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Some applications of Cartan's theory on three-dimensional Cauchy-Riemann geometry

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Part I. Real hypersurfaces with hermitian shape operators

Theoretically we learn from Cartan-Chern's theory ([1] for n=1, [3] in general) how to verify whether a nondegenerate real hypersurface in $\mathbb{C}^{n+1}$ is spherical, i.e. locally *CR* (Cauchy-Riemann)-equivalent to the unit sphere in $\mathbb{C}^{n+1}$. In practice, it is not easy to construct any class of spherical real hypersurfaces with certain geometric conditions.

In the first part of the paper, we consider real hypersurfaces with hermitian shape operators. Namely, the shape operator (or the second fundamental form) is required to be compatible with the induced complex structure. (See §2 for more details) The above condition is natural from the viewpoint of Riemannian geometry. In particular, we are dealing with real hypersurfaces defined by

$$\psi_n = [\exp(x_1 + y_1)^2]/4 + [\exp(x_1 - y_1)^2]/4 + \sum_{k=2}^{n+1}(x_k^2 + y_k^2) - 1 = 0, \quad n = 1, 2, \ldots.$$

The real hypersurface defined by $\psi_2 = 0$ in $\mathbb{C}^3$ is not locally *CR*-equivalent to the unit sphere in $\mathbb{C}^3$. (See the argument in the end of §2) However, the real hypersurface defined by $\psi_1 = 0$ in $\mathbb{C}^2$ is actually spherical.

As a matter of fact, it just happens that

Theorem. *Every nondegenerate (or, say, strictly pseudoconvex) real hypersurface in $\mathbb{C}^2$ with hermitian shape operator is spherical.*

In §§1 and 3, following a procedure developed in [3] to determine the *CR* connection forms and the curvatures, we carry out the lengthy algebraic manipulation. Meanwhile, we see how the shape operator plays its role in Cartan-Chern's

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theory for an embedded CR manifold. In §2 we are devoted to a discussion of real hypersurfaces with hermitian shape operators.

1 Interaction between induced Riemannian structure and induced CR structure

Let us start with a general setting although only the case $n = 1$ will be needed. Let M be a real hypersurface in a Kaehler manifold $\widetilde{M}$ of dimension $n + 1$. Let J be the complex structure of $\widetilde{M}$ and T_xM be the tangent space to M at x. We choose a local field of orthonormal frames $e_0, e_{0^*} = Je_0, e_1, e_{1^*} = Je_1, ..., e_n, e_{n^*} = Je_n$ in M such that, restricted to $M, e_0, e_1, e_{1^*}, ..., e_n, e_{n^*}$ are tangent to M and $e_1, e_{1^*}, ..., e_n, e_{n^*}$ are contained in the maximal complex subspace $T_xM \cap JT_xM$ at each point x of M. Let $\omega^0, \omega^{0^*}, \omega^1, \omega^{1^*}, ..., \omega^n, \omega^{n^*}$ be the field of coframes with respect to $e_0, e_{0^*}, e_1, e_{1^*}, ..., e_n, e_{n^*}$. The corresponding Riemannian connection forms are denoted by ${\omega_B}^A, {\omega_B}^{A^*}$, etc. Let

$$\begin{aligned} \theta^A &= \omega^A + \sqrt{-1}\omega^{A^*}, \theta^{\bar{A}} = \omega^A - \sqrt{-1}\omega^{A^*}, \\ {\theta_B}^A &= {\omega_B}^A + \sqrt{-1}{\omega_B}^{A^*}, A, B = 0, 1, ..., n. \end{aligned}$$

We need a field of admissible coframes for the induced CR structure on M. (For CR structures, see [3]) Take $\theta = \omega^0$ and $\theta^\beta = \omega^\beta + \sqrt{-1}\omega^{\beta^*}$, $1 \leq \beta \leq n$. Since $\omega^{0^*} = 0$ on M, $\theta^0 = \theta$ on M. It follows that

$$d\theta = d\theta^0 = -\sum_{\beta=1}^{n} {\theta_\beta}^0 \wedge \theta^\beta - {\theta_0}^0 \wedge \theta \text{ on } M \tag{1.1}$$

In the subsequent discussion, we are always working on M unless otherwise stated. The summation convention is adopted and the small Greek and English indices run from 1 to n. First, we express the Riemannian connection forms ${\theta_\beta}^0$ in terms of θ^α, $\theta^{\bar{\alpha}}$ and θ,

$${\theta_\beta}^0 = \Gamma_{\beta\alpha}\theta^\alpha + \Gamma_{\beta\bar{\alpha}}\theta^{\bar{\alpha}} + \Gamma_{\beta*}\theta. \tag{1.2}$$

Since ${\omega_0}^0 = 0$ by skew symmetry, it follows that

$${\theta_0}^0 = i{\omega_0}^{0^*} \quad (i = \sqrt{-1}). \tag{1.3}$$

Substituting (1.2) and (1.3) into (1.1) and noting that θ is real, we obtain

$$d\theta = \Gamma_{\beta\bar{\alpha}}\theta^\beta \wedge \theta^{\bar{\alpha}} + \theta \wedge (-\Gamma_{\beta*}\theta^\beta + i{\omega_0}^{0^*}) \tag{1.4}$$

and the symmetry relations

$$\Gamma_{\beta\alpha} = \Gamma_{\alpha\beta} \text{ and } \Gamma_{\beta\bar{\alpha}} = -\bar{\Gamma}_{\alpha\beta}. \tag{1.5}$$

Write ${\omega_\beta}^{0^*} = S_{\beta r}\omega^r + S_{\beta r^*}\omega^{r^*} + S_{\beta 0}\theta$ and ${\omega_\beta}^0 = {\omega_{\beta^*}}^{0^*} = S_{\beta^* r}\omega^r + S_{\beta^* r^*}\omega^{r^*} + S_{\beta^* 0}\theta$ where $S_{AB}, A, B = 0, 1, ..., n, 1^*, ..., n^*$ are components of the second fundamental form. Now the coefficients $\Gamma_{\beta\alpha}, \Gamma_{\beta\bar{\alpha}}, \Gamma_{\beta *}$ in (1.2) can be expressed in terms of S_{AB}'s by observing the corresponding coefficients on both sides of the identity $\theta_{\beta}{}^{0} = \omega_{\beta}{}^{0} + i\omega_{\beta}{}^{0^*}$. Thus we obtain

$$\begin{cases} \Gamma_{\beta\alpha} = (S_{\alpha\beta^*} + S_{\beta\alpha^*})/2 + i(S_{\beta\alpha} - S_{\beta^*\alpha^*})/2 \\ \Gamma_{\beta\bar{\alpha}} = (S_{\alpha\beta^*} - S_{\beta\alpha^*})/2 + i(S_{\beta\alpha} + S_{\beta^*\alpha^*})/2 \\ \Gamma_{\beta *} = S_{\beta^* 0} + iS_{\beta 0}. \end{cases} \tag{1.6}$$

Note that S_{AB} is symmetric, i.e. $S_{AB} = S_{BA}$. In order for (1.4) to fit the fundamental equation (4.10) in [3] for the *CR* structure ($\omega = \theta$, $\omega^{\alpha} = \theta^{\alpha}$ therein), we take

$$g_{\beta\bar{\alpha}} = -i\Gamma_{\beta\bar{\alpha}} \quad \text{and} \quad \phi = -\Gamma_{\beta *}\theta^{\beta} + i\omega_0{}^{0^*} - iS_{00}\theta \tag{1.7}$$

where S_{00} is the θ-coefficient of the expression $\omega_0{}^{0^*} = S_{0\beta}\omega^{\beta} + S_{0\beta^*}\omega^{\beta^*} + S_{00}\theta$. Note that ϕ is real and actually equal to $-S_{0\beta^*}\omega^{\beta} + S_{0\beta}\omega^{\beta^*}$. By (1.5), $g_{\beta\bar{\alpha}}$ is hermitian symmetric. Throughout the paper we assume that

$$det(g_{\beta\bar{\alpha}}) \neq 0.$$

Namely, the induced *CR* structure is nondegenerate. Since one of the Kaehlerian structure equations reads

$$d\theta^{\alpha} = -\theta_{\beta}{}^{\alpha} \wedge \theta^{\beta} - \theta_0{}^{\alpha} \wedge \theta,$$

it is reasonable to take the "first approximation" of *CR* connection forms as below:

$$\phi_{\beta}{}^{\alpha(1)} = \theta_{\beta}{}^{\alpha} \quad \text{and} \quad \phi^{\alpha(1)} = \theta_0{}^{\alpha}.$$

Now, $A_{\alpha\bar{\beta}\gamma}$, $B_{\alpha\bar{\beta}\bar{\gamma}}$ and $C_{\alpha\bar{\beta}}$ are to be determined from the following expression

$$dg_{\alpha\bar{\beta}} - \phi_{\alpha}{}^{\gamma(1)}g_{\gamma\bar{\beta}} - g_{\alpha\bar{\gamma}}\phi_{\bar{\beta}}{}^{\bar{\gamma}(1)} + g_{\alpha\bar{\beta}}\phi = A_{\alpha\bar{\beta}\gamma}\theta^{\gamma} + B_{\alpha\bar{\beta}\bar{\gamma}}\theta^{\bar{\gamma}} + C_{\alpha\bar{\gamma}}\theta. \tag{1.8}$$

A direct computation shows that

$$\begin{aligned} A_{\alpha\bar{\beta}\gamma} &= -i\Gamma_{\alpha\bar{\beta}\gamma} + i\Gamma_{\alpha\bar{\beta}}\Gamma_{\gamma *}/2 \\ C_{\alpha\bar{\beta}} &= -i\Gamma_{\alpha\bar{\beta}*} \end{aligned} \tag{1.9}$$

where $d\Gamma_{\alpha\bar{\beta}} - \theta_{\alpha}{}^{\gamma}\Gamma_{\gamma\bar{\beta}} - \Gamma_{\alpha\bar{\gamma}}\theta_{\bar{\beta}}{}^{\bar{\gamma}}$ is written as

$$\Gamma_{\alpha\bar{\beta}\gamma}\theta^{\gamma} + \Gamma_{\alpha\bar{\beta}\bar{\gamma}}\theta^{\bar{\gamma}} + \Gamma_{\alpha\bar{\beta}*}\theta.$$

On the other hand, substituting (1.2) into the following formula for the curvature on $\widetilde{M}$

$$d\theta_{\alpha}{}^{0} + \theta_{\gamma}{}^{0} \wedge \theta_{\alpha}{}^{\gamma} + \theta_0{}^{0} \wedge \theta_{\alpha}{}^{0} = \widetilde{\Omega}_{\alpha}{}^{0}$$

$$\widetilde{\Omega}_{\alpha}{}^{0} = \widetilde{R}_{\alpha}{}^{0}{}_{CD}\theta^{C} \wedge \theta^{D}, \ C, D = 0, 1, \ldots, n$$

we obtain

$$\begin{aligned}\widetilde{R}_\alpha{}^0{}_{CD}\theta^C\wedge\theta^D &= (d\Gamma_{\alpha\bar{\ell}}-\Gamma_{\alpha\bar{\gamma}}\theta_{\bar{\ell}}{}^{\bar{\gamma}}-\Gamma_{\gamma\bar{\ell}}\theta_\alpha{}^\gamma+i\omega_0{}^{0^*}\Gamma_{\alpha\bar{\ell}})\wedge\theta^{\bar{\ell}}\\ &+ (d\Gamma_{\alpha*}-\Gamma_{\alpha\bar{\ell}}\theta_0{}^{\bar{\ell}}-\Gamma_{\ell*}\theta_\alpha{}^\ell+i\omega_0{}^{0^*}\Gamma_{\alpha*}-\Gamma_{\alpha*}\phi)\wedge\theta\\ &+ (d\Gamma_{\alpha\ell}-\Gamma_{\alpha\gamma}\theta_\ell{}^\gamma-\Gamma_{\gamma\ell}\theta_\alpha{}^\gamma+i\omega_0{}^{0^*}\Gamma_{\alpha\ell})\wedge\theta^\ell\\ &+ \Gamma_{\alpha*}\Gamma_{\beta\bar{\gamma}}\theta^\beta\wedge\theta^{\bar{\gamma}}-\Gamma_{\alpha\ell}\theta_0{}^\ell\wedge\theta. \end{aligned}\tag{1.10}$$

Define the suitable derivatives of $\Gamma_{\alpha*}$, $\Gamma_{\alpha\ell}$ by the following expressions:

$$\begin{aligned}&d\Gamma_{\alpha*}-\Gamma_{\alpha\bar{\ell}}\theta_0{}^{\bar{\ell}}-\Gamma_{\alpha\ell}\theta_0{}^\ell-\Gamma_{\ell*}\theta_\alpha{}^\ell\\ =\ &\Gamma_{\alpha*\gamma}\theta^\gamma+\Gamma_{\alpha*\bar{\gamma}}\theta^{\bar{\gamma}}+\Gamma_{\alpha**}\theta\\ &d\Gamma_{\alpha\ell}-\Gamma_{\alpha\gamma}\theta_\ell{}^\gamma-\Gamma_{\gamma\ell}\theta_\alpha{}^\gamma\\ =\ &\Gamma_{\alpha\ell\beta}\theta^\beta+\Gamma_{\alpha\ell\bar{\beta}}\theta^{\bar{\beta}}+\Gamma_{\alpha\ell*}\theta.\end{aligned}\tag{1.11}$$

In deducing (1.9), we have used the identity

$$\begin{aligned}(-i)(d\Gamma_{\alpha\bar{\beta}} &- \Gamma_{\gamma\bar{\beta}}\theta_\alpha{}^\gamma-\Gamma_{\alpha\bar{\gamma}}\theta_{\bar{\beta}}{}^{\bar{\gamma}}-\Gamma_{\alpha\bar{\beta}}\Gamma_{\gamma*}\theta^\gamma\\ &+ i\Gamma_{\alpha\bar{\beta}}\omega_0{}^{0^*})-\Gamma_{\alpha\bar{\beta}}S_{00}\theta\\ &= A_{\alpha\bar{\beta}\gamma}\theta^\gamma+B_{\alpha\bar{\beta}\bar{\gamma}}\theta^{\bar{\gamma}}+C_{\alpha\bar{\beta}}\theta.\end{aligned}\tag{1.12}$$

After plugging (1.11) and (1.12) in (1.10) and equating the corresponding terms on both sides, we obtain

$$\begin{aligned}iA_{\alpha\bar{\gamma}\beta} &= \widetilde{R}_\alpha{}^0{}_{\beta\bar{\gamma}}-\Gamma_{\alpha\bar{\gamma}}\Gamma_{\beta*}-\Gamma_{\alpha*}\Gamma_{\beta\bar{\gamma}}+\Gamma_{\alpha\beta\bar{\gamma}}-\frac{1}{2}\Gamma_{\alpha\beta}\bar{\Gamma}_{\gamma*},\\ iC_{\alpha\bar{\gamma}} &= \widetilde{R}_\alpha{}^0{}_{0\bar{\gamma}}+\Gamma_{\alpha*\bar{\gamma}}-i\Gamma_{\alpha\bar{\gamma}}S_{00}.\end{aligned}\tag{1.13}$$

Now, we see that things get complicated. In order to go further, we have to make certain assumptions. It is natural to take $\widetilde{M}=\mathbb{C}^{n+1}$ with the Euclidean metric. Therefore the curvature tensor $\widetilde{R}$ vanishes. The basic assumption for M in $\widetilde{M}=\mathbb{C}^{n+1}$ we like to make is

$$\Gamma_{\alpha\beta}=0 \text{ for all } \alpha,\beta=1,\dots,n.\tag{1.14}$$

Under the assumptions above, (1.13) takes the simpler form

$$\begin{aligned}iA_{\alpha\bar{\gamma}\beta} &= -\Gamma_{\alpha\bar{\gamma}}\Gamma_{\beta*}-\Gamma_{\alpha*}\Gamma_{\beta\bar{\gamma}}\\ iC_{\alpha\bar{\gamma}} &= \Gamma_{\alpha*\bar{\gamma}}-i\Gamma_{\alpha\bar{\gamma}}S_{00}.\end{aligned}\tag{1.15}$$

Note that the condition (1.14) reduces the order of derivatives in the expression for $A_{\alpha\bar{\gamma}\beta}$ by one degree if, say, we start with a defining function for M in $\mathbb{C}^{n+1}$. In terms of the second fundamental form, the condition (1.14) is equivalent to the following conditions

$$S_{\alpha\beta^*} = -S_{\beta\alpha^*},\ S_{\beta\alpha} = S_{\beta^*\alpha^*} \tag{1.16}$$

by (1.6).

The geometric meaning of (1.16) will be explained in the next section. (We will see that (1.16) is equivalent to the shape operator being hermitian) Meanwhile, we will show that the condition (1.16) is not trivial by giving some examples other than spheres.

2 Real hypersurfaces with hermitian shape operators: examples

Throughout this section, we shall consider vectors in column form. Consider the real hypersurface M in $\mathbb{C}^{n+1}$ defined by $\psi = 0$ with grad ψ (gradient of ψ) $\neq 0$. The unit normal to M is given by

$$e_{0^*} = grad\ \psi/\|grad\ \psi\|.$$

Let $\tilde{\nabla}$ denote the covariant derivative in $\mathbb{C}^{n+1}$ (Of course, it is just the usual differentiation). And let $\langle , \rangle$ denote the Euclidean metric in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. At $p \in M, X \in T_pM$, the shape operator A is, up to a sign, given by

$$A(X) = \tilde{\nabla}_X e_{0^*}. \tag{2.1}$$

Now,

$$\begin{aligned} \tilde{\nabla}_X e_{0^*} &= X(1/\|grad\ \psi\|)grad\ \psi + (1/\|grad\ \psi\|)\tilde{\nabla}_X grad\ \psi \\ &= -e_{0^*}\langle e_{0^*}, (\tilde{\nabla} grad\ \psi)/(\|grad\ \psi\|)(X)\rangle + (\tilde{\nabla} grad\ \psi)/(\|grad\ \psi\|)(X) \\ &= (I - e_{0^*}{}^t e_{0^*})(Hess\ \psi/\|grad\ \psi\|)(X) \end{aligned} \tag{2.2}$$

where $Hess\ \psi$, ${}^t e_{0^*}$ and I denote the Hessian of ψ, the transpose of e_{0^*}, and the identity matrix respectively.

It is easy to see that the condition (1.16) can be reformulated as follows:

$$\langle AX, X\rangle = \langle AJX, JX\rangle \ \text{ for } X \in T_pM \cap JT_pM. \tag{2.3}$$

We say that the shape operator A is compatible with the induced complex structure if (2.3) holds. And call such A hermitian. Since $\langle e_{0^*}, X\rangle = 0$ for $X \in T_pM$, then

$$\langle e_{0^*}{}^t e_{0^*} Hess\ \psi(X), X\rangle = 0.$$

By (2.1), (2.2), a direct computation puts (2.3) in the following form:

$$\langle Hess\ \psi(X), X\rangle = \langle Hess\ \psi(JX), JX\rangle,\ X \in T_pM \cap JT_pM. \tag{2.4}$$

Let $X = (a^L)_{L=1,2,\ldots,2n+2}, JX = (b^L)$, i.e. $b^1 = -a^2, b^2 = a^1$, etc.. The condition (2.4) can be rewritten as

$$
\begin{aligned}
\psi_L a^L &= 0, \\
\psi_L b^L &= 0 \\
\psi_{LN} a^L a^N &= \psi_{LN} b^L b^N
\end{aligned} \tag{2.5}
$$

where ψ_L, ψ_{LN} denote the first and second derivatives respectively.

Now, consider the case $n = 1$. We see that $a^1 = 0$, $a^2 = \psi_3{}^2 + \psi_4{}^2$, $a^3 = \psi_4\psi_1 - \psi_2\psi_3$, $a^4 = -\psi_2\psi_4 - \psi_1\psi_3$ satisfy both conditions $\psi_L a^L = 0$ and $\psi_L b^L = 0$. So a direct substitution in $\psi_{LN} a^L a^N = \psi_{LN} b^L b^N$ gives

$$
\begin{aligned}
(\psi_3{}^2 &+ \psi_4{}^2)[(\psi_3{}^2 + \psi_4{}^2)(\psi_{22} - \psi_{11}) + 2(\psi_{23} - \psi_{14})(\psi_1\psi_4 - \psi_2\psi_3) \\
&+ 2(\psi_{13} - \psi_{24})(\psi_2\psi_4 + \psi_1\psi_3)] + 4\psi_{34}(\psi_2\psi_3 - \psi_4\psi_1)(\psi_2\psi_4 + \psi_1\psi_3) \\
&+ (\psi_{33} - \psi_{44})[(\psi_1\psi_4 - \psi_2\psi_3)^2 - (\psi_2\psi_4 + \psi_1\psi_3)^2] = 0.
\end{aligned} \tag{2.6}
$$

It is easy to see that the ψ given by $\psi_{11} = \psi_{22}$, $\psi_{23} = -\psi_{14}$, $\psi_{13} = \psi_{24}$, $\psi_{34} = 0$ and $\psi_{33} = \psi_{44}$ solves the equation above. Denote the real coordinates in $\mathbb{C}^2$ by (x_1, y_1, x_2, y_2). The general solution of the wave equation $\psi_{11} = \psi_{22}$ in two variables takes the form $f(x_1 + y_1) + g(x_1 - y_1)$. Therefore $\psi = f(x_1 + y_1) + g(x_1 - y_1) + x_2{}^2 + y_2{}^2 - 1$ satisfies the equation (2.6). In particular,

$$
\psi = [\exp(x_1 + y_1)^2]/4 + [\exp(x_1 - y_1)^2]/4 + x_2{}^2 + y_2{}^2 - 1
$$

is a solution of (2.6) which defines a bounded strictly convex domain $\{\psi < 0\}$ in $\mathbb{C}^2$. Similarly, we can deal with the case $n > 1$. For $n = 2$, the previous a^1, a^2, a^3, a^4 with $a^5 = a^6 = 0$ still satisfy both conditions $\psi_L a^L = 0$ and $\psi_L b^L = 0$. Since the maximal complex subspace at a point has the complex dimension 2, we need one more $X = (\widetilde{a}^L)_{L=1,\dots,6}$ which is independent of (a^L) and $(b^L) = J(a^L)$. It is easy to see that X given by $\widetilde{a}^1 = 0$, $\widetilde{a}^2 = 0$, $\widetilde{a}^3 = 0$, $\widetilde{a}^4 = \psi_5{}^2 + \psi_6{}^2$, $\widetilde{a}^5 = \psi_3\psi_6 - \psi_4\psi_5$, $\widetilde{a}^6 = -\psi_4\psi_6 - \psi_3\psi_5$ is a qualified candidate. Now expanding $\langle Hess\ \psi(X), X\rangle = \langle Hess\ \psi(JX), JX\rangle$ gives

$$
\begin{aligned}
(\psi_5{}^2 &+ \psi_6{}^2)[(\psi_5{}^2 + \psi_6{}^2)(\psi_{44} - \psi_{33}) + 2(\psi_{45} + \psi_{36})(\psi_3\psi_6 - \psi_4\psi_5) \\
&+ 2(\psi_{35} - \psi_{46})(\psi_4\psi_6 + \psi_3\psi_5)] + 4\psi_{56}(\psi_4\psi_5 - \psi_3\psi_6)(\psi_4\psi_6 + \psi_3\psi_5) \\
&+ (\psi_{55} - \psi_{66})[(\psi_3\psi_6 - \psi_4\psi_5)^2 - (\psi_4\psi_6 + \psi_3\psi_5)^2] = 0.
\end{aligned} \tag{2.7}
$$

Observe that the ψ given by $\psi_{11} = \psi_{22}, \psi_{23} = -\psi_{14}$, $\psi_{13} = \psi_{24}$, $\psi_{34} = 0$ and $\psi_{33} = \psi_{44}$, $\psi_{46} = \psi_{35}$, $\psi_{36} = -\psi_{45}$, $\psi_{56} = 0$, $\psi_{55} = \psi_{66}$ satisfies both equations (2.6) and (2.7). In particular,

$$
\psi = [\exp(x_1 + y_1)^2]/4 + [\exp(x_1 - y_1)^2]/4 + x_2{}^2 + y_2{}^2 + x_3{}^2 + y_3{}^2 - 1 \tag{2.8}
$$

is a solution of (2.6) and (2.7) which defines a bounded strictly convex domain $\{\psi < 0\}$ in $\mathbb{C}^3$.

It can be shown that the real hypersurface $\psi = 0$ in $\mathbb{C}^3$ given above is not locally CR-equivalent to the unit sphere in $\mathbb{C}^3$. In fact, if we invoke a theorem on p.72 in S. Webster's thesis, Berkeley, 1975, it suffices to compute the curvature K for the domain $\{x_1 + iy_1 \in \mathbb{C} : [\exp(x_1 + y_1)^2]/4 + [\exp(x_1 - y_1)^2]/4 - 1 < 0\}$

with certain metric and see if it is not a negative constant -2. But this is easy by the formula (1.20) on p.73 of the thesis mentioned above. However the real hypersurface defined by $[\exp(x_1+y_1)^2]/4+[\exp(x_1-y_1)^2]/4+x_2^2+y_2^2-1=0$ in $\mathbb{C}^2$ is actually spherical. (This follows from our theorem) Note that the theorem mentioned above in Webster's thesis is not applicable for the case $n=1$.

3 Proof of the theorem

We know that in the study of CR structures the case for M of dimension 3 is often quite different from the general situation. In this section we continue §1 to follow the procedure developed in [3] to compute the Cartan curvature tensor Q_{11} for the case $n=1$.

The "second approximation" of CR connection forms takes the following form:

$$\begin{aligned} \phi_1{}^{1(2)} &= \theta_1{}^1 + A_1{}^1{}_1\theta^1 + \frac{1}{2}C_1{}^1\theta \\ \phi^{1(2)} &= \theta_0{}^1 + \frac{1}{2}C_1{}^1\theta^1 \end{aligned} \tag{3.1}$$

where $A_1{}^1{}_1$, $C_1{}^1$ are given by (1.15) and $g_{1\bar{1}} = -i\Gamma_{1\bar{1}}$ and its inverse $g^{1\bar{1}}$ are used to raise or lower indices. Then

$$d\phi = i\phi^{1(2)}\wedge\theta_1 + i\theta_{\bar{1}}\wedge\phi^{\bar{1}(2)} + \theta\wedge\widetilde{\psi} \text{ for some real 1-form } \widetilde{\psi}.$$

Let

$$\Phi_1{}^{1(1)} \equiv d\phi_1{}^{1(2)} - i\theta_1\wedge\phi^{1(2)} + i\phi_1{}^{(2)}\wedge\theta^1 + i\{\phi_1{}^{(2)}\wedge\theta^1\} + \frac{1}{2}\widetilde{\psi}\wedge\theta. \tag{3.2}$$

Skew hermitian symmetry of $\theta_0{}^1$ gives

$$\theta_0{}^1 = \Gamma_{1\bar{1}}\theta^1 - \bar{\Gamma}_{1*}\theta \tag{3.3}$$

by (1.2) with the assumption $\Gamma_{11}=0$. It follows that

$$d\theta_1{}^1 = \Gamma_{1\bar{1}}{}^2\theta^{\bar{1}}\wedge\theta^1 mod\ \theta \tag{3.4}$$

Substituting (3.1) in (3.2) and using (3.3) and (3.4) give

$$\Phi_1{}^{1(1)} = 2\Gamma_{1\bar{1}}{}^2\theta^1\wedge\theta^{\bar{1}} + dA_1{}^1{}_1\wedge\theta^1 - A_1{}^1{}_1\theta_1{}^1\wedge\theta^1 + C_1{}^1\Gamma_{1\bar{1}}\theta^1\wedge\theta^{\bar{1}} + C_{\bar{1}}{}^{\bar{1}}\Gamma_{1\bar{1}}\theta^{\bar{1}}\wedge\theta^1 mod\ \theta.$$

Define $A_1{}^1{}_{1\bar{1}}$, $A_1{}^1{}_{11}$ and $A_1{}^1{}_{1*}$ by

$$dA_1{}^1{}_1 - A_1{}^1{}_1\theta_1{}^1 = A_1{}^1{}_{1\bar{1}}\theta^{\bar{1}} + A_1{}^1{}_{11}\theta^1 + A_1{}^1{}_{1*}\theta$$

and $S_{11}{}^1{}_{\bar{1}}{}^{(1)}$ by $\Phi_1{}^{1(1)} = S_{11}{}^1{}_{\bar{1}}{}^{(1)}\theta^1\wedge\theta^{\bar{1}}$ mod θ. Then we have

$$S_{11}{}^1{}_{\bar{1}}{}^{(1)} = 2\Gamma_{1\bar{1}}{}^2 - A_1{}^1{}_{1\bar{1}} + C_1{}^1\Gamma_{1\bar{1}} - C_{\bar{1}}{}^{\bar{1}}\Gamma_{1\bar{1}} \tag{3.5}$$

It is easy to obtain

$$A_1{}^1{}_{1\bar{1}} = -2\Gamma_{1*\bar{1}} + 2\Gamma_{1\bar{1}}{}^2. \tag{3.6}$$

Using (3.6) and (1.15), we can reduce (3.5) to

$$S_{11}{}^1{}_{\bar{1}}{}^{(1)} = 2\Gamma_{1*\bar{1}} \tag{3.7}$$

Following (3.7), we have

$$\begin{aligned} S_{1\bar{1}}{}^{(1)} &= S_{11}{}^1{}_{\bar{1}}{}^{(1)} = 2\Gamma_{1*\bar{1}} \\ S^{(1)} &= S_1{}^{1(1)} = S_{1\bar{1}}{}^{(1)} g^{1\bar{1}} = 2i\Gamma_{1*\bar{1}}\Gamma^{1\bar{1}} \quad \text{and} \\ 3D_{1\bar{1}} &= iS_{1\bar{1}}{}^{(1)} - (i/4)S^{(1)}g_{1\bar{1}} = (3i/2)\Gamma_{1*\bar{1}} \end{aligned} \tag{3.8}$$

where $\Gamma^{1\bar{1}} = \Gamma_{1\bar{1}}{}^{-1}$. On the other hand

$$\phi^{1(2)} = \theta_0{}^1 + \frac{1}{2}(\Gamma_{1*\bar{1}}\Gamma^{1\bar{1}} - iS_{00})\theta^1. \tag{3.9}$$

by (1.15) and (3.1).

An easy computation shows that

$$\phi = -S_{1^*0}\omega^1 + S_{10}\omega^{1^*} \tag{3.10}$$

by (1.6) and (1.7). It follows that $d\phi = \theta \wedge [(-S_{1^*00} + S_{00}S_{01})\omega^1 + (S_{100} + S_{00}S_{01^*})\omega^{1^*}]$. Therefore using (3.9) and (3.10), we obtain

$$d\phi - i\phi^{1(2)} \wedge \theta_1 - i\theta_{\bar{1}} \wedge \phi^{\bar{1}(2)} = \theta \wedge \widetilde{\psi}$$

where

$$\widetilde{\psi} = (-\frac{1}{2}\Gamma_{1**} - \Gamma_{1*}\Gamma_{1\bar{1}})\theta^1 - (\frac{1}{2}\bar{\Gamma}_{1**} - \bar{\Gamma}_{1*}\Gamma_{1\bar{1}})\theta^{\bar{1}}. \tag{3.11}$$

Note that it is easy to deduce $\Gamma_{1**} = S_{1^*00} - S_{00}S_{01} + i(S_{100} + S_{00}S_{01^*})$.

Let $\phi^{1(3)} = \phi^{1(2)} + D_1{}^1\theta^1$ and $\phi_1{}^1 = \phi_1{}^{1(2)} + D_1{}^1\theta$. Then by (3.9) and (3.8), we have

$$\phi^{1(3)} = \theta_0{}^1 - \frac{1}{2}iS_{00}\theta^1. \tag{3.12}$$

while (3.1), (1.15) and (3.8) imply

$$\phi_1{}^1 = \theta_1{}^1 - 2\Gamma_{1*}\theta^1 - \frac{1}{2}iS_{00}\theta. \tag{3.13}$$

Let

$$\Phi_1{}^{1(2)} \equiv d\phi_1{}^1 - i\theta_1 \wedge \phi^{1(3)} + 2i\phi_1{}^{(3)} \wedge \theta^1 + \frac{1}{2}\widetilde{\psi} \wedge \theta. \tag{3.14}$$

Substituting (3.13), (3.12) and (3.11) in (3.14) and carrying out the computation, we finally obtain

$$\Phi_1{}^{1(2)} = 0. \tag{3.15}$$

Here we have used the identity:

$$dS_{00} - \phi S_{00} = -i(\frac{1}{2}\Gamma_{1**} + \Gamma_{1\bar{1}}\Gamma_{1*})\theta^1 + i(\frac{1}{2}\bar{\Gamma}_{1**} - \Gamma_{1\bar{1}}\bar{\Gamma}_{1*})\theta^{\bar{1}} \quad (mod\ \theta)$$

which can be easily deduced.

Now, (3.15) says that $V_1{}^1{}_1 = 0$. Therefore $E_1 = (\frac{2}{3i})V_1{}^1{}_1 = 0$ and $\phi^1 = \phi^{1(3)} + E^1\theta = \phi^{1(3)}$. Then a lengthy but straightforward computation gives

$$d\phi_1 - \phi_{1\bar{1}} \wedge \phi^{\bar{1}} + \frac{1}{2}\psi \wedge \theta_1 = Q_{11}\theta^1 \wedge \theta \ (mod \ \theta^{\bar{1}} \wedge \theta, \ \theta^1 \wedge \theta^{\bar{1}})$$

where

$$Q_{11} = i\Gamma_{1\bar{1}}[\Gamma_{1*1} + (\Gamma_{1*})^2]. \tag{3.16}$$

Note that (1.8) has been used to deal with the term $d\Gamma_{1\bar{1}}$.

Recall that $\Gamma_{1*\bar{1}}$ is defined by (cf. (1.11))

$$d\Gamma_{1*} - \Gamma_{1\bar{1}}\theta_0{}^{\bar{1}} - \Gamma_{1*}\theta_1{}^1 = \Gamma_{1*1}\theta^1 + \Gamma_{1*\bar{1}}\theta^{\bar{1}} + \Gamma_{1**}\theta. \tag{3.17}$$

The covariant derivatives S_{A0B} of the second fundamental form are given by

$$dS_{A0} - S_{C0}\omega_A{}^C - S_{AC}\omega_0{}^C = S_{A0B}\omega^B$$

where A, B, C range over $0, 1, 1^*$.

Using (1.16), (1.6), (1.2) and $\theta_1{}^1 = \omega_1{}^1 + i\omega_1{}^{1^*}$, $\omega_{1^*}{}^{1^*} = \omega_1{}^1$, $\omega_{1^*}{}^1 = \omega_{1^*}{}^1$, we can reduce the left hand side of (3.17) to

$$\begin{aligned}[(S_{1^*01} - S_{00}S_{11}) &+ iS_{101}]\omega^1 + [S_{1^*01^*} \\ &+ i(S_{101^*} + S_{00}S_{1^*1^*})]\omega^{1^*} \\ &+ [(S_{1^*00} - S_{00}S_{10}) + i(S_{100} + S_{00}S_{1^*0})]\theta.\end{aligned}$$

where we have used $S_{11^*} = S_{1^*1} = 0$ by (1.16). Thus it follows that

$$\begin{cases} \Gamma_{1*1} + \Gamma_{1*\bar{1}} = S_{1^*01} - S_{00}S_{11} + iS_{101} \\ i(\Gamma_{1*1} - \Gamma_{1*\bar{1}}) = S_{1^*01^*} + i(S_{101^*} + S_{00}S_{1^*1^*}). \end{cases} \tag{3.18}$$

Now it is easy to see from (3.18) that

$$\Gamma_{1*1} = S_{11^*0} + i(S_{110} - S_{1^*1^*0})/2 \tag{3.19}$$

by (1.16) and symmetry of S_{ABC}. From (1.16), the following "commutation" relations

$$\begin{cases} S_{1^*1^*A} - S_{11A} = 2(S_{10}S_{1^*A} + S_{01^*}S_{1A}) \\ S_{11^*A} + S_{1^*1A} = 2(S_{10}S_{1A} - S_{01^*}S_{1^*A}), \quad A = 0, 1, 1^* \end{cases} \tag{3.20}$$

hold. In particular, $S_{11^*0} = (S_{10})^2 - (S_{1^*0})^2$ and $S_{110} - S_{1^*1^*0} = -4S_{01}S_{01^*}$. Substituting these equalities in (3.19) and comparing with $\Gamma_{1*} = S_{1^*0} + iS_{10}$, we find that $\Gamma_{1*1} + (\Gamma_{1*})^2$ vanishes identically. It follows that Q_{11} vanishes identically by (3.16). Therefore M is spherical according to Cartan's theory ([1], [3]). We have proven our theorem.

As a consequence, we can easily deduce by Hartogs' theorem

Corollary. *The bounded strictly convex domain defined by* $\psi = [\exp(x_1+y_1)^2]/4 + [\exp(x_1-y_1)^2]/4 + x_2{}^2 + y_2{}^2 - 1$ *less than zero in* $\mathbb{C}^2$ *is biholomorphically equivalent to the unit ball in* $\mathbb{C}^2$.

Recall that the shape operator of the real hypersurface given by the above $\psi = 0$ in $\mathbb{C}^2$ is hermitian as shown in §2 and also note that ψ is real analytic.

Part II. Invariant elements of surface area in 3-dimensional CR geometry

In §1 of this part, we construct some invariant area elements on a non-characteristic surface in a *CR* 3-space. Thus we have the notion of surface area which is invariant under ambient *CR* transformations. Moreover, an invariant Lorentzian metric on the surface is also obtained. So in particular light rays of this invariant Lorentzian metric form invariant curves on the surface. In §2 we deduce the equation of minimal surfaces with respect to the simplest invariant area element *dA*. The actual computation for surfaces in the Heisenberg group is carried out in §3. There is a potential application that *dA* may give rise to a characterization of the "mass" in *CR* geometry [4] as the Willmore integrand does in the geometry of asymptotically flat manifolds [5]. Also *CR*-invariant area elements might be of use for giving quantitative description of objects in contact 3-topology.

1 Invariant area elements on a non-characteristic surface

We are going to deal with the local *CR* geometry. The ambient 3-space is endowed with a strictly pseudoconvex *CR* structure. (see [1], [3]) Let $(\theta, \theta^1, \theta^{\bar{1}})$ be an admissible coframe with respect to this *CR* structure, where θ is a contact form and θ^1 is complex, such that

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} + \theta \wedge \phi \tag{1.1}$$

for some real one-form ϕ. Call such coframe (or just θ^1) unitary. Given an admissible unitary coframe $(\theta, \theta^1, \theta^{\bar{1}})$, there associate uniquely determined connection forms ϕ_1^1, ϕ^1, ψ satisfying certain structure equations (e.g. [3]) with

$$\phi_1^1 + \phi_{\bar{1}}^{\bar{1}} - \phi = 0. \tag{1.2}$$

Two admissible unitary coframes and their associated connection forms are related according to the formula

$$\widetilde{\Pi} = dh \cdot h^{-1} + h\Pi h^{-1} \tag{1.3}$$

where

$$\Pi = \begin{pmatrix} \pi_0^0 & \theta^1 & 2\theta \\ -i\phi^{\bar{1}} & \phi_1^1 + \pi_0^0 & 2i\theta^{\bar{1}} \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^1 & -\overline{\pi_0^0} \end{pmatrix}$$

with $\pi_0^0 = (-1/3)(\phi_1^1 + \phi)$ and

$$h = \begin{pmatrix} t & 0 & 0 \\ t_1 & t_1^1 & 0 \\ \tau & \tau^1 & \bar{t}^{-1} \end{pmatrix}$$

subject to

$$\begin{cases} t_1 = -2itt_1^1\tau^{\bar{1}} \\ t\bar{t}^{-1}t_1^1 = 1 \\ | t_1^1 |= 1 \\ | \tau^1 |^2 +(i/2)(\bar{\tau}\bar{t}^{-1} - \tau t^{-1}) = 0. \end{cases} \tag{1.4}$$

In particular, the change of admissible unitary coframes reads

$$\begin{cases} \widetilde{\theta} = u\theta \\ \widetilde{\theta}^1 = \theta^1 u_1^1 + \theta v^1 \end{cases} \tag{1.5}$$

where

$$\begin{cases} u =| t |^2 > 0, u_1^1 = t(t_1^1)^{-1}, v^1 = -2 \mid t \mid^2 \tau^1 (t_1^1)^{-1} \text{ and} \\ | t_1^1 |= 1 \text{ or } u =| u_1^1 |^2 \text{ (by unitarity).} \end{cases} \tag{1.6}$$

Write $\theta^1 = \omega^1 + i\omega^2, \widetilde{\theta}^1 = \widetilde{\omega}^1 + i\widetilde{\omega}^2, u_1^1 = u_{1r}^1 + iu_{1c}^1$ and $v^1 = v_r^1 + iv_c^1$ for $\omega^1, \omega^2, \widetilde{\omega}^1, \widetilde{\omega}^2$, $u_{1r}^1, u_{1c}^1, v_r^1, v_c^1$ real. Let M be a piece of surface defined by $\omega^2 = 0$. Note that it is not possible to have a surface defined by $\theta = 0$ due to non-integrability (1.1). A point p of a surface is usually called characteristic if θ vanishes when restricted to this surface at p. In this terminology our surface M is non-characteristic. If $\widetilde{\omega}^2$ is another choice of defining M, then $\widetilde{\omega}^2$ has to be proportional to ω^2. It follows from (1.5) that $u_{1c}^1 = v_c^1 = 0$ and

$$\begin{cases} \widetilde{\theta} = u\theta \\ \widetilde{\omega}^1 = \omega^1 u_{1r}^1 + \theta v_r^1 \\ \widetilde{\omega}^2 = \omega^2 u_{1r}^1 \end{cases} \tag{1.7}$$

with $u_{1r}^1 = +\sqrt{u}$ (Here we assume that the change preserves the orientation of M. See the latter context.). The existence of ω^2 such that it is the imaginary part of an admissible unitary 1-form θ^1 and $\omega^2 = 0$ defines (non-characteristic) surfaces can be easily shown. (see the actual computation in §3)

By the third equality of (1.4), we can write $t_1^1 = e^{i\lambda}$. Since $u_{1r}^1 = t(t_1^1)^{-1}(u_{1c}^1 = 0)$ by (1.6) is real, it follows that $\lambda = 2k\pi/3$ for some integer k due to the second equality of (1.4). So t_1^1 is constant. (this fact will be used in the latter computation)

Equating real and imaginary parts of both sides of the structure equation $d\theta^1 = \theta^1 \wedge \phi_1^1 + \theta \wedge \phi^1$ gives

$$d\omega^1 = \omega^1 \wedge \phi^1_{1r} - \omega^2 \wedge \phi^1_{1c} + \theta \wedge \phi^1_r \tag{1.8.1}$$

$$d\omega^2 = \omega^1 \wedge \phi^1_{1c} + \omega^2 \wedge \phi^1_{1r} + \theta \wedge \phi^1_c \tag{1.8.2}$$

where we write $\phi^1_1 = \phi^1_{1r} + i\phi^1_{1c}$ and $\phi^1 = \phi^1_r + i\phi^1_c$ for $\phi^1_{1r}, \phi^1_{1c}, \phi^1_r, \phi^1_c$ real. On M, (1.8.2) is reduced to

$$0 = \omega^1 \wedge \phi^1_{1c} + \theta \wedge \phi^1_c.$$

By Cartan's lemma it follows that on M

$$\phi^1_{1c} = h_{10}\theta + h_{11}\omega^1 \tag{1.9.1}$$

$$\phi^1_c = h_{00}\theta + h_{01}\omega^1 \tag{1.9.2}$$

with $h_{10} = h_{01}$. The quadratic differential form

$$\Pi = \omega^1\phi^1_{1c} + \theta\phi^1_c = h_{11}(\omega^1)^2 + 2h_{10}\omega^1\theta + h_{00}\theta^2$$

is usually called the second fundamental form of M (with respect to the ambient CR structure). Now it is natural to ask what would be the transformation law of $h_{ij}, 0 \le i,j \le 1$, under (1.7), the change of admissible coframes. A lengthy but straightforward computation based on (1.3) using (1.4) and (1.6) with u^1_1, v^1 real (in particular, we use the fact that t^1_1 is constant as mentioned above) shows that

$$\begin{cases} \tilde{\phi}^1_{1c} = \phi^1_{1c} + 6\rho\omega^1 + 6\rho^2\theta \\ \sqrt{u}\tilde{\phi}^1_c = \phi^1_c - 2\rho\phi^1_{1c} - 6\rho^2\omega^1 - 4\rho^3\theta \end{cases} \tag{1.10}$$

where $\rho = v^1_r/2u^1_{1r}$ (note that $u^1_{1r} = \sqrt{u}$). It follows from (1.9.1), (1.9.2) and (1.10) that

$$\sqrt{u}\tilde{h}_{11} = h_{11} + 6\rho \tag{1.11.1}$$

$$u\tilde{h}_{10} + 2\sqrt{u}\rho\tilde{h}_{11} = h_{10} + 6\rho^2 \tag{1.11.2}$$

$$u^{\frac{3}{2}}\tilde{h}_{00} + 4u\rho\tilde{h}_{10} + 4\sqrt{u}\rho^2\tilde{h}_{11} = h_{00} + 8\rho^3. \tag{1.11.3}$$

Solving (1.11.1) for ρ and then substituting in (1.11.2) and (1.11.3), we obtain

$$u(\tilde{h}_{10} + \frac{1}{6}\tilde{h}^2_{11}) = h_{10} + \frac{1}{6}h^2_{11} \tag{1.12.1}$$

$$u^{\frac{3}{2}}(\tilde{h}_{00} + \frac{2}{3}\tilde{h}_{10}\tilde{h}_{11} + \frac{2}{27}\tilde{h}^3_{11}) = h_{00} + \frac{2}{3}h_{10}h_{11} + \frac{2}{27}h^3_{11}. \tag{1.12.2}$$

Since $\tilde{\theta} \wedge \tilde{\omega}^1 = u^{\frac{3}{2}}\theta \wedge \omega^1$ by (1.7), the following 2-forms

$$\begin{cases} \mid H \mid^{\frac{3}{2}} \theta \wedge \omega^1 \text{ (with } H = h_{10} + \frac{1}{6}h_{11}^2) \\ (h_{00} + \frac{2}{3}h_{10}h_{11} + \frac{2}{27}h_{11}^3)\theta \wedge \omega^1 \end{cases} \tag{1.13}$$

are invariant under (1.7). Similarly the second fundamental form transforms as below:

$$\widetilde{\Pi} = \sqrt{u}\{\Pi + 6\rho(\omega^1)^2 + 12\rho^2\omega^1\theta + 8\rho^3\theta^2\}.$$

Then it is a direct verification that

$$\begin{aligned} \Pi^* &= \mid H \mid^{\frac{1}{2}} \{\Pi - h_{11}(\omega^1)^2 + \frac{1}{3}h_{11}^2\omega^1\theta - \frac{1}{27}h_{11}^3\theta^2\} \\ &= \mid H \mid^{\frac{1}{2}} \{2H\omega^1\theta + (h_{00} - \frac{1}{27}h_{11}^3)\theta^2\} \end{aligned}$$

changes invariantly.

Now let X and $\hat{X}$ be (local) CR 3-spaces. They are oriented according to the order of $\theta, \omega^1, \omega^2$ ($\hat{\theta}, \hat{\omega}^1, \hat{\omega}^2$ respectively). Let M and $\hat{M}$ be non-characteristic surfaces in X and $\hat{X}$ with orientations in the order of (θ, ω^1) and $(\hat{\theta}, \hat{\omega}^1)$ respectively. Let $\varphi : X \to \hat{X}$ be an orientation-preserving CR diffeomorphism which maps M onto $\hat{M}$ and also preserves orientations on M and $\hat{M}$. Then $\widetilde{\theta} = \varphi^*\hat{\theta}, \widetilde{\omega}^1 = \varphi^*\hat{\omega}^1$, $\widetilde{\omega}^2 = \varphi^*\hat{\omega}^2$ are related to $\theta, \omega^1, \omega^2$ in (1.7). Let dA denote either 2-form of (1.13) on M and $d\hat{A}$ denote the corresponding 2-form on $\hat{M}$. Thus the previous argument gives the following result.

Theorem. *The 2-forms of (1.13) (called area elements if not* 0*) are invariant under ambient orientation-preserving CR diffeomorphisms which also preserve orientations of corresponding surfaces, i.e.*

$$\varphi^*(d\hat{A}) = dA.$$

Moreover, suppose $H \not\equiv 0$. Then Π^* is an invariant Lorentzian metric on the surface M, i.e.

$$\varphi^*(\hat{\Pi}^*) = \Pi^*$$

where $\hat{\Pi}^*$ is the corresponding Lorentzian metric on the surface $\hat{M}$.

2 The equation of CR-invariant minimal surfaces

In this section we deduce the equation of minimal surfaces with respect to the CR-invariant area element

$$dA = \mid H \mid^{\frac{3}{2}} \theta \wedge \omega^1$$

where we recall that $H = h_{10} + \frac{1}{6}h_{11}^2$ ($\neq 0$ by assumption). The idea is similar as in [2], for instance, for the affine geometry.

Let Ω be a small domain of M with boundary $\partial\Omega$. Its area is

$$A(\Omega) = \int_\Omega dA.$$

We need to compute the first variation $\delta A(\Omega)$ under an infinitesimal displacement of Ω with $\partial\Omega$ kept fixed. Analytically let $\pi : M \times I_\varepsilon \to X$ (I_ε denotes the interval $-\varepsilon < t < \varepsilon$) be a smooth mapping with $\pi(x,0) = x \in M \subset X$ and $\pi(x,t) = x$ for all $x \in \partial\Omega$ and all $t \in I_\varepsilon$ such that the restriction of π to $\Omega \times I_\varepsilon$ is an embedding for simplicity. Also assume we have an admissible coframe $\theta, \omega^1, \omega^2$ on $\pi(\Omega \times I_\varepsilon)$ for small ε such that

$$\left\{\begin{array}{l} \pi^*\theta = \hat{\theta} + a^0 dt \\ \pi^*\omega^1 = \hat{\omega}^1 + a^1 dt \\ \pi^*\omega^2 = adt \end{array}\right. \tag{2.1}$$

with a^0, a^1, a smoothly extended to be zero on $\partial\Omega$. (It is not difficult to achieve the above admissible coframe once we figure out the existence of ω^2 from the actual computation in section 3.) It follows from (2.1) that

$$\pi^*(\theta \wedge \omega^1) = \hat{\theta} \wedge \hat{\omega}^1 + dt \wedge (a^0\hat{\omega}^1 - a^1\hat{\theta}) \tag{2.2}$$

and also

$$\begin{aligned} &| H |^{-\frac{3}{2}} d(| H |^{\frac{3}{2}} \theta \wedge \omega^1) = | H |^{-\frac{3}{2}} d(| H |^{\frac{3}{2}}) \wedge \theta \wedge \omega^1 + d(\theta \wedge \omega^1) \\ &= \frac{3}{2} d \log | H | \wedge \theta \wedge \omega^1 - \frac{3}{2}\phi \wedge \theta \wedge \omega^1 + h_{11}\theta \wedge \omega^2 \wedge \omega^1. \end{aligned} \tag{2.3}$$

by (1.1) (note that $i\theta^1 \wedge \theta^{\bar{1}} = 2\omega^1 \wedge \omega^2$), (1.2), (1.8.1) and (1.9.1).

The operator d on $M \times I_\varepsilon$ can be decomposed as

$$d = d_M + dt\frac{\partial}{\partial t}.$$

Multiplying $| H |^{\frac{3}{2}}$ on both sides of (2.2) and taking exterior differentiation gives

$$\begin{aligned} &\frac{\partial}{\partial t}(| H |^{\frac{3}{2}} \hat{\theta} \wedge \hat{\omega}^1) \wedge dt \\ = \quad & dt \wedge d_M\{| H |^{\frac{3}{2}} (a^0\hat{\omega}^1 - a^1\hat{\theta})\} + | H |^{\frac{3}{2}} \{\frac{3}{2}(dlog | H | - \pi^*\phi) \wedge \pi^*(\theta \wedge \omega^1) \\ & -ah_{11}dt \wedge \pi^*(\theta \wedge \omega^1)\} \end{aligned} \tag{2.4}$$

by (2.3) and the third equality of (2.1). Next we deal with the term $dlog \mid H \mid -\pi^*\phi$. Exterior differentiation of $\omega^2 = adt$ (the pull-back π^* is omitted in the following context for simplicity of notation) implies

$$\omega^1 \wedge \phi^1_{1c} + \theta \wedge \phi^1_c + dt \wedge (a\phi^1_{1r} + da) = 0 \tag{2.5}$$

by (1.8.2). Express ϕ^1_{1c} and ϕ^1_c as follows:

$$\phi^1_{1c} = h_{11}\omega^1 + h_{10}\theta + h_1 dt \tag{2.6.1}$$

$$\phi^1_c = h_{01}\omega^1 + h_{00}\theta + h_0 dt. \tag{2.6.2}$$

From (2.5), we also have the expression for the last term:

$$a\phi^1_{1r} + da = h_0\theta + h_1\omega^1 + hdt. \tag{2.7}$$

Taking exterior differentiation of (2.6.1) and using the structure equations (1.1), (1.8.1) and $d\phi^1_{1c} = 3\omega^1 \wedge \phi^1_r + 3\omega^2 \wedge \phi^1_c$ (see [3]) give

$$\begin{aligned}(dh_{11} + 3\phi^1_r - h_{11}\phi^1_{1r}) \wedge \omega^1 \quad &+ \quad (dh_{10} - h_{11}\phi^1_r - h_{10}\phi) \wedge \theta \\ &+ \quad \{dh_1 + a(3\phi^1_c + h_{11}\phi^1_{1c} + 2h_{10}\omega^1)\} \wedge dt = 0.\end{aligned}$$

Therefore we can write

$$dh_{11} + 3\phi^1_r - h_{11}\phi^1_{1r} = h_{111}\omega^1 + h_{110}\theta + p_{11}dt \tag{2.8.1}$$

$$dh_{10} - h_{11}\phi^1_r - h_{10}\phi = h_{101}\omega^1 + h_{100}\theta + p_{10}dt \tag{2.8.2}$$

$$dh_1 + a(3\phi^1_c + h_{11}\phi^1_{1c} + 2h_{10}\omega^1) = p_{11}\omega^1 + p_{10}\theta + q_1 dt \tag{2.8.3}$$

with $h_{110} = h_{101}$. It follows from (2.8.1) and (2.8.2) that

$$\begin{aligned} dH &= dh_{10} + \frac{1}{3}h_{11}dh_{11} \\ &= H\phi + (\frac{1}{3}h_{111}h_1 + h_{101})\omega^1 + (\frac{1}{3}h_{110}h_{11} + h_{100})\theta + (\frac{1}{3}h_{11}p_{11} + p_{10})dt. \end{aligned} \tag{2.9}$$

That is to say,

$$d \log \mid H \mid \equiv \phi + H^{-1}(\frac{1}{3}h_{11}p_{11} + p_{10})dt (mod\ \omega^1, \theta). \tag{2.10}$$

By substituting (2.10) in (2.4), we obtain

$$\frac{\partial}{\partial t}(|H|^{\frac{3}{2}}\hat{\theta}\wedge\hat{\omega}^1)$$
$$= d_M\{|H|^{\frac{3}{2}}(a^0\hat{\omega}^1 - a^1\hat{\theta})\} + |H|^{\frac{3}{2}}\{\frac{3}{2}H^{-1}(\frac{1}{3}h_{11}p_{11} + p_{10}) - ah_{11}\}\hat{\theta}\wedge\hat{\omega}^1. \tag{2.11}$$

Now compute

$$\begin{aligned} A'(0) &= \frac{\partial}{\partial t}\int_\Omega |H|^{\frac{3}{2}}\hat{\theta}\wedge\hat{\omega}^1|_{t=0} \\ &= \int_\Omega\{\frac{3}{2}H^{-1}(\frac{1}{3}h_{11}p_{11} + p_{10}) - ah_{11}\}dA \end{aligned} \tag{2.12}$$

by (2.11) and Stokes' theorem (noting that $a^0 = a^1 = 0$ on $\partial\Omega$).

To deal with the first term in the above integrand, we observe for t =constant that after a straightforward computation

$$\begin{aligned} d\{|H|^{\frac{1}{2}}h_1(3\omega^1 - h_{11}\theta)\} &= \{|H|^{\frac{1}{2}}(h_{11}p_{11} + 3p_{10}) + \frac{1}{2}sign(H)|H|^{-\frac{1}{2}} \\ &\times (2h_{10}h_{111} + \frac{2}{3}h_{11}^2h_{111} + 2h_{11}h_{110} + 3h_{100})h_1 \\ &- a|H|^{\frac{1}{2}}(9h_{00} + 8h_{11}h_{10} + h_{11}^3)\}\theta\wedge\omega^1 \end{aligned} \tag{2.13}$$

by (2.9), (2.8.3) (with ϕ_c^1 and ϕ_{1c}^1 replaced by h_{ij} using (1.9.1) and (1.9.2)), (2.8.1) and structure equations (1.1), (1.8.1).

On the other hand, from (2.7) we get

$$a\phi_{1r}^1\wedge\theta + da\wedge\theta = h_1\omega^1\wedge\theta \text{ (note that } t = \text{constant).} \tag{2.14}$$

Multiplying both sides of (2.14) by $|H|^{\frac{1}{2}}f$ and integrating give rise to

$$\int_\Omega a\{d(|H|^{\frac{1}{2}}f) - \frac{3}{2}|H|^{\frac{1}{2}}f\phi\}\wedge\theta = \int_\Omega |H|^{\frac{1}{2}}fh_1\theta\wedge\omega^1. \tag{2.15}$$

Here we have used (1.1), (1.2) and Stokes' theorem with $a = 0$ on $\partial\Omega$. Applying (2.15) with $f = |H|^{-1}(h_{10}h_{111} + \frac{1}{3}h_{11}^2h_{111} + h_{11}h_{110} + \frac{3}{2}h_{100})$ and then substituting in the integration of (2.13), we finally reduce (2.12) to

$$A'(0) = \int_\Omega a\sum\wedge\theta \tag{2.16}$$

where the 1-form

$$\sum = -\frac{1}{2}d(|H|^{\frac{1}{2}}f) + \frac{3}{4}|H|^{\frac{1}{2}}f\phi - \frac{1}{2}sign(H)|H|^{\frac{1}{2}}(9h_{00} + 6h_{11}h_{10} + \frac{2}{3}h_{11}^3)\omega^1.$$

Since minimal surfaces are critical points of A, i.e., $A'(0) = 0$ for all a with $a = 0$ on $\partial\Omega$, the equation of minimal surfaces is therefore given by

$$\sum \wedge\theta = 0$$

which is a nonlinear fourth-order differential equation of the defining function for the surface.

In the appendix we do further reduction. Actually, we can choose suitable coframes such that $h_{11} = 0$ and h_{100} or $h_{111} = 0$.

Remark. Although we have the equation of minimal surfaces as above, we do not know any solution yet. Of course, the analogue of the Plateau problem can be asked.

3 Actual computation in the Heisenberg group

In this section we will carry our the actual computation for the CR-invariant area element $dA =\mid H \mid^{\frac{3}{2}} \theta \wedge \omega^1$ in the 3-dimensional Heisenberg group $\mathbb{H}^1$.

The usual coframe in $\mathbb{H}^1$ is given by

$$\theta = \frac{1}{2}dt + xdy - ydx \tag{3.1.1}$$

$$\theta_0^1 = dz, \quad z = x + iy \tag{3.1.2}$$

for $(t,x,y) \in \mathbb{H}^1$. An admissible unitary θ^1 (i.e. $d\theta = i\theta^1 \wedge \theta^{\bar{1}} mod\ \theta$) has the following form

$$\theta^1 = e^{i\lambda}dz + v^1\theta \tag{3.2}$$

for λ real. Now suppose our surface M is defined by $y = f(x,t)$. Set $\omega^2 = gd(y - f)$ with a function g to be determined. It is clear that $\omega^2 = 0$ on M. Take

$$\omega^1 = (\cos\lambda)dx - (sin\lambda)dy.$$

Then by (3.2), $\theta^1 = \omega^1 + i\omega^2$ is equivalent to the following equalities:

$$\begin{cases} \cos\lambda + xv_c^1 = g \\ \sin\lambda - yv_c^1 = -f_x \cdot g \\ v_c^1 = -2f_t \cdot g \text{ (and } v_r^1 = 0) \end{cases} \tag{3.3}$$

(f_x, f_t denote the partial derivatives of f in x, t respectively). From (3.3), we obtain

$$\cos\lambda = ga, \sin\lambda = gb \tag{3.4}$$

with $a = 1 + 2xf_t, b = -2yf_t - f_x$. So we must have

$$g = (a^2 + b^2)^{-\frac{1}{2}}$$

by assuming that $a^2 + b^2$ does not vanish. To satisfy (1.1), we choose

$$\phi = 2v_c^1 \omega^1 = -4f_t g^2 (adx - bdy). \tag{3.5}$$

Following §6 in [3], we can take the "first approximation" of ϕ_1^1, ϕ^1 satisfying (1.8.1), (1.8.2) as follows:

$$\begin{cases} {}_0\phi_{1c}^1 = -d\lambda + 3v_c^1 \omega^2 \\ {}_0\phi_r^1 = -v_c^1 d\lambda \\ {}_0\phi_c^1 = -dv_c^1 + 2(v_c^1)^2 \omega^1 \end{cases} \tag{3.6}$$

(note that $\phi_{1r}^1 = \frac{1}{2}\phi = v_c^1 \omega^1$ by (1.2)). According to [3]

$$\phi_{1c}^1 = {}_0\phi_{1c}^1 + D_{1c}^1 \theta \tag{3.7}$$

where D_{1c}^1 is determined by

$$\text{Imaginary part of } \{d({}_0\phi_1^1) - i\theta^{\bar{1}} \wedge ({}_0\phi^1) + 2i({}_0\phi^{\bar{1}}) \wedge \theta^1\} \equiv -8D_{1c}^1 \omega^1 \wedge \omega^2 (mod\ \theta). \tag{3.8}$$

Substituting (3.6) in (3.8) gives

$$D_{1c}^1 = (-\frac{3}{2})(v_c^1)^2. \tag{3.9}$$

Thus we obtain

$$\phi_{1c}^1 = -d\lambda + 3v_c^1 \omega^2 - \frac{3}{2}(v_c^1)^2 \theta \tag{3.10}$$

by (3.7), (3.6) and (3.9). On $M, \omega^2 = 0$. So h_{10} and h_{11} are determined by the following equality

$$-d\lambda - \frac{3}{2}(v_c^1)^2 \theta = h_{10}\theta + h_{11}\omega^1. \tag{3.11}$$

due to (1.9.1). Further computation goes on for the specific surface given by the graph

$$y = f(x,t) = [(1-t^2)^{\frac{1}{2}} - x^2]^{\frac{1}{2}} > 0 \tag{3.12}$$

over $\{t^2 + x^4 < 1\}$. The above surface is a portion of the following closed surface S:

$$(x^2 + y^2)^2 + t^2 = 1.$$

It is easy to see that there are only two characteristic points on S, namely $(0,0,1)$ and $(0,0,-1)$. For f given by (3.12) we have precise formulas for f_t, f_x:

$$f_t = -\frac{t}{2ys}, f_x = -\frac{x}{y}$$

where $s = x^2 + y^2$. Then we can easily compute a, b, and g. The result is as follows:

$$\begin{cases} a = 1 - \frac{xt}{ys} \\ b = \frac{t}{s} + \frac{x}{y} \\ g = ys^{\frac{1}{2}}. \end{cases}$$

Therefore by (3.4) and the above formulas, we get

$$\begin{cases} \cos\lambda = ga = (ys - xt)/s^{\frac{1}{2}} \\ \sin\lambda = gb = (yt + xs)/s^{\frac{1}{2}}. \end{cases} \tag{3.13}$$

Now noting that $ds = 2xdx + 2ydy$, we can compute

$$d\lambda = \frac{d\sin\lambda}{\cos\lambda} = \{(s + x^2 - xyts^{-1})dx + (t + xy - y^2ts^{-1})dy + ydt\}/(ys - xt). \tag{3.14}$$

Solving dt, dx, dy with $\theta, \omega^1, \omega^2$ previously given, we obtain

$$\begin{cases} dt = 2(\theta - xdy + ydx) \\ dx = (\cos\lambda)\omega^1 - (sin\lambda)v_c^1\theta \\ dy = -(\sin\lambda)\omega^1 - (\cos\lambda)v_c^1\theta \end{cases} \tag{3.15}$$

on the surface: $\omega^2 = 0$. Equating corresponding coefficients of both sides of (3.11) give

$$\begin{cases} h_{11} = (xyts^{-1} - s - x^2 - 2y^2)\cos\lambda + (t - xy - y^2ts^{-1})\sin\lambda \\ h_{10} = -2y + v_c^1\{(s + x^2 + 2y^2 - xyts^{-1})\sin\lambda + (t - xy - y^2ts^{-1})\cos\lambda\} - \frac{3}{2}(v_c^1)^2 \end{cases}$$

by (3.14) and (3.15). Expressing h_{11} and h_{10} in x, y, s, t by (3.13) and the formula for v_c^1 in (3.3), we get

$$\begin{cases} h_{11} = 3s^{\frac{1}{2}}(xt - ys) \\ h_{10} = \frac{1}{2}s^{-1}(6yst^2 + 6xts^2 - 2xt - 4ys - 3t^2). \end{cases} \tag{3.16}$$

Note that in the above computation we keep using $s^2 + t^2 = 1$ and y is implicitly given by (3.12).

Similarly for the graph contained in S and defined by $y < 0$, we still have our dA. By symmetry, dA is also well defined for $x > 0$ and $x < 0$. Altogether dA makes sense for everywhere on S except characteristic points $(0, 0, 1)$ and $(0, 0, -1)$. However dA vanishes wherever $H = 0$. And we do have points at which $H = 0$. For instance, H vanishes at $(x, y, t) = (\pm 1, 0, 0)$ and $(\pm\xi, 0, 0)$ for at least one ξ between 0 and 1. In fact, $H = t[\frac{3}{2}x^4t + 3x^3 - \frac{1}{x} - \frac{3}{2}\cdot\frac{t}{x^2}]$ for $y = 0$ by

(3.16) and continuity. It is interesting to give a complete description of the zero set of dA on S. Also near the characteristic point $(0,0,1)$, say, $h_{11} \sim 0, h_{10} \sim s^{-1}$. Hence $dA \sim s^{-2}y^{-1}dx \wedge dt$. (note that $y^{-1}dx \wedge dt = -x^{-1}dy \wedge dt$ whenever $xy \neq 0$) For $y > 0, y^{-1} \geq s^{-\frac{1}{2}}$. On the other hand, near $(0,0,1), s^{-\frac{5}{2}} \sim (1-t)^{-1-\frac{1}{4}}$ and it is easy to see that the integral

$$\int_0^\delta \int_{1-\varepsilon}^{\sqrt{1-x^4}} (1-t)^{-1-\frac{1}{4}} dt dx$$

diverges for any small $\delta, \varepsilon > 0$. Therefore the integral of dA over the whole surface S diverges.

A Appendix

It will be shown that by suitable choices of coframes we can make $h_{11} = 0$ and h_{100} or $h_{111} = 0$.

First we examine the change of h_{111} defined by (2.8.1) under (1.7). It is easy to see that from (1.3) we have

$$\tilde{\phi} = \phi - d\log u - 4\rho\omega^2 + s\theta \tag{A.1}$$

where $s = 4\mathbb{R}e(\tau\bar{t})$ (note that $t = \sqrt{u}e^{-i\lambda}$ with $\lambda = 2n\pi/3$ for some integer n). Since we learn $u\tilde{H} = H$ by (1.12.1), it follows from (A.1) that on M $(\omega^2 = 0)$,

$$\tilde{\phi} - d\log|\tilde{H}| = \phi - d\log|H| + s\theta. \tag{A.2}$$

On the way to search for the change of ϕ_r^1 under (1.7) via the formula (1.3), we obtain that on M,

$$\begin{aligned} & d\tilde{h}_{11} + 3\tilde{\phi}_r^1 - \tilde{h}_{11}\tilde{\phi}_{1r}^1 \\ = \;& u^{-\frac{1}{2}}(dh_{11} + 3\phi_r^1 - h_{11}\phi_{1r}^1) + \frac{3}{2}su^{-\frac{1}{2}}\omega^1 + (\rho s u^{-\frac{1}{2}} - \frac{1}{6}s\tilde{h}_{11})\theta \end{aligned} \tag{A.3}$$

by (1.2), (1.11.1) and (A.1) through a lengthy computation. Thus comparing (A.3) with (2.8.1) gives (t = constant)

$$2u\tilde{h}_{111} = 2h_{111} + 3s \tag{A.4.1}$$

$$2\sqrt{u}(u\tilde{h}_{110} + v_r^1\tilde{h}_{111}) = 2h_{110} - sh_{11} \tag{A.4.2}$$

by (1.7) and (1.12.1). So we can choose $s = -\frac{2}{3}h_{111}$ to annihilate the right-hand side of (A.4.1), i.e. to make $\tilde{h}_{111} = 0$.

Taking exterior differentiation of (2.6.2) gives

$$
\begin{aligned}
(dh_{01} - h_{11}\phi_r^1 &- h_{10}\phi) \wedge \omega^1 + (dh_{00} - 2h_{01}\phi_r^1 - 3h_{00}\phi_{1r}^1 + Q_c\omega^1) \wedge \theta \\
&+ \{dh_0 + a(h_{01}\phi_{1c}^1 + 2h_{00}\omega^1 + \frac{1}{2}\psi - Q_r\theta)\} \wedge dt = 0
\end{aligned}
$$

where $Q = Q_r + iQ_c$ with Q_r, Q_c real is called the Cartan (curvature) tensor (it is written as Q_{11} in [3]). Therefore we can write

$$dh_{01} - h_{11}\phi_r^1 - h_{10}\phi = h_{011}\omega^1 + h_{010}\theta + p_{01}dt \tag{A.5.1}$$

$$dh_{00} - 2h_{01}\phi_r^1 - 3h_{00}\phi_{1r}^1 + Q_c\omega^1 = h_{001}\omega^1 + h_{000}\theta + p_{00}dt \tag{A.5.2}$$

$$dh_0 + a(h_{01}\phi_{1c}^1 + 2h_{00}\omega^1 + \frac{1}{2}\psi - Q_r\theta) = p_{00}\theta + p_{01}\omega^1 + q_0 dt \tag{A.5.3}$$

with $h_{010} = h_{001}(= h_{100}$ and $p_{01} = p_{10}$ also by (2.8.2) with $h_{10} = h_{01})$. Now from (1.11.1) we can choose $v_r^1 = -\frac{1}{3}\sqrt{u}h_{11}$ such that $\widetilde{h}_{11} = 0$. Suppose this be done, i.e., suppose $h_{11} = 0$. Then ϕ_r^1 and ϕ_c^1 transform as follows:

$$\sqrt{u}\widetilde{\phi}_r^1 = \phi_r^1 + \frac{1}{2}s\omega^1 \tag{A.6.1}$$

$$\sqrt{u}\widetilde{\phi}_c^1 = \phi_c^1 \tag{A.6.2}$$

according to (A.3) and the second equality of (1.10). Hence

$$h_{00} = \widetilde{h}_{00}u^{\frac{3}{2}} \tag{A.7}$$

by (A.6.2). On the other hand, h_{110} changes "tensorially ":

$$u^{\frac{3}{2}}\widetilde{h}_{110} = h_{110} \tag{A.8}$$

due to (A.4.2) since $h_{11} = v_r^1 = 0$ by assumption. Also we can write

$$3\phi_r^1 = h_{111}\omega^1 + h_{110}\theta \tag{A.9}$$

by (2.8.1) (t = constant). Now a direct computation starting with exterior differentiation of (A.7) gives

$$u^2\widetilde{h}_{001} = h_{001} - h_{10}s \tag{A.10.1}$$

$$u^{\frac{5}{2}}\widetilde{h}_{000} = h_{000} - \frac{3}{2}h_{00}s \tag{A.10.2}$$

by (A.5.2), (A.2), (A.9), (A.8), (A.4.1) and noting that $u\widetilde{h}_{10} = h_{10}$ and $Q_c = u^2\widetilde{Q}_c$. It is clear now that from (A.10.1) and (A.10.2) we can choose s to make either right-hand side vanish (by assuming $h_{10} \neq 0$ or $h_{00} \neq 0$), i.e., $\widetilde{h}_{100} = \widetilde{h}_{001} = 0$ or $\widetilde{h}_{000} = 0$.

Note that there are 3 degrees of freedom for us to choose admissible coframes $(\theta, \omega^1, \omega^2, \phi)$ subject to the equation (1.1) under (1.7) and (A.1), namely, u, v_r^1 and s. After choosing v_r^1 and s to make $h_{11} = 0$ and h_{111} or h_{100} or $h_{000} = 0$ respectively, we have only one degree of freedom, u, left over.

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