienting by $SO(3)$, i.e. the set of fixed points in $\tilde{F}_{2k+2}(S^2)$ which maps into the point sequence $(x_1, \dots, x_{2k})$. Such point sequences are determined by x_{2k+1}, which misses $2k+2$ points (including the two poles of the rotation), and so the fibre we want is simply a sphere with $(2k+2)$ deleted points $\simeq$ wedge of $(2k+1)$ circles.

This argument does not apply to give us $E(2)$. Here we choose the poles of the rotation to be $0, \infty$, and fix x_1 on the positive real axis; then $x_2 = -x_1$. We are left with a choice of x_3 in $\mathbf{R}^2$ with three deleted points, but we still have one further symmetry possible arising from interchanging the poles. This acts on $\mathbf{R}^2 \setminus \{0\}$ as inversion. Identify the punctured plane with the wedge of three circles; it can be seen that the inversion is homotopic to a rotation on two of them, and to reflection in the axis in the third. Hence the quotient of the punctured plane by an inversion can, up to homotopy equivalence, be identified with the connected union of two circles and an arc.

The end result, then, is that in this case the Poincaré polynomial of $E(k+1)$ (which is the one we are looking for, the invariants of (σ, ϵ) on BK^n) is

$$(1-2t)(1-5t)(1-7t)\dots(1-(2k+1)t)$$

and the Euler characteristic is $-(-2)^{k-1}.k!$

To sum up the results of this section, we have:

Proposition 3.5

(i) *If n is odd, $n = 2k+3$, the fixed point set of (σ, ϵ) on BK^n is empty unless σ is a product of k disjoint two-cycles. In this case the fixed point set is connected and its Euler characteristic is $(-2)^k.k!$*

(ii) *If n is even, $n = 2k+2$, the fixed point set is empty unless σ is a product of $(k-1)$ or k disjoint two-cycles. In both cases the fixed point set is connected; its Euler characteristic is $(-2)^{k-1}.(k-1)!$ in the first case and $-(-2)^{k-1}.k!$ in the second.*

3.5 The dihedral invariants - twisted

The last fixed point sets which we need to find are the twisted sets for pairs (σ, ϵ), where σ and ϵ have the same meaning as in the previous section. The problem, then, reduces to finding points (up to $SO(3)$ action) $(x_1, \dots, x_n)$ such that for some τ of determinant -1 in $O(3)$,

$$\tau(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n-2)}, x_n, x_{n-1})$$

A first observation is that any such τ is composed of reflection in some plane T and rotation through some angle (perhaps zero) about the axis orthogonal to T; and since τ has finite order, so does the rotation.

Lemma. *If the angle of rotation of τ is not* 0*, then the axis must have x_{n-1}, x_n as its poles.*

Proof. In fact, τ fixes $x_{n-1} + x_n$. If the angle of rotation is not 0, then 1 is not an eigenvalue of τ, so $x_n = -x_{n-1}$. Hence x_{n-1}, x_n are eigenvectors for the eigenvalue (-1), and so poles of the rotation. (We include the case of the map $-Id_{S^2}$, whose angle is π, and whose poles are arbitrary.)

The lemma implies that we need to consider the cases of zero and nonzero angle separately. The first we have, in the main, considered already. In fact, τ and hence σ has order two, and we can find the fixed points of (σ, ϵ) using the results of section 3.3. These imply that if σ is the identity, then there are $(n-1)!$ contractible components, and that if the number of two-cycles in σ is > 0, then $\chi = 0$ provided that *one* of the n points is fixed under σ. (This was always true in section 3.3.) However, if $n = 2k+2$ is even and σ consists of k two-cycles, we can't do any of the 'gauge fixing' we used earlier; τ, the reflection in T, exchanges the points in pairs. For example, in the case of $\sigma = (1,2)\ldots,(2k-1,2k)$, $\tau(x_{2i-1}) = x_{2i}$ and conversely (for $i = 1, \ldots, k$).

We proceed as follows. Take T to be the equator and x_1 in the upper hemisphere. Then each x_{2i-1} determines x_{2i}.There are 2^k ways of distributing $x_3, x_5, \ldots, x_{2k+1}$ between the hemispheres; they give rise as usual to 2^k homeomorphic components of the fixed point set. The component in which all are in the upper hemisphere is now familiar; it consists of sequences $(x_1, \ldots, x_{2k+1})$ in the upper hemisphere, quotiented by the action of $SO(2)$. This is $\tilde{F}_{k+1}(\mathbf{R}^2)/SO(2) = BK^{k+2}$. Hence its Euler characteristic is $\chi = -(-2)^k.(k-1)!$

We now consider the case where the angle of rotation of τ is not zero. Suppose it is $\frac{2k\pi}{m}$, where $(k,m) = 1$. Then τ has order $2m$ if m is odd, and m if m is even; but in either case, the restriction of τ to the equatorial plane has order m.

We therefore have the following two possibilities for m and σ:

(i) m is even and σ is a product of disjoint m-cycles
(ii) m is odd, and σ contains either m-cycles (for points on the equator) or $2m$-cycles (for points not on the equator) or a mixture.

The last possibility seems unnecesarily complicated. Fortunately, we can dispose of it as follows:

Lemma. *In case (ii) above, if σ contains both m-cycles and $2m$-cycles, then the Euler characteristic is zero:* $\chi((BK^n)^{(\sigma,\epsilon)\tau}) = 0$.

Proof. Suppose we are looking for the fixed points $[x]$ of a permutation such as

$$(1,2,\ldots,m)(m+1,m+2,\ldots,3m)(\ldots)(n-1,n)$$

Fix x_{n-1}, x_n at the poles; by the preceding remarks, $x_1, \ldots, x_m$ must be on the equator, and if we fix x_1, there are no further degrees of freedom. We then find that x_{m+1} can be anywhere in one of the hemispheres, except at the poles. Hence, it can be chosen in a punctured disk, which as usual leads to Euler characteristic zero.

There are therefore (to rephrase our previous statement in the light of the lemma) three possibilities for σ. It is in any case a product of q m-cycles, where $qm = n - 2$; but

(i) If m is odd, the angle of rotation is $\frac{2k\pi}{m}$, and all the points x_i are on the equator;
(ii) If $m = 4s$, the angle of rotation is $\frac{2k\pi}{m}$, and the points x_i can be anywhere (except at the poles);
(iii) If $m = 4s + 2$, then *either* the angle is $\frac{2k\pi}{m}$, and the points x_i are as in (ii), *or* the angle is $\frac{2k\pi}{(2s+1)}$, and the points x_i are neither at the equator nor at the poles.

In each case, we have $\phi(m)$ distinct choices for k.

Note. The case $m = 2$ is special, although it falls nominally under (iii). Here, the first possibility is of an angle π, i.e. a non-trivial rotation; but the second has angle zero. This therefore is contained in our earlier discussion, where we assigned to it the Euler characteristic $\chi = (-2)^{k-1}.(k - 1)!$ We shall see that this agrees with the general formula.

We consider these three possibilities in relation to the Euler characteristic of the fixed point set. In case (i), we are in a familiar situation. We can fix x_1 to be, say, at $(1,0,0)$. Then we have arbitrary choices for $x_{m+1}, \ldots, x_{(q-1)m+1}$, all on the equator. We find that the fixed point set for a given k has $m^{q-1}(q - 1)!$ contractible components, and the total Euler characteristic is $\phi(m).m^{q-1}(q - 1)!$ This includes $m = 1$ which we have already considered as a special case.

In case (ii), we can fix x_1 so that its x-coordinate is positive and its y-coordinate is 0. Then x_{rm+1} for $1 < r < q$ is in a sphere with $(rm + 2)$ points deleted. By the usual argument we find that each component of the fixed point set has Poincaré polynomial equal to

$$(1 - (m + 1)t)(\ldots)(1 - ((q - 1)m + 1)t)$$

and the total Euler characteristic is $\phi(m).(-m)^{q-1}(q - 1)!$

The first of the two situations in case (iii) is just the same as case (ii) and gives us the same answer, $\phi(m).(-m)^{q-1}(q - 1)!$ for the Euler characteristic. In the second, each orbit (for example $(x_1, \ldots, x_m)$) consists half of points in the upper hemisphere and half of points in the lower — $(2s + 1)$ of each. We fix x_1 as in case (ii); then we must choose x_{m+1} in a disc with $(2s + 2)$ points removed, and generally x_{rm+1} in a disc with $r(2s + 1) + 1$ points removed. Furthermore, for each of the q orbits we have two possibilities for its hemisphere. So the fixed point set consists of $\phi(m).2^q$ components, each with Poincaré polynomial

$$(1 - (2s + 1)t)\ldots(1 - (q - 1)(2s + 1)t)$$

and its Euler characteristic is $\phi(m).(2^q)(-(2s + 1))^{q-1}(q - 1)!$ which is the same as $2\phi(m).(-m)^{q-1}(q - 1)!$ Here, we note that the 'special' case of $m = 2$ and zero angle gives us the same result, as calculated above.

We summarize all these results as follows:

Proposition 3.6 *The invariants of* $(\sigma, \epsilon)\tau$ *acting on* $B\Gamma'_n$ *are non-empty only if* σ *is a product of* q m*-cycles, where* $n - 2 = qm$ *and* $m = 1, 2, \ldots$. *In this case,*

$$\chi = \begin{cases} \phi(m).m^{q-1}(q-1)! & \text{if } m \text{ is odd} \\ \phi(m).(-m)^{q-1}(q-1)! & \text{if } m \equiv 0 \ (\text{mod } 4) \\ 3\phi(m).(-m)^{q-1}(q-1)! & \text{if } m \equiv 2 \ (\text{mod } 4) \end{cases}$$

3.6 Lefschetz numbers and characters

We next want to put the information on fixed point sets to work to find information on the characters of Σ_{n-2} on $K^*(BK^n)$. To simplify matters, we shall consider rational coefficients $\mathbf{Q}$, where the K-theory and the cohomology are naturally identified (see [2]). We can easily extend to any field of characteristic zero, in particular to $\mathbf{C}_p$ by tensoring. We consider the following situation: X is a finite complex, and $f : X \to X$ is a simplicial homeomorphism, which necessarily has finite order. Subdividing if necessary, we can suppose that for any simplex $s, f(s)$ meets s in a face, so that the fixed point set X^f is a subcomplex. We *don't* assume that X^f is discrete. The Lefschetz number is defined in one of two equivalent ways. The K-theoretic version, which is what we eventually need, is that if

$$t_\alpha(f) = \mathrm{Tr}\big[f^* : K^\alpha(X) \to K^\alpha(X)\big] \qquad (\alpha = 0, 1)$$

then

(8) $$\mathcal{L}(f) = t_0(f) - t_1(f)$$

while the equivalent $\mathbf{Q}$-cohomology version, which we shall use for the next proof, is the usual alternating sum of traces on cochain or cohomology groups. The result we require is

Proposition 3.7 *Under the above conditions, the Lefschetz number* $\mathcal{L}(f)$ *of* f *equals the Euler characteristic* $\chi(X^f)$ *of the fixed point set.*

Note. This is a well-known result, and is an exercise in Tony Armstrong's 'Basic Topology'. However, it's no work to supply a proof here, and I can't find an accessible one. (In particular, it isn't easy to unearth in Lefschetz's book.)

Proof. The map f, being simplicial, induces an endomorphism of the cochain complexes ($\mathbf{Q}$ coefficients) $C^*(X)$, $C^*(X^f)$, $C^*(X, X^f)$, and of the short exact sequence which connects them. However, under these conditions, the Lefschetz number, like the Euler characteristic, is additive:

(9) $$\mathcal{L}(C^*(X)) = \mathcal{L}(C^*(X^f)) + \mathcal{L}(C^*(X, X^f))$$

But now clearly the Lefschetz number is zero for $C^*(X, X^f)$, since there are no fixed simplexes; while for $C^*(X^f)$, f is the identity, so its Lefschetz number is the Euler characteristic of X^f. This, using (9), establishes the proposition.

Unfortunately for the application of this result, at the moment both our space $X = BK^n$ and its fixed point sets are far from finite,although they are up to homotopy. However, this is not a serious problem. We know that $\tilde{F}_n(S^2)$ is contained in the finite complex $\prod^n S^2$. We can make the action of $SO(3) \times \Sigma_n$ on this simplicial, and so give the quotient $Y = (\prod^n S^2)/SO(3)$ the structure of a finite complex in which X, or $\tilde{F}_n(S^2)/SO(3)$ is contained as the complement of a subcomplex Y_0. The action of Σ_n on everything is also simplicial by construction. Take a fine enough (equivariant) subdivision, and let $\bar{X}$ be the complement of an open regular neighbourhood of Y_0 in Y. Then not only is the inclusion of $\bar{X}$ in X a homotopy equivalence, but the same is true of the fixed point sets for the action of Σ_n on $\bar{X}$. We can therefore substitute the Euler characteristics of fixed point sets derived in 2.2., 2.3., into the formula for the Lefschetz number which we just obtained for finite complexes. We shall express this in terms of characters as follows.

Proposition 3.8 *Let* $A = K^0(BK^n; \mathbf{Q})$, $B = K^1(BK^n; \mathbf{Q})$. *Then, for* $\sigma \in \Sigma_n$,

$$\mathrm{Tr}_A(\sigma) - \mathrm{Tr}_B(\sigma) = \chi((BK^n)^\sigma) \tag{10}$$

$$\mathrm{Tr}_A(\sigma) + \mathrm{Tr}_B(\sigma) = \chi((BK^n)^{\sigma\tau}) \tag{11}$$

where τ is the involution defined in 2.3., and the fixed subspaces are the ones we have already studied.

Proof. Equation (10) is simply a restatement of what we have shown already. Equation (11) adds a new ingredient, essentially that the trace of $\sigma\tau$ on $K^\alpha(BK^n)$ is $(-1)^\alpha$ times the trace of σ. This in turn follows from:

Lemma. *The induced homomorphism τ^* in cohomology of BK^n is $(-1)^q$ on H^q.*

Proof. Use induction on n. The forgetful map π from BK^n to BK^{n-1} commutes with τ, as does, by good luck, the cross section we constructed. It's therefore enough to look at the action of τ^* on a fibre which is fixed under τ, say at

$$\pi^{-1}[x_1, \ldots, x_{n-1}] \subset \tilde{F}_n(S^2)/SO(3)$$

with all the x's on the equator. Here τ is the reflection in the equator, acting on the complement of the $(n-1)$ points. Identify this space as usual with the wedge of $(n-2)$ circles and we see that $\tau^* = -1$ on H^1 of the fibre. Since this is all the fibre cohomology, the lemma follows and hence so does Proposition 3.8.

We conclude this section by recalling a basic result from character theory. Suppose G is a finite group, and M a finite rank module over kG, where k is a field of characteristic zero. Then the rank of the invariant submodule M^G is equal to

$$\text{(12)} \qquad \frac{\sum_{\gamma \in G} \mathrm{Tr}_M(\gamma)}{|G|}$$

where $|G|$ is the order of the group. (For a proof of this see e.g. [10].)

3.7 *Proof of the theorem*

We now have the material necessary to prove Theorem 2, with a little computation. We begin with a result which relates the numbers r_n, s_n, $\tilde{r}_n$, $\tilde{s}_n$ to Euler characteristics:

Proposition 3.9 *Let* r_n, s_n, $\tilde{r}_n$, $\tilde{s}_n$ *be as in Theorem 2. Then*

$$r_n - s_n = \frac{1}{(n-2)!} \sum_{\sigma \in \Sigma_{n-2}} \chi((BK^n)^\sigma) \tag{13}$$

$$r_n + s_n = \frac{1}{(n-2)!} \sum_{\sigma \in \Sigma_{n-2}} \chi((BK^n)^{\sigma\tau}) \tag{14}$$

$$\tilde{r}_n - \tilde{s}_n = \frac{1}{2(n-2)!} \sum_{\sigma \in \Sigma_{n-2}} (\chi((BK^n)^\sigma) + \chi((BK^n)^{(\sigma,\epsilon)}) \tag{15}$$

$$\tilde{r}_n + \tilde{s}_n = \frac{1}{2(n-2)!} \sum_{\sigma \in \Sigma_{n-2}} \chi((BK^n)^{\sigma\tau}) + \chi((BK^n)^{(\sigma,\epsilon)\tau}) \tag{16}$$

Proof. First, equation (12) identifies r_n and s_n with the sums over Σ_n of the appropriate traces, divided by $|\Sigma_{n-2}| = (n-2)!$ The result for $\tilde{r}_n$ and $\tilde{s}_n$ is similar, with the group Σ_{n-2} replaced by $\Sigma_{n-2} \times \Sigma_2$, whose order is $2.(n-2)!$ Next, proposition 3.6. shows that the difference (resp. sum) of the traces of any σ on K^0, K^1 is equal to the Euler characteristic of the fixed point set of σ (resp. $\sigma\tau$) on BK^n. This establishes the proposition.

We now need to compute the right hand sides in the four formulae (13)-(16) using what we have already found about the Euler characteristics of the fixed point sets. The first result is:

Proposition 3.10 *We have*

$$\sum_{\sigma \in \Sigma_{n-2}} \chi((BK^n)^\sigma) = \begin{cases} (n-2)! & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Proof. Write S_n for the required sum. Then we know that the only σ's which contribute to S_n are products of k-cycles, where k divides $(n-2)$. For each such k, we add a term equal (in the terms of section 3.2.) to $\chi(E(\frac{n-2}{k},k))$.

We return for the moment to writing q for $\frac{n-2}{k}$. We have by the corollary to Proposition 3.3:

$$\chi(E(q,k)) = \phi(k).(-k)^{(q-1)}(q-1)! \tag{17}$$

This must be supplemented by a calculation of the number of conjugacy classes of the given type. The standard formula (see [9]) gives us this in the form $\frac{(n-2)!}{k^q.q!}$ Substituting this in (15) and cancelling, we find that the total contribution from products of k-cycles is $\phi(k).(-1)^{(q-1)}(n-3)!$ Hence dividing by $(n-2)!$

$$\sum_{\sigma\in\Sigma_{n-2}} \chi((BK^n)^\sigma) = -(n-3)! \sum_{k|n-2} \phi(k)(-1)^{\frac{n-2}{k}} \tag{18}$$

The $-$ sign arises because we have $(-1)^{q-1}$ in (15).

Now if n, and hence $n-2$, is odd, so are all factors k and every term in (18) is positive. Using the well-known formula $\sum_{k|n}\phi(k) = n$, we obtain proposition 3.10 in this case.

Next suppose that n is even, say $n-2 = 2^t.m$ with m odd. Then every factor $k|m$ gives a sequence of factors $k, 2k, \dots, 2^t.k$ of $n-2$. Now note that $\frac{n-2}{2^i.k}$ is even for $i < t$ and odd for $i = t$; so the sign in the sum (18) is negative in the first case and positive in the second. Hence the contribution of these factors to the sum is $-\phi(k) - \phi(2k) - \dots - \phi(2^{t-1}.k) + \phi(2^t.k)$. This can be easily seen to be zero from general properties of ϕ. This completes the proof of the proposition. Putting propositions 3.9. and 3.10. together, we have the formula of theorem 2 for $r_n - s_n$.

Next, we prove the formula for $r_n + s_n$. Considering the fixed point sets of $\sigma\tau$ (equation (14)), we know that the only nonzero Euler characteristics occur when:

(a) $\sigma = 1$, when the fixed point set is homotopy equivalent to $\frac{1}{2}(n-1)!$ points and
(b) σ is a 2-cycle, when it is homotopy equivalent to $(n-3)!$ points.

It is now easy to compute using proposition 3.9. and the fact that there are $\binom{n-2}{2}$ 2-cycles:

$$r_n + s_n = \frac{1}{(n-2)!}\left[\frac{1}{2}(n-1)! + \frac{(n-2)(n-3)}{2}(n-3)!\right] = n-2 \tag{19}$$

This formula completes the proof of theorem 2 in the 'ordinary' case; we now proceed to the dihedral case. The formula for $\tilde{r}_n - \tilde{s}_n$ will follow when we have shown

Proposition 3.11 *The sum*

$$\sum_{\sigma\in\Sigma_{n-2}} \chi((BK^n)^{(\sigma,\epsilon)})$$

is equal to 0 *when n is even and to* $(-1)^k(2k+1)!$ *when* $n = 2k+3$ *is odd.*

Proof. By Proposition 3.5, if $n = 2k+2$ is even, then there are two cases to consider. Either σ is a product of $(k-1)$ disjoint two-cycles — and then $\chi = (-2)^{k-1}(k-1)!$ — or σ is a product of k such cycles, and $\chi = -(-2)^{k-1}k!$ Now, there are $\frac{(2k)!}{2^k(k-1)!}$ elements of the first type, and $\frac{(2k)!}{2^k.k!}$ of the second; multiplying, and adding, we get zero for the sum of all Euler characteristics.

If $n = 2k+3$, then σ must be a product of k disjoint two-cycles, and $\chi = (-2)^k.k!$ Since there are $\frac{(2k+1)!}{2^k.k!}$ such elements, we obtain the result of Proposition 3.11.

Putting this together with proposition 3.10, we find that $\tilde{r}_n - \tilde{s}_n$ is zero if n is even, and $\frac{1}{2}(1+(-1)^k)$ if $n = 2k+3$ is odd. This gives the formula of theorem 2 immediately.

Lastly we have to prove the formula for the sum $\tilde{r}_n + \tilde{s}_n$. This is slightly more complicated. The important factor in differentiating between cases is the sign for even values of m in proposition 3.6.

We first make a simple observation. Let (σ) be the conjugacy class of σ, and suppose its elements to be made up of q disjoint m-cycles, where $qm = n-2$. The number of elements in (σ) is $\frac{(n-2)!}{m^q.q!}$ Hence, if this is multiplied by $m^{q-1}(q-1)!$, the result is $\frac{(n-2)!}{mq} = (n-3)!$ Now suppose in the first place that n is odd, the easiest case. Then so are all its factors, and each factor m contributes (using proposition 3.6 and the above remark) an amount $\phi(m).(n-3)!$ to the sum of Euler characteristics in equation (16). Hence, as in the proof of proposition 3.10, we obtain $(n-2)!$ for the sum required. This gives

$$\tilde{r}_n + \tilde{s}_n = \frac{1}{2.(n-2)!}((n-2)+1)(n-2)! = \frac{n-1}{2}$$

for n odd, and this is the formula of theorem 2 in the odd cases.

Next, suppose $n = 4k$, $n-2 \equiv 2 \pmod 4$. Say $n-2 = 2n'$ where n' is odd. For each factor m of n' we have two factors m, $2m$ of n. By proposition 3.6 and the remark above, these contribute $\phi(m).(n-3)!$ and $3.\phi(2m).(n-3)! = 3.\phi(m).(n-3)!$ (The sign in the second case is positive because $\frac{n}{2m}$ is odd.) Since the sum of all $\phi(m)$ is n', the contribution from dihedral elements is $2.(n-2)!$ Hence,

$$\tilde{r}_n + \tilde{s}_n = \frac{1}{2.(n-2)!}((n-2)+2)(n-2)! = \frac{n}{2}$$

i.e. $2k$ as required.

Lastly, let $n = 4k+2$, $n-2 \equiv 0 \pmod 4$. Then write $n-2 = 2^t.n'$, with $t > 1$. Again let m be a factor of n'; it determines factors $m, 2m, \ldots, 2^t.m$ of

$n-2$. For each of these except the last, $\frac{n}{2^t.m}$ is even. We find using the previous methods the following contribution to the sum:

$$(n-3)!\phi(m)(1-3\phi(2)-\phi(4)-\ldots-\phi(2^{t-1})+\phi(2^t))$$

which it is easy to see is zero. Hence,

$$\tilde{r}_n+\tilde{s}_n=\frac{1}{2.(n-2)!}((n-2)+0)(n-2)!=\frac{n-2}{2}$$

which is $2k$ again. This completes the proof of theorem 2.

3.8 Some examples

Theorem 1 is a complicated result, while theorem 2 is very simple. Together, in any case, they add up to give us an effective method of computing the p-adic or infinite part of $K^*(B\Gamma^n)$ for any $n>2$. To make this clear, let us look at some simple examples.

Suppose first that n has a simple odd prime factor p; say $n=pq$ where q is prime to p. Let us find the p-adic part of $K^*(B\Gamma^n)$. We have from theorem 1:

$$K^*(B\Gamma^n)\otimes \mathbf{C}_p=\mathbf{C}_p\oplus\binom{p-1}{2}\left(K^*(B\Gamma^{q+2})\right)^{\Sigma_q} \tag{20}$$

Theorem 2 requires us to distinguish between q (and hence n) odd and even. Let us write the ranks of K^0, K^1 as an ordered pair of integers (r,s). We have:

$$\operatorname{rk}(K^*(B\Gamma^n)\otimes\mathbf{C}_p)=\begin{cases}(1+(\frac{p-1}{2}).k,(\frac{p-1}{2}).k) & \text{if } q=2k\\ (1+(\frac{p-1}{2}).(k+1),(\frac{p-1}{2}).k) & \text{if } q=2k+1\end{cases} \tag{21}$$

The answer is of course exactly the same for the p-adic part of $K^*(B\Gamma^{n+2})$, while in the case of $B\Gamma^{n+1}$, we have only to replace $(\frac{p-1}{2})$ by $(p-1)$.

We finally give a brief demonstration of the method by computing the quotients $\tilde{K}^*(B\Gamma^n)/\text{Torsion}$ for $3\le n\le 10$. We know that the groups in question are profinite, and hence a sum of copies of $\hat{\mathbf{Z}}_p$ (the p-adic numbers) for a finite set of primes p. Theorem 1 tells us which primes; and theorems 1 and 2 together make it possible to compute how many copies of $\hat{\mathbf{Z}}_p$ appear for any p. Formula (21) gives us the ranks for $p=5,7$, and for $p=3$ if $n<9$. For the 3-adic part when $n=9$ we use theorem 1 (i), (ii) to give the summands

$$(3.(K_3^*(BK^3)^{\Sigma_1})\oplus(K_3^*(BK^5)^{\Sigma_3})$$

which has rank $(3,0)+(2,1)=(5,1)$ using theorem 2. The corresponding part when $n=10$ must then simply be twice this, and so has rank $(10,2)$.

It remains to compute the 2-adic part, which is the complicated case. Here we have to invoke theorem 1 cases (ii), (iii) or (iv). Using the notation of theorem 2, the ranks are the following.

$n =$:

3	(r_3, s_3)
4	$(r_3, s_3) + (\tilde{r}_4, \tilde{s}_4) + (\tilde{r}_3, \tilde{s}_3)$
5	$2.(r_3, s_3) + (r_4, s_4)$
6	$(r_3, s_3) + (\tilde{r}_4, \tilde{s}_4) + (\tilde{r}_5, \tilde{s}_5)$
7	(r_5, s_5)
8	$2.(r_3, s_3) + (r_4, s_4) + (\tilde{r}_6, \tilde{s}_6) + (\tilde{r}_5, \tilde{s}_5)$
9	$4.(r_3, s_3) + 2.(r_4, s_4) + (r_6, s_6)$
10	$2.(r_3, s_3) + (r_4, s_4) + (\tilde{r}_6, \tilde{s}_6) + (\tilde{r}_7, \tilde{s}_7)$

(Note (a) that $\tilde{r}_n, \tilde{s}_n$ play the role of some sort of half of r_n, s_n; (b) a broad pattern is beginning to emerge.)

We can now derive the values of these from theorem 2. The end result is:

Proposition 3.12 *The following are the groups* $\tilde{K}^*(B\Gamma^n)/\text{Torsion}$ *for* $3 \le n \le 10$.

$B\Gamma^3$	$(\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3, 0)$
$B\Gamma^4$	$(3\hat{\mathbf{Z}}_2 + 2\hat{\mathbf{Z}}_3, \hat{\mathbf{Z}}_2)$
$B\Gamma^5$	$(3\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3 + 2\hat{\mathbf{Z}}_5, \hat{\mathbf{Z}}_2)$
$B\Gamma^6$	$(3\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3 + 4\hat{\mathbf{Z}}_5, 2\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3)$
$B\Gamma^7$	$(2\hat{\mathbf{Z}}_2 + 2\hat{\mathbf{Z}}_3 + 2\hat{\mathbf{Z}}_5 + 3\hat{\mathbf{Z}}_7, \hat{\mathbf{Z}}_2 + 2\hat{\mathbf{Z}}_3)$
$B\Gamma^8$	$(5\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3 + 6\hat{\mathbf{Z}}_7, 4\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3)$
$B\Gamma^9$	$(8\hat{\mathbf{Z}}_2 + 5\hat{\mathbf{Z}}_3 + 3\hat{\mathbf{Z}}_7, 4\hat{\mathbf{Z}}_2 + \hat{\mathbf{Z}}_3)$
$B\Gamma^{10}$	$(6\hat{\mathbf{Z}}_2 + 10\hat{\mathbf{Z}}_3 + 2\hat{\mathbf{Z}}_5, 3\hat{\mathbf{Z}}_2 + 2\hat{\mathbf{Z}}_3 + 2\hat{\mathbf{Z}}_5)$

The values for $n = 3, 4, 6$ agree (I am relieved to say) with those obtained by completely different methods in [8].

A Appendix. The existence of torsion

The few examples of $K^*(B\Gamma^n)$ known so far — basically, $n = 3, 4$ and 6 —are all torsion free, so that theorems 1 and 2 tell us all there is to know. This (and the similar results for finite groups) prompted me in [8] to conjecture that this was always the case. This is in fact quite untrue, as we shall see. We have in fact:

Theorem 3 *Let* $p > 3$ *be a prime. Then if* $2p+2 < n < 3p$, *we have in* p*-localized* K*-theory*

$$K^1(B\Gamma^n)_p = \mathbf{Z}/p;$$

$$\tilde{K}^0(B\Gamma^n)_p = 0$$

In consequence, $K^*(B\Gamma^n)$ *has* p*-primary part equal to the torsion group* $\mathbf{Z}/p$.

The reason for the above choice of example is that it is small enough to be (relatively) easily computable and large enough so that the first Dyer-Lashof operations in homology lead to something non-trivial when translated across. Of course the given values of n, p are such that there is no p-adic part in the

K-theory according to theorem 1. Accordingly, if there is anything non-trivial in the p-part it must be torsion.

The theorem is an easy consequence of the structure of the cohomology of $B\Gamma^n$. Specifically, we use the following result:

Proposition A.1 *If $2p + 2 < n < 3p$, then the mod p cohomology Poincaré polynomial of $B\Gamma^n$ is $1+t^{2p-2}+t^{2p-1}$; and the mod p Bockstein is an isomorphism from H^{2p-2} to H^{2p-1}.*

This result immediately implies theorem 3. For the reduced p-local cohomology of $B\Gamma^n$, being finitely generated, must be just $\mathbf{Z}/p$ in dimension $2p-1$; and the result on the K-theory follows from the (trivial) Atiyah-Hirzebruch spectral sequence.

For the proof of Proposition A.1, I appeal to the paper of Bödigheimer, Cohen and Peim [5], which computes $H^*(B\Gamma^n; \mathbf{F}_p)$ for all n and all odd primes p, and so contains this result as a rather trivial example of the general calculation. However, as the theorem which describes the homology for primes $p > 3$ is complicated, I'll explain explicitly how the deduction is made. The 'formula' of [5] is that (for $n > 2$ and $p > 3$) $H^k(B\Gamma^n; \mathbf{F}_p)$ is the component of degree $k + 2qn$ and weight n in a bigraded (degree and weight) $\mathbf{F}_p$-vector space

$$[A_{2q} \oplus (H_*(BS^3; \mathbf{F}_p) \otimes U_{2q})] \otimes BW_{2q+1} \tag{22}$$

where A_{2q}, U_{2q} are specified $\mathbf{F}_p$-vector spaces. BW_{2q+1} is an algebra related to the cohomology of $\Omega^2 S^{4q+3}$; for our purpose it is enough to know that in weight less than $4p$ it is generated by:

1 (degree and weight 0)

λ_1 (degree $(4q+2)p - 2$ and weight $2p$)

$\beta(\lambda_1)$ (degree $(4q+2)p - 1$ and weight $2p$)

where β is the Bockstein. We now have:

Lemma. *If n is congruent mod p to one of $3, \dots, p-1$, then the homogeneous subspace of A_{2q}, U_{2q} of weight n has dimension 1, 0 respectively over $\mathbf{F}_p$. A_{2q} is generated by Γ_n of degree $2qn$.*

This follows easily from the criterion given, that $(n, 3) \not\equiv 0 \bmod p$; this is satisfied precisely for the given values of n. (We therefore find our geometrical distinction between values of n mod p agreeing with that arrived at by homotopy theory in [5].) We now deduce that for $2p + 2 < n < 3p$ the terms of weight n in the expression (22) consist of $\gamma_n \otimes 1$, $\gamma_{n-2p} \otimes \lambda_1$, $\gamma_{n-2p} \otimes \beta(\lambda_1)$. Subtracting $2qn$ to get back to the cohomology of $B\Gamma^n$, these have dimensions equal to $1, 2p-2, 2p-1$ respectively. This is the statement of proposition A.1.

Note 1. Theorem 3 provides us with p-torsion only in cases where, according to theorems 1 and 2, there is no p-adic part. It would be interesting to know that it can occur in the other places; specifically, to know something of 2- and 3-torsion. (Also, we can observe that the first case of theorem 3 is that of 5-torsion in $K^*(B\Gamma^{13})$. Is there any torsion in $B\Gamma^n$ for smaller n?)

Note 2. We have, in the above results, enough information on the K-theory of $B\Gamma^n$ and its relation to the group theory to ask how the classes coming from representations might enter. There are also a variety of results on representations of mapping class groups, such as those of Kohno [9] derived from the monodromy of the Knizhnik-Zamolodzhnikov equation. It appears rather hard, but could be rewarding, to relate some of these to classes in K-theory.

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